SUPPORTING STUDENTS' CONCEPTIONS OF ALGEBRAIC EQUATIONS AND EXPRESSIONS USING REALISTIC MATHEMATICS EDUCATION (RME) DESIGN THEORY

by

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ABSTRACT

MELISSA KIMEL MILLER. Supporting students' conceptions of algebraic equations and expressions using realistic mathematics (RME) design theory. (Under the direction of DR. MICHELLE STEPHAN.)

This study investigated ways to support students' development of mathematical meaning making as they learned to simplify and solve algebraic expressions and equations. The literature addresses the occurrence of specific conceptions by students of all ages through prior research (Blanton, 2008; Blanton & Kaput, 2005; Herscovics & Linchevski, 1994; Kaput, 2008; Kieran, 1992; Linchevski & Herscovics, 1996; NCTM< 2000; Philipp, 1992; Seng, 2010; Sfard, 1991; Underwood Gregg & Yackel, 2002; Warren & English, 2000). This qualitative design research study explored the use of an instructional design theory, Realistic Mathematics Education (RME) with the implementation of *The Candy* Shop instructional sequence by Underwood Gregg & Yackel (2002) to support students' learning of this content area. The overall goal of the study was to observe what classroom mathematical practices emerged to support students' in their initial conceptualizations of these algebraic concepts and use those practices to make revisions to the instructional sequence.

DEDICATION

There are many people in my life that without their support and love during this process I would not have been able to complete this milestone.

First and foremost, I would like thank God for giving me the abilities needed to be able to do this.

To my husband, Daniel, who made many sacrifices as I spent multiple days and nights at the computer. Thank you for your continuous love, support, and encouragement along the way.

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LIST OF ACRONYMS/ABBREVIATIONS

CHLT	Classroom Hypothetical Learning Trajectory
CTE	Classroom Teaching Experiment
HLT	Hypothetical Learning Trajectory
IRB	Institutional Review Board
RME	Realistic Mathematics Education
SWE	System(s)-Wide Experiment
TDE	Teacher Development Experiment
TE	Teaching Experiment

CHAPTER ONE: INTRODUCTION

Furner, Yahya and Duffy (2005) argue that it is becoming imperative for all citizens to be confident in their mathematics abilities; for knowledge of mathematical skills are necessities for success in today's world. It is understood that students gain more understanding of the content when they are actively engaged in the subject matter. Mathematics, exposing new abstract perspectives into the world, can be a very strict game with strange regulations about what is allowed and what is not. The Jasper Project (Cognition and Technology Group at Vanderbilt, 1990; 1997) was one of the earliest large actions in the United States to "fix" mathematics instruction within real-life context, utilizing technology. Jasper included video-based adventures, in which the students actively participate in solving problems from the videos. The results of the Jasper project revealed that it was not adequate for the students to only watch the video; the most significant experience for the students is to be involved in finding the solution. Even with the increasing emphasis of technology in the field of algebra, students' comprehension and capability of manipulating variables and basic algebra concepts in a necessity (Kieran, 1996). In order for a child to gain access into this new subculture, of transferring from everyday knowledge and communication to a more abstract mathematical understanding, they must be engaged in active participation (Cobb & Bauersfeld, 1995).

Statement of the Problem

Algebra is one of the most important topics in mathematics, as the concepts within algebra are applicable in many existing fields of mathematics and science (Seng, 2010). Much research has been compiled on how students struggle with grasping algebraic concepts and the concept of how to define and utilize variables (Blanton, 2008; Blanton & Kaput, 2005; Kaput, 2008; Philipp, 1992) when simplifying and solving algebraic expressions and equations (Blanton, 2008; Blanton & Kaput, 2005; Herscovics & Linchevski, 1994; Kieran, 1992; Linchevski & Herscovics, 1996; NCTM, 2000; Seng, 2010; Sfard,1991; Underwood Gregg & Yackel, 2002; Warren & English, 2000). The majority of research completed before the 1990's focused just on the students that were moving from arithmetic into algebra in search of the learning of concepts that strengthen student success in algebra, including unknowns and variables (NCTM, 2007).

As students transition from arithmetical to algebraic thinking they struggle to move from procedural to structural conceptualizations as well. Most students initially reason procedurally, executing arithmetical operations on numbers, because they lack structural conceptions (Kieran, 1992). The textbook trend of building students' procedural, rather than structural, conceptions when implementing instruction (Eisenberg & Dreyfus, 1988) often results in students' misinterpreting algebraic conceptions. The transition from procedural to structural reasoning takes time and much effort; nevertheless it is beneficial for students to have a thorough understanding of the algebraic ideas.

The instruction of algebraic concepts is often implemented straight from textbooks that capitalize on procedural approaches, presenting exercises that lack realistic context and focus on arithmetical operations, such as substituting numbers in expressions. Limited studies have been dedicated to finding ways of facilitating the transition from procedural to structural conceptions (Kieran, 1992), or implementation of real-world situations to support students' application and conceptualization of the mathematics content. This is especially important for the elementary and middle school students as they build their prior knowledge to advance to higher levels of math. Some studies have established that there are prerequisites within elementary mathematics to provide foundations for algebraic development, emphasizing a significant need for more research to be attentive to integrating both procedural and structural conceptions in elementary settings (Blanton & Kaput, 2003; Slavitt, 1999).

Because there were many fundamentals that I had limited or no experience with prior to the study, there were recognizable challenges that I encountered. I had never participated in a study as 1) both the teacher and the researcher, 2) part of the research team, or 3) the facilitator of inquiry lessons while working with norms. These challenges were analyzed and are discussed in detail in Chapter Four.

Significance of Study

This study was created to add to the limited research documenting students' development of algebraic conceptualizations through real-world implementation. The purpose of this research is to observe how students conceptualize algebraic expressions, equations, and variables, through the support of real-world context such as in the implementation of *The Candy Shop Sequence*, an instructional sequence designed using Realistic Mathematics Education (RME) theory. The Design Research methodology used in this study investigated possible educational improvements within the learning that took place in the classroom.

There is limited research on supporting young students' algebraic learning with the implementation of a RME model. Many students experience difficulty when moving from regular arithmetic to algebra, not having the opportunity to grasp full comprehension or make connections (Warren & English, 2000). In the field of education, prior research is presented through what would be considered a traditional instructional method. However, "RME designers view learning as situations within a classroom activity system" (Gravemejier, Bowers, & Stephan, 2003). The basis of Realistic Mathematics Education (RME) is the structure of instructional materials should take place as students "reinvent conventional mathematics" (Gravemeijer, 2004). RME allows students to work from their prior knowledge and build schema as they mathematize contextual situations. Students are then given opportunities to discover and construct knowledge of the content.

This study was a classroom teaching experiment conducted with fifth grade students. The class met for approximately 45-60 minutes for each of the sixteen class sessions.

The Classroom Teaching Experiment (CTE) assisted in the goal of supporting students' learning in a content-specific domain, developing theories on how students learn independently and in a collective group while engaging in instructional activities that guide this process (Cobb et al., 2003; Rasmussen & Stephan, 2008). Analyzing how the mathematical concept of algebraic expressions and equations were elaborated in the mathematical textbooks, the struggles that students' encountered within the algebra content area, and reflecting on how and what the students should learn informed a hypothetical learning trajectory (HLT). The results of these analyses were used to form short-term mathematical goals to structure a course of learning, as well as choosing instructional tasks from *The Candy Shop*, and altering them to correspond with the fifth grade class that will be participating in the study.

Focus of Research

The focus of this study was to document the collective learning of the classroom community as they engage in the *Candy Shop* sequence. The emergent perspective, which characterizes learning as both social and individual processes (Cobb & Yackel, 1996; Stephan & Cobb, 2003), served as the interpretative framework for this study's data.

The social perspective, including 1) classroom social norms, 2) sociomathematical norms, and 3) classroom mathematical practices was documented through analysis of the video-taped classroom sessions and field notes made by the research team. Ways in which the implementation of the RME sequence supports classroom mathematical practices was determined through discussions between students and me, as the teacher, in whole-group setting, as well as students' work samples. Students' contributions to the classroom activities and their restructuring of knowledge, as they participate in these activities and the whole group discussions, were analyzed through their classroom discussions and work. For this purpose, this study addressed the following research question:

What are the classroom mathematical practices that emerge as students engage in The Candy Shop sequence?

The RME model has a considerable impact on mathematics education in the Netherlands. When RME is implemented the students are not given ready-made models that represent particular mathematical concepts. They are confronted with context problems presented in such a way that they extract modeling activities, which in turn lead to the emergence of models (Van den Heuvel-Panhuizen, 2003). A goal of this research was to "design an insightful instructional sequence (because) students should be given the opportunity to ground their understanding in their own informal knowledge" (Gravemeijer & Doorman, 1999, p. 115). For example, the *Candy Shop* instructional sequence approaches the ideas of algebraic expressions, equations and variables by building onto students' prior knowledge of basic arithmetic skills and operations.

Chapter Two presents a synthesis of research, including an overview of algebra as generalized arithmetic. This is followed by a discussion of algebraic reasoning, variables, and operations and equality. The chapter concludes with an explanation of the need for more research and possible implications for this particular study.

Chapter Three includes the research methodology using a diagram that is introduced initially in the chapter that maps out the structure of Design Research and the route that will specifically be used for this study. The chapter entails the dissection of Design Research methodology, highlighting the various approaches that could and were taken for research. Next is a comprehensive discussion of Realistic Mathematics Education (RME) as the instructional design theory, including its significant heuristics and components, as well as how they are achieved in this study. This is followed by a discussion of the theoretical framework, interpretive framework, and instructional sequence that was implemented in this study. The chapter concludes with how data was collected and analyzed.

CHAPTER TWO: LITERATURE REVIEW

The methods for solving and simplifying algebraic equations and expressions can be challenging mathematical procedures to comprehend. Algebraic procedures are traditionally implemented through memorizing rules and applying methods of arithmetic to solve algebra problems efficiently (Kieran, 1992). Several research studies have conveyed that when students are presented algebra through procedural processes, their knowledge of algebra is disjointed and inaccurate (Kieran, 1992, 2007; NCTM, 2000; Underwood Gregg & Yackel, 2002). Therefore, it is necessary for research to develop an inclusive algebra curriculum that will support students' conceptual understanding of many algebraic concepts, which many have claimed to have already developed.

Though more research is needed with students of all ages, this study seeks to add to the research of upper-elementary students' development of algebraic expressions and variables through a classroom teaching experiment using an instructional sequence designed under the principles of Realistic Mathematics Education (RME). Research with elementary students is limited. Research primarily focuses on secondary students in algebra courses (Warren, 2003), rather than supporting entry level students' conceptualizations though real-world instructional sequences. Exploring the role of real-world instructional sequences in supporting students' conceptual learning and development of algebraic expressions and variables is therefore a viable contribution to research in algebra.

Much of the research conducted before the 1990s focused mainly on the transitions essential for thirteen and fourteen year olds to shift from arithmetic to algebra (Kieran, 1992). The persistent development of this large body of research has studied the learning of concepts (i.e. unknowns and variables, expressions and equations, and the development of the definition given to the equal and minus signs) that support students' achievement in algebra (Kieran, 2007).

Prior research has also found that many difficulties that arise with algebraic concepts appear to be related to the textbooks that teachers are using for implementation (Kieran, 1992; Seng, 2010). Before the insufficiencies in the textbooks can be eradicated, there is a need for a more inclusive awareness of how students' conceptualizations of algebra can be supported through more effective ways of learning.

This chapter includes an overview of the three unique components of algebra: 1) generalization, 2) equality and operations, and 3) thinking about unknown quantities that are noticeable features of algebraic thinking that can be found in elementary mathematics instruction. These components offer a basis for identifying how fifth grade students reason algebraically and if the characteristics of word problems provide opportunities for students to interpret them algebraically. The chapter starts with an overview of algebra, as generalized arithmetic. This is then followed by a discussion of algebraic thinking as procedural-structural conceptions. The next section continues with an explanation of variables, highlighting common mistakes that students make while learning how to manipulate variables, and some implications for instruction of variables in the classroom. Following is a section that informs about common misinterpretations of operations and equality; offering implications for implementation of instruction that could support students' conceptualizations in algebraic expressions and variables.

Algebra: Generalized Arithmetic

School algebra is many times referred to as generalized arithmetic (Blanton, 2008; Booth, 1988; Philipp, 1992), or symbolizing general statements through arithmetical regulations and operations (Blanton, 2008; Kieran, 1992). For example a child may first learn that "any time you add zero to a number you will get the same number back" as they work through different problems, such as 2 apples plus 0 apples equals 2 apples. As the students' mathematical understandings advance, they may learn how to express this same generalization in a symbolic way, such as "$a + 0 = a$, where a is any real number" (Blanton, 2008, p.3).

According to Blanton (2008) "*generalized arithmetic* refers to building generalizations about operations on and properties of numbers" (p. 5). Through this, and students' experiences with adding, subtracting, multiplying and dividing, students are able to generalize significant mathematical ideas and begin to notice systems of number behaviors and how operations affect the numbers (Blanton, 2008). Algebra, considered a strand of content that begins as early as pre-kindergarten and continues through grade twelve, is often conceived as manipulating letters and symbols to solve equations or to simplify algebraic expressions (NCTM, 2000). However, algebra and algebraic thinking is much more (Blanton, 2008).

Consistent with the NCTM *Principles and Standards for School Mathematics* (2000) which outlines the content and processes guiding teaching practices, algebra and algebraic thinking is acknowledged as one of the main features of elementary school mathematics. The ability to construct, communicate and clarify relationships, or generalizations, of mathematics are essentials of algebra (Kaput, 2008). When taking into consideration the role of algebra in elementary grades, a more dynamic perspective of algebra thinking is necessary. Blanton and Kaput (2005) state algebraic reasoning is a "process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse, and express them in increasingly formal and age-appropriate ways (Kaput, 1995; 1999)" (p. 413).

According to NCTM (2000) the transition from arithmetic to algebraic reasoning has proven to be a struggle for students, which is why it is acknowledged that students need to be given opportunities to engage in algebra earlier on in their education (Driscoll, 1999; Kaput, 1998, 1999; NCTM, 1998; 2000; 2007). These experiences should challenge instruction to support students' transition to formal algebra by exploiting algebraic symbols used by students while constructing comprehension of patterns and functions, the ability to represent and analyze mathematical situations, using mathematical models and analyzing change in various contexts (Blanton, 2008; Lannin, 2003; NCTM, 2000). The objective for beginning algebra students, in this required transition, is to make the shift from finding numerical answers to engaging in tasks that promote reasoning about numerical relations in given situations discussing them explicitly in everyday language with the ultimate goal of learning how to represent these relations with letters (Booth, 1988; Herscovics & Linchevski, 1994; Warren, 2003; Warren & English, 2000). When shifting through this transformation two crucial characteristics to consider are: 1) using letters to symbolize numbers and 2) being aware of the mathematical method being symbolized by the use of numbers and letters (Kieran, 1992), which are addressed in the following sections.

Constructing generalizations in arithmetic can help students generate strong associations between the mathematical content components of number, operation, and algebra (Blanton, 2008; Lannin, 2003). These generalization tasks (Lannin, 2003) along with having opportunities to discuss and write about the relationships among quantities (Blanton, 2008) could be the key to supporting students' comprehensive knowledge of algebraic concepts.

There is an obvious gap in the transition between arithmetic and algebra. Although many significant efforts have been made to support student progress there is still not a definite distinction between arithmetic and algebra (Herscovics & Linchevski, 1994). Local and national standards have pushed "algebra for all" as they have seen the need for algebraic concepts to be introduced at an earlier age. Recently there have been more attempts in researching issues of what Cai and Knuth (2011) identify as "algebraization" in elementary grades. *Algebraization* is the essence of basic thinking, associated with the conceptual areas of algebra within elementary and middle school mathematics where the structure of ideas takes place. Algebraic thinking in the earlier grades must require the development of specific ways of thinking, exploiting innovative practices of the mind, as well as new tools, to comprehend mathematical relationships. Since a vast amount of curricula, professional development, and instructional practices, that are intended to support the development of students' algebraic thinking, depend on what has already been acknowledged about students' algebraic thinking and its development, it is critical to examine issues related to students' comprehension in algebra learning (Cai & Knuth, 2011). Therefore there is a need for instructional practices that require content within the identified areas of difficulties in algebraic thinking to support students' doing, thinking, discussing and interacting. Investigating the application of this type of instructional practice as well as observing how it supports students' conceptualization of algebra could lead to ideas of reform for this area of mathematics concepts.

Algebraic Reasoning

Sfard (1991) noted that abstract mathematical ideas are conceived in two primarily different ways, operationally (as processes) or structurally (as objects); double understandings of algebraic expressions that generate critical problems for numerous students (Seng, 2010). Sfard (1991) characterizes these two conceptions in the following:

> "There is a deep ontological gap between operational and structural conceptions… seeing mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. In contrast interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed." (p.4)

Sfard (1991) asserts that when initially acquiring new mathematical skills, many start with the operational, also known as the procedural (Kieran, 1992), conception and later shift to the structural conception. This transition is neither immediate nor effortless; however, it is beneficial to have deep understanding of the mathematical ideas in these different ways, as both conceptions have significant functions in mathematical activities. Although it will improve their conceptualizations of understanding complex algebra ideas, it does not necessarily mean that their overall thinking has become sophisticated; they are able to manipulate algebraic ideas more fluently (Sfard & Linchevski, 1994).

Mathematics textbooks are renowned for introducing algebraic ideas giving the impression that only procedural approaches are adequate for solving. They typically present exercises involving numerical substitution in algebraic expressions and other arithmetic methods for solving algebraic equations. Kieran (1992) states that these *procedural* approaches refer to arithmetical operations executed on numbers to generate numbers, such as replacing the variables in the algebraic expression $2a + b$, as $a = 3$ and $b = 8$, to get the result of 14. This, in essence, compels students to use procedural interpretations rather than the structural ones necessary (Kieran, 1992) causing them many times to circumvent the algebraic symbolism (Kieran, 1992; Underwood Gregg & Yackel, 2002).

There is much evidence that students' structural conceptions are lacking (Kieran, 1992). The NCTM Algebra Working Group (NCTM, 1998) and the Early Algebra Group (Kaput, Carraher, & Blanton, 2008) show efforts of researchers collaborating to develop methods to incorporate algebraic reasoning in grades K-12 with the overall objective to emphasize this shift from discovering rules for symbol manipulation toward developing algebraic reasoning in contrast to forcing existing high school curriculum onto the lower grade levels (Jacobs et al., 2007). However, students are posed with the task of simplifying expressions almost immediately in their algebra courses. Even though these activities begin with simple expressions that can be related to initial numerical, procedural conceptions they cannot continue at this novice level. As the nature of the simplification become more complex, the tasks are impossible for students to perform unless they are able to develop the knowledge that the expression is to be perceived as a mathematical object (Kieran, 1992).

The trend in algebra to present procedural conceptions rather than structural, often results in many students considering algebra as a conspectus of algorithms (Eisenberg & Dreyfus, 1988), which hinders their process of generalizing and applying their knowledge (Kieran, 1992). Two themes that have emerged while research has greatly focused on the learning aspect of algebra are: 1) the impartiality of procedural interpretations over structural and 2) the difficulty in acquiring a structural conception of algebra (Kieran, 1992). In order for procedural conceptions to be transformed into structural conceptions, it is essential to allow extended experiences (Sfard, 1991). It is also imperative for students not to eliminate procedural conceptions when they acquire structural conceptions, such as considering and operating on expressions, equations and function as objects. Procedural conceptions are essential building blocks for the construction of structural conceptions, yet both conceptions have significant roles in mathematical activities (Sfard, 1991; Kieran, 1992). Sfard (1991) labels the great ontological jump that one makes between procedural and structural conceptions as reification, or “the act of turning computational operations into permanent object-like entities.” For example, when one perceives x + 4 procedurally, they see this as the process of adding 4 more to the number x; however, if they use structural conceptions they may now see x + 4 as one object, understanding that it can be used in a variety of different ways.

Kieran’s (1992) research findings show that a majority of students, “at worst memorize a pseudo-structural content; at best, develop and continue to rely on procedure conceptions” (p.412), not fully reaching the structural piece of algebra’s procedural-structural cycle. Evidence in research implies more effort is required in classroom instruction, devoting additional time with procedural conceptions to establish a foundation for developing structural algebraic conceptions. Research also specifies that since procedural conceptions are attainable, it would be beneficial to utilize these conceptions to create comprehensive algebraic activities. There appears to be a need for further, more thorough, reflection of the transition from procedural to structural conception than what currently exists in textbooks; however, there have been few studies dedicated to finding ways of facilitating this transition (Kieran, 1992).

Variables

One of the most essential concepts in mathematics to understand in a meaningful way is variables (Davis 1964; Hirsh & Lappan, 1989; Philipp, 1992), particularly used as representations for learning algebra and being successful in higher mathematics (Philipp, 1992; Pollack, 2012; National Mathematics Advisory Panel [NMAP], 2008; Rosnick, 1982; Schoenfeld & Arcavi, 1999). However, many times students misinterpret the true symbolization of a variable (i.e. quantity, abbreviation, etc.) when applying the variables in algebraic mathematics (Pollack, 2012).

As mathematical rules continue to be unchanged and used inappropriately this often leads to the introduction of new rules being misconstrued. Students that use the procedural conception when learning new concepts in mathematics, such as algebraic expressions, often encounter difficulties as they prefer to apply arithmetical rules (i.e. finding the solution) that they feel are relevant (Seng, 2010). Students that approach an algebra expression, that indicates both the process of adding the terms and an object, frequently comprehend the expression to only be performed as process. For example, in arithmetic the equation, 2 + 4 can be replaced by 6; however, several researchers (Crowley, Thomas & Tall, 1994; Davis, et al., 1978; Sfard, 1995) have revealed that students struggle with the structural conception to accept that symbols in an algebraic expression, such as $5x + 9$, as mental objects, or processes that cannot be "completed" because they do not know what x represents (Chalouh & Herscovics, 1983; Tirosh, Even & Robinson, 1998). In this expression, $5x + 9$, students want to add the two terms together for the result of $13x$. Even though this answer is still considered an open expression that represents the multiplication operation, students feel this answer is "complete" and that there is no need for further simplification (Tirosh, Even & Robinson, 1998).

Underwood Gregg and Yackel (2002) proposed that students are not able to make sense of the processes they are using because they do not conceptualize algebraic expressions as *composite units*. They refer to Steffe (1992) to explain that as the notion of composite units is acquired by a child then they are able to manage units of various levels. Underwood Gregg and Yackel (2002) explain their idea with the following example:

> "the student can treat a number, such as 23, as a single unit made up of 23 ones or as a group of 23 individual units and can move back and forth between these conceptions and coordinate them in flexible ways. He or she can think of a group of 23 combined with a group of 25 as two groups of 20 and 8 more or, possibly, as 2 fewer than two groups of 25. This type of flexibility is required to understand place-value numeration. In the same way, conceptualizing an unknown as a composite allows a student to think of $x + x$ as $2x$, where $2x$ is the result of counting x units, followed by counting x units again. That is, 1, 2, 3, …, x, $x + 1$, $x +2$, …, $x + x$. To continue the analogy, the algebraic expression $3x – 2$ can, depending on one's current needs, be viewed as a single composite unit, as $3x – 2$ individual units, as 2 fewer than 3 units of size x, or as 2 units of size x and

> 2 fewer than another unit of size *x*. In this sense, the conceptualization of algebraic expressions is similar to conceptions involved in place-value numeration"
> (p. 492).

As students advance in their mathematical thinking, they are able to shift from using natural language descriptors, such as "the number of apples," to simplifying their thoughts into symbolic language (i.e. using the variable *a* for the number of apples). Classroom studies have found this shift to using letters, and other non-literal symbols, as representations of unknown or varying quantities usually happens during second or third grade, or sometimes earlier. However, although students are capable of using letters to symbolize unknown quantities this does not indicate that they have full comprehension of variables (Blanton, 2008). Students may initially identify unknown values of literal symbols (i.e., *x*) or non-literal symbols (i.e.,◇) in equations (such as $9 + 4 = x + 2$, or $5 + \diamond = 12$), however unfolding ideas of variables as quantities, which assumes a range of values (such as the number of apples), is difficult as the students construct the concept of variable (Blanton, 2008). Gaining full comprehension of a variable, explained further in the following, is not simple (Blanton, 2008); time and experience are needed for one to build a deep comprehension of this concept (NCTM, 2000).

Some research affirms that the multiple definitions and operations that variables encompass within the world of mathematics may be partially liable for the difficulties that students encounter with the concept of a variable (Philipp, 1992; Rosnick, 1982; Schoenfeld & Arcavi, 1999; Wagner, 1983). Dolciani et al. (1967) states that within the last thirty years, textbooks characteristically define a variable as "a symbol which may represent any of the members of a specified set, called the replacement set or domain of the variable" (Philipp, 1992, p. 557), while Chalouh and Herscovics (1988) state that quite often a variable is introduced as “a letter that stands for one or more numbers” (p.33). The definition that Chalouh and Herscovics (1988) offer limits the definition of variable as numbers being symbolized by only letters while the textbook definition (Dociani et al.,1967) has no limit on symbols or what they represent. Even though these definitions may be acceptable for educators, students do not have opportunities to make sense of the diversity of the meaning of a variable (Chalouh & Herscovics, 1988). Many mathematicians, mathematics educators, and textbook writers institute definitions that corrupt the true meaning of a variable by including insufficient or profuse vague information, as these two above. It would be nearly impossible to create a “perfect” definition of variable that carries one meaning and includes all of a variable’s uses. Therefore, it is crucial that math educators understand the many ways that variables are used in mathematical contexts and allow their students the opportunities to consider the assorted functions (Philipp, 1992). Typically a variable is indicated by a literal symbol, a letter from the alphabet, which represents a numerical value or other algebraic objects.

Many students struggle with distinguishing the multiple ways in which letters operate as variables in algebra (Kieran, 1992). For example, literal symbols can be manipulated in various ways throughout a range of algebraic equations. Philipp (1992) uses the equation $C = kg$ as an example of utilizing literal symbols as constants, varying quantities and parameters to illustrate the process of purchasing gasoline for an automobile. In this equation, C and g are varying quantities that change together, where C is the cost of gasoline given the number of liters pumped, g. Both of these varying quantities are dependent on the variable k, the parameter, determined at the outset by choosing the specific pump at a certain gas station, being placed for the particular problem. This example shows how letters can be so diverse and have many concepts to apply in just one equation. In addition to how variables are applied in Philipp's (1992) given equation, literal symbols are also used as "unknowns ($x + 8 = 19$), as labels ($3f = 1y$, where f represents "feet" and y represents "yards"), as generalized numbers ($2x + 3x = 5x$), and as abstract symbols in mathematical systems ($e * a = a$, where e is an identity for the operation *)" (Philipp, 1992, p. 558). In algebra, it is essential for the student to know how to identify and utilize the diverse roles of a variable for the given circumstances, as well as the teacher to know the common mistakes made by students in order to find ways to support the students' understandings.

An unknown establishes the goal of solving an equation to find the value(s) of x to make the equation valid (Philipp, 1992). For example to solve the problem $x^2 – 4 = 16$, the two true values of x would have to be determined. Many students may initially think the problem cannot be "solved" (referring to their prior understanding of arithmetic equations). However, many students are successful expressing unknowns by assigning values to the variable, constructing tables (Blanton, 2008), or even using inverse operations. Functional thinking, an essential way to integrate early algebra, initiates students' comprehension of unique relationships and correspondence between two or more quantities through creating tables and graphs. Functional thinking is "a process of building, describing, and reasoning with and about functions" (Blanton, 2008, p. 31). For example, if students are trying to figure out how many eyes that seven dogs have they can create a function table where they place a dependent variable (in this case, the number of eyes) and an independent variable (in this case, the number of dogs) and use this to construct a computation (i.e. $E = 2 \text{ x } n$ or $E = 2n$, where E is the number of eyes and n is the number of dogs). This computation will allow students to determine the value of a quantity at any point without knowing previous values. There are two steps that educators and students use to create these tables, using recursive patterns and finding correspondence between the quantities. Using recursive patterns involves looking for relationships within a sequence of values, such as focusing on the number of eyes to figure out the pattern is doubling, or counting by two (i.e. 2,4, 6…) to find the correspondence between the quantities, or thinking about the operation needing to be applied on the independent variable to acquire the dependent variable (i.e. 1 dog acquires 2 eyes, 2 dogs acquires 4 eyes, etc…. seeing that the pattern is to double, or multiply the independent variable by 2 to acquire the dependent variable). The function tables are used as tools, aiding students' organization of information about various quantities verifying significant relationships (Blanton, 2008). The representation that is used for the expression of the functional relationship helps clarify procedural interpretations (Kieran, 1992); allowing students to construct their knowledge of variable use as they observe algebraic notions.

Labels are variables used as specific letters to represent a specific object, such as in the formula for area (l x w = area) where l = length and w = width (Philipp, 1992). Students have mistaken variables as qualitative symbols, using variables as abbreviations rather than quantities. When approaching a problem, such as $4c + 3d$, stating the problem as "4 cats + 3 dogs" may not supports students' conceptualization of variables. It not only promotes an invalid view of the meaning of the letters but also gives students justification for simplifying the expression to $7cd$. Students tend to rationalize that when you add 4 cats and 3 dogs then it is equivalent to having (4 + 3 =) 7 cats *and* dogs, therefore they do not comprehend why the expression $4c + 3d$ cannot be further simplified (Booth, 1988). Another example that originates from students' arithmetical background, is considering the letters as representations of units, such as *l* for liters or *m* for meters, rather than the number of liters or meters (Booth, 1988).

Generalized numbers denote identity properties that allow specific literal symbols to be put together and replaced by another value, producing a true statement. For example, when simplifying the expression $2t + 3t – 9$, it is understood that the identity of $2t + 3t$ can be conjoined into $5t$; therefore, creating a new true statement of $5t – 9$ (Philipp, 1992).

Many children cannot comprehend the notion of a letter as a variable (Booth, 1988) and are apt to interpret letters as having specific numbers with independent values (Collis, 1975; Küchemann, 1981). For example, some students identify a variable's value as its location in the alphabet (i.e., $a = 1$, $b = 2$, and $z =26$) or always being equal to the same number (i.e., *x* is always 5). In a study carried out by Küchemann (1978, 1981), observing students' interpretations of literal symbols, he discovered that although the interpretation that students chose to use was partially contingent with the disposition and difficulty of the question a small percentage of the 13 through 15 year olds assessed were able to regard the letter as a generalized number. Küchemann's identified and categorized six ways that students interpreted letters in the context of arithmetic problems, using Collis's (1975) original classifications. The six characterizations included:

1) Letter Evaluated – "the letter is assigned a numerical value from the outset" (Kieran, 1992, p. 396). For example, the expression $3x + 1$ would be calculated by the *x* being equal to some random number chosen by the student,

resulting an answer that is dependent on the student's choice of number to plug in for the variable.

2) Letter not considered – "the letter is ignored or its existences is acknowledged without giving it a meaning" (Kieran, 1992, p. 396). For example, if the student was given the computation $2x + 4y$, the student would solve this as $6xy$ not understanding that the two different terms could not be combined through addition.

3) Letter considered as a concrete object – "the letter is regarded as shorthand for a concrete object or as a concrete object in its own right" (Kieran, 1992, p. 396). An example of this would be variables in $8a + 4b$ could refer to 8 apples and 4 bananas when the literal translation should be 'eight times the number of apples plus four times the number of bananas.'

4) Letter considered as a specific unknown – "the letter is regarded as a specific unknown number" (Kieran, 1992, p. 396). With this conception, students perceive each variable as a distinct unknown number, such as in the equation $5x – 3 = 7$, the variable x represents the single digit (10) that corresponds with the equation.

5) Letter considered as a generalized number – "the letter is seen as representing, or at least as being able to take on, several values rather than just one" (Kieran, 1992, p. 396). A model for this conception would be inequality, such as $x > 9$, where x denotes a set of numbers greater than nine.

6) Letter considered as a variable – "the letter is seen as representing a range of unspecified values and a systematic relationship is seen to exist between two

> such sets of values" (Kieran, 1992, p. 396). Linear equations, such as $x = 3y$, illustrate this conception; contingent on the value of the variable y there could be an unlimited quantity of values for x.

Küchemann (1978; 1981) discovered that few students were able to interpret letters as variables. A greater number of students were able to interpret letters as unknowns. However, a majority of these students considered letters as concrete objects or simply disregarded them. These findings corresponded with Sfard's (1991) theorized evolutionary development of generating structural conceptions of algebra expressions (more detail in the section on Algebra Expressions), implying several of the students evaluated were not yet proficient in recognizing literal expressions as numerical input-output procedures (Kieran, 1992).

Pollack (2012) suggests that defining a variable and the role it plays in given problems demands attention when evaluating the types of algebra problems to motivate algebra students. Many times the operations of a variable are intertwined, as in Philipp (1992) example of purchasing gasoline used above, which increases the complexity of a student's ability to discriminate in which role the variable should be used (Kieran, 1992; Schoefeld & Arcavi, 1999). Pollack (2012) addresses the possibility of lessening students' difficulty and motivating students to learn how to recognize the altering roles of a variable through specific algebra problems, aiding students to link the variable to its quantitative referent. Her study focuses on variables that symbolize numerical quantities utilized in various ways, such as generalized numbers (i.e., $a + b = b + a$ when all numbers, a and b are real), specialized constants (i.e., $\prod = 3.141$), unknowns (i.e., $3x + 9 = 12$, where the variable has a specific value), and varying quantities (an argument of an function, such as f(x) = 2x where the student must characterize 1/x as x increases). Pollack (2012) found that the students struggled mostly with problems that included additional steps or computation with concepts, such as fractions and decimals (NMAP, 2008), and when the variable's quantitative referent was not found until the end of the problem.

Therefore, in order to support these students that pervasively struggle to differentiate variables (Trigureros & Ursini, 2003) it is necessary to put more time and effort into planning instruction. Some researchers (Pollack, 2012) imply that it is necessary to implement deliberately formed algebra tasks, scaffolding students' prior knowledge (i.e. eliminating information for students to discover unknowns, building an algebraic view of equality, etc.), to encourage students to partake in opportunities to construct and express generalizations and newly gained conceptualizations of diverse variable roles.

There is ongoing debate of where arithmetic procedures end and algebra procedures begin that has yet to be resolved. The focus of research on algebra tends to focus on beginning algebra students' difficulties and considerations of teaching interventions that could facilitate this cognitive gap (Herscovis & Linchevski, 1994; Filloy & Rojano, 1989). More recent research disputes the source of students' subsequent struggles may possibly be caused by the lack of algebraic thinking introduced in earlier school curricula (Carraher, Schliemann & Brizuela, 2001). There are studies (Blanton & Kaput, 2003; Slavitt, 1999) that have established the belief that it is the responsibility of elementary arithmetic instruction to provide a strong mathematical foundation in order for algebraic thinking to develop successfully. Many elementary algebra studies that have taken place often consider the structural conception of simplifying expressions with symbols (referring to a letter as an object) and the procedural conception of considering substitution in expressions (referring to replacing a letter with a number) as two independent activities rather than discovering ways the two are related. For this reason there should be more research designated to integrating the procedural and structural conceptions in an elementary setting.

Operations and Equality

Much research has reported students' misconceptions and errors when solving and simplifying algebraic expressions (Linchevski & Herscovics, 1996; Seng, 2010; Underwood Gregg & Yackel, 2002; Warren & English, 2000). This research concludes that students are not capable of processing that an algebraic expression such as $4x + 5$ may be considered an object, where there is not necessarily a way of adding the two together, unlike that of arithmetic where the student is used to getting an object from the process of adding the two terms together (Kieran, 1992). When using the literal symbols, Booth (1988) found misinterpreted notions regarding the meaning of the operation and equals symbols children obtain during early arithmetical experiences. In arithmetic, symbols such as the + and = are characteristically interpreted as actions to be performed, where + means to perform the operation and = means to "write down the answer" (Behr, Erlwanger, & Nichols, 1976; 1980; Ginsburg, 1977). Fillroy and Rojano (1989) elaborate with what they call the "arithmetical" notion of equality by stating that "in arithmetical terms, the left side of the equation corresponds to a sequence of operations performed on numbers (known or unknown); the right side represents the consequence of having performed such operations" (p.19). For example, in the equation $Ax + B = C$ the left side can be ‘undone’ with operations starting with the number C. However, in the equation $Ax + B = Cx + D$ the operations needed to solve this are out of the realm of arithmetic. Mevarech & Yitschak (1983) found that algebra students in college, despite being successful solving single-variable equations, had inadequate understandings of the definition of the equal sign, being a separator symbol rather than a symbol for equivalence (Kieran, 1992). If students are struggling in college with comprehending the = symbol as equivalence this illustrates the need for improvement in instruction earlier in students’ mathematical education for building onto later concepts, such as algebra. This also illustrates the need for students to be introduced to the concepts through a context in which they can connect (Fillroy & Rojano, 1989) in order to make a seamless transition from arithmetic to algebra.

For the transition to algebra, students should be made aware that the idea of the addition symbol may signal the result of addition (i.e., an expression such as $3x + 4$) as well as the action (i.e., a “closed” single-term solution such as $3a + 2a = 5a$) (Booth, 1988). Many times when students do not understand both conceptions of addition they will approach an algebra expression, such as $5a + 9n$, through procedural conceptions conjoining the terms as $14an$. To prepare for algebra it is necessary for students to grasp comprehension of the generality of the operations, rather than only being able to extract the “procedural rules” and properties of arithmetic. Students need to be able to recognize and relate to situations in given problems, using operations appropriately to generate solutions and a symbol system to express the real meaning of the operations (Warren, 2003), in order to create generalizations (Booth, 1988).

The equal sign (=) is one of the symbols whose meaning is established early in students mathematical journey (Blanton, 2008). Many students characterize the = as the symbol that operates as a barrier between the left-side which holds the equation to be solved and the right-side which holds the answer that is recorded. When students encounter expressions, such as 3x + 3y, most sense that the expression is incomplete and that the expression has to "equal something." This suggests "that a procedural interpretation of an algebraic expression requires that part of the representation indicates the result of carrying out the procedure" (Kieran, 1992, p. 397). For the most part, when students use their early arithmetical understanding they fail to understand the function of = in algebraic expressions (Blanton, 2008; Booth, 1988; Kieran, 1992). For example, researchers have found that many algebra students felt that the given tasks of simplifying expressions (Wagner, Rachlin & Jensen, 1984), expressing the area of a rectangle (Chalouh & Herscovics, 1988), and finding the value of a variable (Kieran, 1983) were impossible because they lacked the = symbol (Tirosh, Even, & Robinson, 1998; Kieran, 1992).

It is important to emphasize the two-way value of the = symbol by reading it as "is equal to" (Booth, 1988) or "the same as" (Blanton, 2008) rather than "makes" to give the students an appropriate experience (Booth, 1988). One suggestion to revise this misinterpretation of the = symbol is to give students appropriate experiences to define the = as a symbol for equality (Matz, 1980; Kieran, 1992). For instance, students could be asked to express their answers as unanswered computations (i.e. $5 + 9 = \square + 3$), or as the sum of two numbers (i.e. 3 + 5 = 1 + 7), as opposed to having them give single-value answers (i.e. 6 + 7 = 13) each time (Blanton, 2008; Kieran, 1994).

There have been many researchers (Bell, Malone, & Taylor, 1987; Kieran, 1981; Mevarech & Yitschak, 1983) to conduct studies to find what misunderstandings students have of the equal symbol and ways to revamp implementation to aid students to the accurate definition and understanding. Kieran (1983) discovered that majority of the 12 and 13-year-olds in his study defined the equal symbol in terms of an "answer." However, in her teaching sessions students constructed arithmetic equalities, balancing one then multiple operations on each side of the equal sign, giving them the name "arithmetic identities" (Kieran, 1992, p. 399). For example, the students constructed 'arithmetic identities' such as 7 x 2 + 3 – 2 = 5 x 2 – 1 +6. These activities supported the students' conceptualization of the equal symbol's definition of equality as well as justify that both side were equal because they had the same value. This also, in turn, allowed students to see the suggestion that the equal sign was a relational symbol in this situation rather than a symbol that signals one to "do something" (Kieran, 1992).

Children are reluctant to record an algebraic expression as an answer, tending to give numerical values to letters or variables (Collis, 1975; Küchemann, 1981). This directly contributes to the operational problem found in entry-level students and their misinterpretation of multiplication represented by conjoined terms, such as 4n. Students tend to want to find the sum (i.e. $4 + n$) or mistake this for representation of place-value (i.e. $4n = 43$… forty three, in which the n can be replaced with any number to form a new number) (Booth, 1988; Chalouh & Herscovics, 1988) because they do not understand that these unknowns ($4 + n$ and $4n$) can be considered as composite units depending on one's needs (Underwood Gregg & Yackel, 2002). For example, $4n$ can be viewed as the result of counting n units, followed by counting n units again, such as n, $n + 1$, $n + 2$, … $n + n$.

There are examples of conjoining terms in the arithmetic curriculum that could repress misconceptions of students that consider initial perceptions of these operations (+ and =) and feeling the need to put sets together as well-formed, closed answers (Booth, 1988; Collis, 1975; Davis, 1975; Matz, 1980). Mixed fractions (i.e. 4 ¼ = 4 + ¼) and place value (i.e. 94 = 9 tens + 4 ones) (Matz, 1980) offer teaching moments that can aid in the teaching of defining both the + and = symbols (Booth, 1988).

Given that numerous entry level students struggle with this concept, Booth (1988) suggests that the introduction of conjoined terms should be deferred, or at least given the expanded form (4 x *n* or *n* x 4) with the abbreviated version during the introduction period. Underwood Gregg and Yackel (2002) also imply imposing potential for implementation of realistic instructional sequences to support students' conceptual development of algebraic expressions and operations. This approach regards the use of symbols as activities for recording and communicating students' mathematical activity, with intensions of creating a solid conceptual foundation for development of additional algebraic concepts.

These suggested transitional tasks within the realistic instruction, if given in earlier mathematics courses, could support students' shift from arithmetic to algebra. With a more comprehensive knowledge of the use of symbolization within arithmetic and algebra students' could have less complication throughout the procedural-structural transition and conceptualization of solving and simplifying algebra equations and expressions.

Conclusion

The need for understanding how instructional tasks can support elementary students learning of algebraic thinking is significant when developing mathematics curriculum for all students to successfully transition between the procedural-structural and arithmetic-algebraic conceptualizations. Since research on instructional implementation in the elementary classroom is limited, research findings illuminating students' interpretations of algebraic thinking and variables was used in conjunction with a research-based realistic mathematics education (RME) instructional sequence to develop conjectures for this study. These are further discussed in the next chapter.

Implications

There is a gap that exists in the literature involving experiments that focus on designing productive instructional materials using realistic contexts for promoting algebraic reasoning with elementary students. This dissertation attempted to explore the implementation of algebraic content in an elementary setting using a Realistic Mathematics Education (RME) instructional sequence to understand the implications of supporting students' conceptualizations of algebraic expressions, equations and the use of variables. Existing literature has been attentive to the individual ideas primarily highlighting students' misconceptions of algebraic content and the lack of students' development in the transition between the procedural and structural conceptions. As these components of research are intertwined into this study, my initial hope was to find specific ideas and themes that can be generalized for development within various mathematical concepts. These ideas and themes would not only provide implications within mathematics education but could also provide implications for further extensive research within other curricular areas where student support is needed.

CHAPTER THREE: METHODOLOGY

This study featured qualitative methods documenting how the implementation of a Realistic Mathematics Education (RME) instructional sequence supports students' conceptualizations of algebraic equations and expressions. I conducted a classroom teaching experiment, as the research-teacher, in my own fifth grade classroom while implementing *The Candy Shop* instructional sequence designed by Dr. Diana Underwood Gregg and Erna Yackel (2002). Underwood Gregg and Yackel (2002) implemented this instructional sequence with college age developmental level (remedial) students and found that this sequence helped support students' conceptualizations of expressions of the form $ax + b$ as a quantity. With the use of a design-based research methodology the data was collected from this sixteen session classroom teaching experiment.

As there are many methodologies that can be used to conduct educational research, design research is gaining prominence because it establishes a link between research and practice through the context of the learning environment (Cobb et.al, 2003). This methodology is cyclical which could involve years of implementation and revision, and it is interventionist implying that the researcher is not simply an observer but also a modifier. Cobb and Steffe (1983) stated that "the activity of exploring children's construction of mathematical knowledge must involve teaching" (p. 83). They explain that as a teacher, one can delve further than the researcher's theoretical analysis, allowing the opportunity to test and revise the understanding of the students. Also the teacher has the ability to influence both the students' construction of mathematical knowledge as well as helping them reconstruct the contexts within which they learn mathematics, by interactions and forming close personal relationships with the students. Therefore, as the researcher in this study, I chose to be the modifier or teacher to take advantage of the opportunity to observe the students' constructive processes firsthand in order to give a thorough reflection of the students' mathematical knowledge. And for that reason, it was appropriate for this study to utilize design research as the methodology.

The following chart (Fig. 1) illustrates the structure of the design research methodology. Within the overall methodology there are multiple types of experiments that may be implemented depending on the overall goal of the study conducted. This will be discussed in more detail later in this chapter. The focus of this particular chart elaborates the Classroom Teaching Experiment (CTE) branch because that is the type of implementation that was chosen for this particular study.

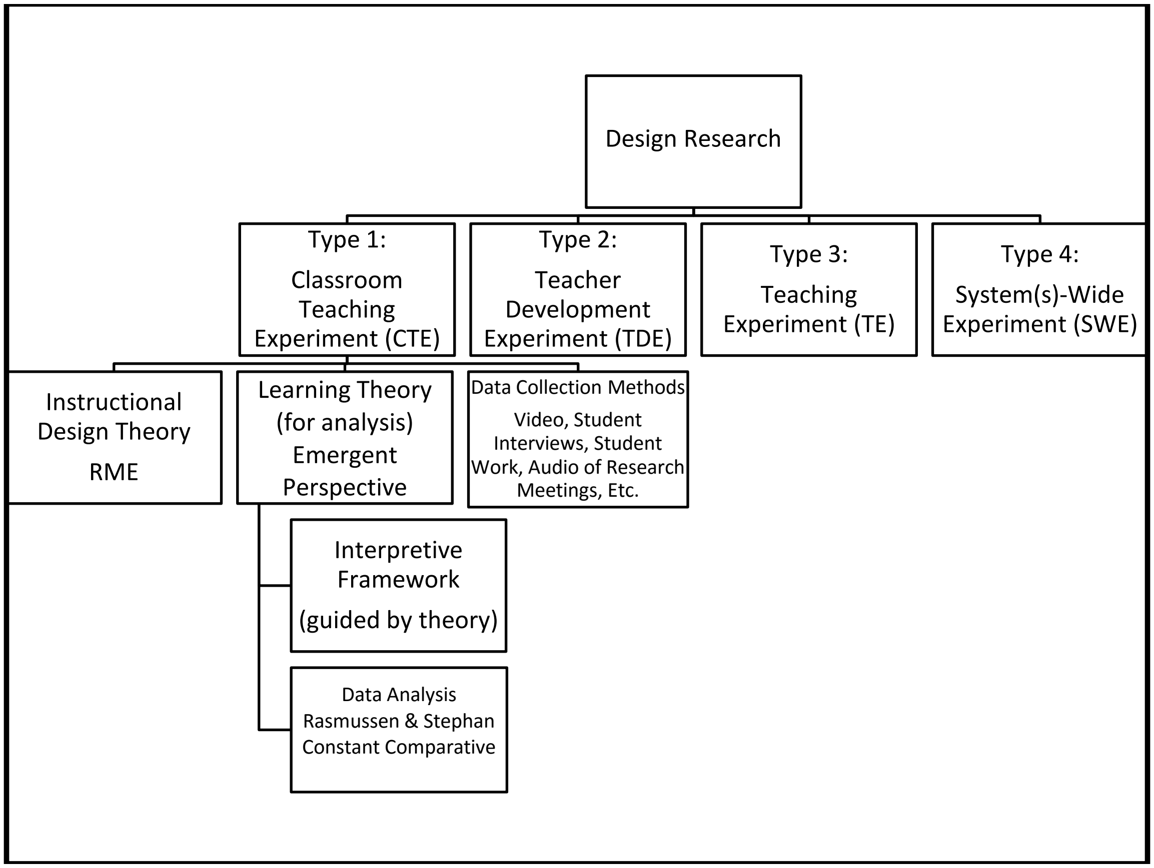


Figure 1: Design research methodology.

This chapter begins with a discussion of design based research. This discussion is then followed by a review of the instructional design theory used to create the hypothetical learning trajectory (HLT) and sequence of tasks for the sessions as well as the theoretical framework, which informed the data collection. The chapter concludes with a description of analysis for the data.

Design Research

Design Research, also referred to as design experiments, (Cobb et al., 2003) has been integrated into studies of mathematics education over previous decades (Steffe and Thompson, 2000). Design experiments typically target domain-specific learning processes to develop and explain theories that identify successive patterns of students’ learning and ways to support these patterns as they surface (Cobb et al., 2003). Design Research entails producing and analytically examining ways of learning, while assessing and revising means of supporting the learning process. Cobb et al. (2003) labels this practice as a *learning ecology* which he defines as “a complex, interacting system involving multiple different types and levels – by designing its elements and anticipating how these elements function together to support learning” (p.9) to stress the significance of the synchronized interaction between all of the required elements of design contexts.

As Design Research is not specific to mathematics, it is recognized as a methodology that advances educational research by using interventions, generally through real-world situations, to offer insight into traditional learning concepts (Joseph, 2004). It encompasses three significant intertwined elements of research, design, and pedagogical practices (Joseph, 2004) to investigate the possibilities of educational improvement (Gravemeijer & Van Erde, 2009). The research and design aspects include planning and creating innovative educational settings for exploring teaching and learning processes (Cobb et al., 2003) that potentially support students’ learning (Gravemeijer & Cobb, 2006).

There are several research methodologies that align with the principles of Design Research in educational settings, in mathematics in particular: the classroom teaching experiment, teaching experiment, teacher development experiment and system… (See Fig. 1). All of these methodologies are consistent with Design Research in that their purpose and goal is to test and revise conjectures, which are updated by continuous and retrospective analyses of the events, including students,’ teachers,’ or other individuals,’ reasoning and learning environments (Stephan & Cobb, 2003; Cobb et al., 2003). To achieve this goal the following three essential stages of Design Research must be met: 1) developing a preliminary design, 2) implementing the innovation, and 3) carrying out a retrospective analysis (Gravemeijer, 2004).

When choosing Design Research, the next step is to select the appropriate methodology, intended for the focus of the research that one is conducting. As there are many settings, four types of Design Research methodologies, which will be discussed in this section, are 1) the Classroom Teaching Experiment (CTE); 2) the Teacher Development Experiment (TDE); 3) the Teaching Experiment (TE); and 4) the System(s)-Wide Experiment (SWE) (see Fig.1).

A Teaching Experiment (TE), also known as the one-on-one or one-on-group approach, is utilized when the purpose of the study is to document cognitive development of a variety of students to map out cognitive learning trajectories. The TE is conducted by a combination of teacher-experimenter and student with the intention of creating a small-scale version of a learning ecology to focus on detail of students' conceptualizations within the given instructional tasks (Cobb et al., 2003; Cobb & Steffe, 1983; Steffe & Thompson, 2000). A teaching experiment involves a sequence of teaching episodes (Steffe, 1983) which consist of "a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episodes" (Steffe and Thompson, 2000, p. 273). As a series of instructional activities are designed, tested and revised during a teaching experiment, it is the researcher's goal to identify how a student best learns the concept (Van den Akker et al., 2006). One of the benefits of a TE is the researcher having the ability to have direct experience, continually interacting with the individual students throughout the teaching episodes, in which it is likely that they will observe the students' restructuring. This is crucial to achieve the goal for this setting (Steffe & Thompson, 2000).

When a study aims to work with teachers to support their growth, a Teacher Development Experiment (TDE) is employed. The TDE methodology targets two groups of educators; the prospective and the experienced teachers. A TDE for preservice teachers involves a research team that facilitates the coordinating and studying of instruction for prospective teachers (Cobb et al, 2003; Simon, 2000), while the TDE for in-service teachers consists of a group of researchers that collaborate with experienced teachers to support professional development (Cobb et al., 2003; Stein, Silver, & Smith, 1998).

A System(s)-Wide Experiment (SWE) is a methodology of Design Research used when the objective is to make improvements in an overall system. In an educational setting this would involve collaboration between a research team and teachers, school administration, and other stake holders in order to provide appropriate modifications necessary for the enhancements (Cobb et al., 2003).

Because the objective of this study was to try to find ways to support the classroom learning and instruction, a Classroom Teaching Experiment (CTE) was the suitable method of Design Research. Classroom teaching experiments are conceptualized with goals of supporting students learning in a content-specific domain. The intention of this type of study was to develop theories of how students learn both independently and in a collective group as they participate in the instructional activities that guide this process (Cobb et al., 2003; Rasmussen and Stephan, 2008). In a typical CTE, a research team collaborates with a teacher (whom may or may not be one of the researchers) that is taking on the responsibility of instruction (Cobb, 2000; Confrey & Lachance, 2000; Gravemeijer, 1994). Innovative instructional designs are implemented and students' interactions are analyzed to make conjectures about students' mathematical practices. The overall objective is to encourage students' learning to gradually become more academically and ethically independent/autonomous (Kamii, 1985) as they participate in specially structured tasks (Nicholls, 1983).

This setting allows researchers to focus on significant moments in students' mathematical development as they create cognitive restructurings. These restructurings in a CTE occur as students cooperate with the teacher and fellow students. This gives researchers the opportunity to attend to other issues, such as embedding students' conceptualizations within social context. For this purpose, the communally designed social norms, which are the teachers' and students' obligations and expectations for participation in the classroom, are examined. The students' emotional acts are also taken into consideration, as they factor, by both hindering and contributing, into the structure and continuous regeneration of these norms (Cobb et al., 2001).

With this study's primary goal of testing and revising *The Candy Shop* instructional sequence as means of supporting students' mathematical conceptualizations in the domain-specific content area of algebraic equations and expressions, it was appropriate to use a Classroom Teaching Experiment. Conducting the CTE involved the research team examining students' continual participation with the given instructional tasks, which in turn promoted forming new conjectures to uncover the best means of supporting the students' conceptualizations and development. An example of a conjecture for this particular study was that the concept of simple arithmetic expressions (i.e. 3 + 2 *x* 5) should be taught and understood before introducing algebraic expressions involving unknown variables (i.e. 3 + *a x* 5). The analyses drove the selection of mathematical activities to utilize from the instructional sequence that best met the students' learning needs. These conjectures, within the instructional sequences, were provisional as they are tested, refined and retested in a daily cyclic manner throughout the study (Cobb et al., 2001), which lead to new conjectures being created, assessed, and revised, as students averted from the course of the initial hypothetical learning trajectory (HLT) (Gravemeijer, Bowers & Stephan, 2003; Steffe & Thompson, 2000), which is discussed in one of the following sections of this chapter. It was important to continually refine and reassess conjectures made throughout the duration of the study, as the HLT and instructional activities were influenced by the students' conceptualizations.

The research team played a significant role in the classroom teaching experiment as we offered a variety of perspectives from our observations of the students' conceptualizations with the instructional tasks. The research team for this study consisted of two people; an assistant professor and chair of my dissertation committee and myself, who was a doctoral student at the dissertation stage. I was the instructor for the sixteen session instructional sequence. The assistant professor/chair participated in the instructional sequence implementation a few times during the study.

Conjectures made by our research team were what drove the design research. The initial design was a conjecture about how to best support the process of learning that was being assessed. However, as the design experiment transpired these conjectures were analyzed to help improve the instructional design. As a result, more conjectures were generated and refuted, as new conjectures were developed and become susceptible to be assessed (Cobb et al., 2003). Therefore the design experiment process was an iterative process, as it was a cycle of revising and assessing the HLT, conjectures, and instructional activities (Cobb et al., 2003) to advise further design experiments (Gravemeijer, 2004). The intension of the iterative design, as it required methodical consideration of learning, was "an explanatory framework that specifies expectations that became the focus of investigation during the next cycle of inquiry" (Cobb et al., 2003, p. 10).

Instructional Design Theory

Since a CTE was chosen as the type of Design Research methodology for this study, with the goal to implement an instructional sequence intended to change the learning of students, an instructional design theory is required (see Fig.1). Therefore, Realistic Mathematics Education (RME), an instructional design theory which has a set of heuristics that are domain specific to mathematics, was chosen for this study.

RME was developed in the Netherlands and presented as an alternative mathematics instruction to traditional teaching methods (Van den Heuvel-Panhuizen, 2003; Streefland, 1991). The objective of RME is to help guide the work of creating the sequence, focusing on advancing in the emergence of formal mathematical knowledge. In order for the students to advance within the content area of algebraic expressions we utilized "three basic RME heuristics, that when taken together, inform designers in their efforts to support students' reasoning with the cycle of design research" (Gravemeijer, Bowers, & Stephan, 2003, p.52). These heuristics included 1) sequences must be experientially real for students, 2) guided reinvention, and 3) emergence of student-developed models.

To ensure the first heuristic, *the sequences are experientially real for students*, was accomplished it was mandatory for the research team to create scenarios, involving mathematical situations, which established students' primary mathematical activity (Gravemeijer, Bowers, & Stephan, 2003). Context problems, or problems of which the problem situation was real for students, played a role from the start (Gravemeijer & Doorman, 1999). For example, *The Candy Shop* sequence approached the concept of creating and simplifying algebraic equations and expressions through a scenario of a familiar fictional cartoon family, The Simpsons™, owning a candy shop. The tasks created for the sequences involved candy shop workers helping this fictional family figure out ways to distribute candy in various quantities to fulfill customer orders. Students had to imagine themselves working in the candy shop, as a worker, helping the family run their candy shop.

The second heuristic, *guided reinvention*, involved the process of our research team creating an HLT for the development of sequential instructional activities. When developing the HLT for this study our effort was to form conjectures about the communal progress of the mathematics community, focused on practices that may surface as the sequence began (Gravemeijer, Bowers, & Stephan, 2003). A key role within the reinvention approach is the problem situations. The students were guided through the mathematical activities with the opportunity to reinvent mathematics (Gravemeijer, 2004). As the students and teachers engaged in the activities of *The Candy Shop* sequence new insights into the types of mathematical practices that emerged outlined the basis for what established a modified HLT for the subsequent tasks in the instructional sequence to keep students' conceptualizations on track in learning the concepts (Gravemeijer, Bowers, & Stephan, 2003). *The Candy Shop* sequence is a scenario of a familiar fictional cartoon family, The Simpsons™, owning a candy shop. The intention of using this fictional cartoon family that is familiar to this age group was to get the students interested so they were engaged in the activities of the sequence. The tasks of *The Candy Shop* sequence were arranged for the students to start with arithmetical operations. For example, in the initial task of *The Candy Shop*, students were presented the situation of the candy shop and the ways that the Simpson family wanted to package various numbers of individual pieces of candy into rolls, depending on the flavor of the candy. The goal for the students was to use prior knowledge of addition, subtraction and even skip counting to find the total number of candy for given visuals and packing rules that the family provided (shown in Fig.2 at the end of this section). All of the instructional tasks were arranged to support the students' knowledge of arithmetic expressions as the basis for creating algebraic expressions and solving or simplifying algebraic equations as they progress through the instructional sequence. However, as they were reviewed by our research team during the duration of the study there were alterations made to better support the students.

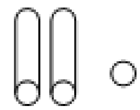
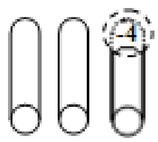
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.

Figure 2: Initial candy shop task. Students are to find the total quantity of candy that a customer receives using given rules.

The third RME heuristic, *emergence of student-developed models*, focused on the use of models as students' conceptualizations progressed from informal to formal mathematical activities. Models were defined as "student-generated ways of organizing their mathematically grounded activity" (Gravemeijer, Bowers, & Stephan, 2003, p. 53). During instructional tasks I tried to help the students model their own informal mathematical activities, with the intent that the students' models would gradually develop into a model for more formal algebraic reasoning (Gravemeijer, 2004).

The Candy Shop instructional sequence initially provided students with visual models within the instructional tasks with the intention of students developing their own visuals or ways to solve the mathematical problems. For example, in the initial tasks that were mentioned previously (see Fig.2) the students were given an illustration of the rolls and individual candies to help support them in finding the total candy according to the Simpsons' packing rules. Also, as *The Candy Shop* sequence continued one of the characters, Maggie, created a candy sales promotion called Maggie's Mystery Rolls. Using various letters as variables to name her unknown number of candies in a roll, Maggie created a mystery roll contest. This allowed students to observe letters as representatives of a quantity. The students were then introduced to Maggie's cousin, Brainy Brian and his use of balances to figure out how many candies are in each mystery roll. Again, the students were provided visuals of the balances with rolls and individual pieces. The balance visuals progressively moved to numbers and words and then to only numbers and variables. As seen in Fig. 3 below, the balance was used to guide students into simple equations by manipulating the individual pieces and rolls of candy on the scale to find the value of the letter variable. As the instructional sequence progresses the visuals were brought into and then replaced with only equations or expressions throughout the sequence, with the intentions of guiding students to generate their own models to demonstrate their conceptualizations of how to create algebraic expressions and solve or simplify algebraic equations.

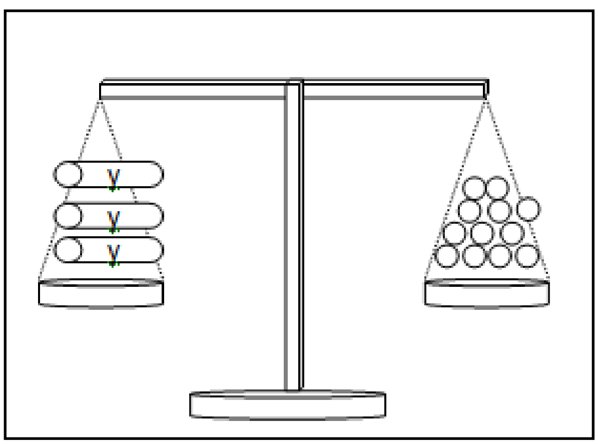


Figure 3: Cousin Brian's balance strategy. This visual illustrated 3 of the Mystery Rolls "y" is balanced to 12 pieces of candy (3y=12).

When developing the preliminary design, a hypothetical learning trajectory (HLT) (Simon, 1995) must be generated as a foundation for the instruction. The HLT consists of the learning goals and instructional sequence. During the design experiment there were "restructuring" moments (revealed in student's language and actions) in which students changed their ways of thinking from their initial functioning mathematical methods (Steffe & Thompson, 2000). These moments demonstrated students' progression and development of mathematical concepts, which lead to the research team modifying processes and revising their conjectures within the design experiment to correspond and enhance proceeding learning objectives. As these modifications were being made, the HLT also needed to be revised (Cobb et al., 2003) to assist in this process.

The three heuristics of RME, listed above, were used to develop the HLT and instructional sequence tasks implemented in this study. A hypothetical learning trajectory is a theoretical model, and "a cognitive tool grounded in constructivism" (Clements & Sarama, 2004, p.85) for the design of mathematical instruction. These learning trajectories are "hypothetical" because there was no way of knowing the actual learning and teaching that occurred prior to administering the instruction (Clements & Sarama, 2004; Simon, 1995). According to Simon (1995), an HLT is a teaching construct where teachers conjecture students' strategies and the activities that will advance students in the content area being taught. Therefore the teacher acts as an agent by hypothesizing learning trajectories for the purpose of planning tasks that connect students' current and future conceptualizations.

There are essential components to consider when creating a HLT, such as learning goals regarding student reasoning, the progression of students' mathematical learning experiences, the sequence of instructional tasks in which the students engaged (Larson, Zandieh, & Rasmussen, 2008; Simon 1995), and the role of the teacher (Larson, Zandieh, & Rasmussen, 2008; Smith and Stein, 1998). Gravemeijer, Bowers, and Stephan (2003) think of this idea of HLT along with RME planning addressed the irony of “on-the-fly planning” by considering four explicit factors differentiating from conventional ways of lesson planning: “1) the socially situated nature of the learning trajectory, 2) the view of planning as an iterative cycle rather than a single-shot methodology, 3) the focus on students’ constructions rather than mathematical content, and 4) the possibility of offering the teacher a grounded theory describing how a certain set of instructional activities might play out in a given social setting” (p. 55). These characteristics of RME planning and development of HLTs are explained in the following:

First, it is important to know your audience. When generating the learning goal, age appropriate “learning models that reflect natural development progressions” (Clements & Sarama, 2004, p. 83) must be identified. However, as all classrooms are made up of various types of students and settings, it is unlikely that one sequence will produce the same results, so learning is considered a “classroom activity system” (Gravemeijer, Bowers, & Stephan, 2003, p.55). Consequently, Stephan (2014) distinguishes between a traditional, cognitive HLT and an HLT constructed for a classroom. The cognitive HLT attempts to predict the cognitive route of an individual student within a particular mathematical domain. However, a classroom HLT (CHLT) anticipates the classroom mathematical practices, discourse, tools and imagery that might be constituted as taken-as-shared within the classroom community that consists of 20 or more students.

When developing a hypothetical learning trajectory, an outline of conjectures about the potential classroom mathematical practices is created keeping in mind the potential cognitive progression of students’ learning within the particular classroom and adjusting the CHLT during instruction. Instructional activities are generated and continuously analyzed by using the conjectured mental constructions and mathematical practices to create key tasks designed to promote learning at a given conceptual level or criterion in the CHLT. The CHLT includes the conjectured ways in which students may conceptualize the ideas that are presented within the learning tasks as well as the mathematical ideas that may become taken-as-shared as students participate in the public discourse. For example when planning the tasks it was imperative to give students tasks that allowed them to build understanding of variables as a quantity and as composite units before they were to build an understanding of how to identify and utilize them within the same instructional task.

To achieve this goal of understanding the audience, the development of this study's CHLT started with analyzing how the mathematical concept of algebraic expressions and equations were elaborated in the mathematical textbooks, the struggles that students' encountered within this content area, and reflecting on how and what the students should learn. Additionally, the instructional designers utilized research on students' conceptions of algebra to form short-term mathematical goals to structure a CHLT, as well as create instructional tasks from *The Candy Shop*, altering them to correspond with fifth grade class that is participating in the study. For example, the conjecture that was made when thinking of the initial instructional task, was that the fifth grade participants of this study should be able to use their informal mathematics knowledge of arithmetical operations, addition and subtraction, as well as skip counting to apply a give rule, or "packing rule," to figure out the total number of candies being packaged. Philipp (1992) confirms a point found in earlier writings of Wagner (1981,

1983) that the activity highlights the significance that context plays in thought determining the role of a symbol. Therefore, the conjecture that built off of this primary was that this initial activity was built off of Philipp's (1992) discussion of what background knowledge students already possess and how this knowledge could interact and possibly support their conceptions of literal symbols. The purpose of giving the pictures for the students to use in the initial tasks was to "provide some visual imagery that students could use for both reasoning about and recording transactions in the candy shop" (Underwood Gregg & Yackel, 2002).

The second characteristic of planning as an iterative cycle rather than a single-shot methodology, is being more lenient when planning and modifying procedures of the instruction. As the research is conducted, the teacher and researchers may discover new forms of mathematical practices that emerge as basis for customizing the foundation of the CHLT for succeeding lessons in the instructional sequence Gravemeijer, Bowers, and Stephan (2003). This begins what Simon (1995) described as the *mathematical teaching cycle* and Gravemeijer, Bowers, and Stephan (2003) extended the idea of this cyclic planning to portray the progression of the teacher modifying stages of the implementation to observe what is happening and getting the instruction back on course. This iterative process of screening the efficiency of the design composition presents accruing instantaneous feedback on the practicality of its CHLT (Kelly, 2004), as well as allowing the teacher to visualize the best course of learning for the students (Gravemeijer, Bowers, & Stephan, 2003).

A third aspect of Design Research experiments is that the CHLT utilizes students' cognitive development as the foundation for design, unlike conventional instruction that focuses on the mathematical content to organize instructional goals. The CHLT forms the basis for creating tasks and conjectures on how the students may respond to each of the tasks given.

The final distinction between Design Research and conventional lesson planning is the objective of Design Research, which is affirmed by Gravemeijer, Bowers, & Stephan (2003) to describe a course of learning by presenting a "grounded theory that describes the tools, imagery, discourse and mathematical practices that emerge" (p. 56) from the implementation of the instructional sequence within the setting of the mathematics classroom. Through the implementation of the instructional sequence, teachers are able to contribute to research and become more conscious of the social disposition within education. These opportunities within Design Research initiate enhancements to conjectured instructional theory with the teachers newly acquired knowledge and a more global course of learning that can be customized to specific classroom situations by classroom teachers (Gravemeijer, Bowers, & Stephan, 2003).

For this study to capture "the power and uniqueness of the learning trajectories construct" (Clements & Sarama, 2004, p.83), classroom learning trajectories were interpreted through descriptions of children's conceptualizations and interactions within the classroom setting through tasks from *The Candy Shop* instructional sequence. The conjectures that our research team created, with intent to support students' understandings, over the duration of this study were used to fulfill the overall objective of this study's design. As stated previously, the overall objective of this HLT was to stimulate cognitive developments or actions to support students' development of algebraic concepts (Clements & Sarama, 2004; Gravemeijer, 1999; Simon, 1995), specifically algebraic equations and expressions.

Theoretical Framework

Up to this point, I have focused mainly on the instructional design that guides the creation of the HLT and associated instructional sequences of tasks. Within the classroom teaching experiment, daily and retrospective analyses of student learning is essential to both the daily decision-making during experimentation as well as retrospective assessments of student growth. In this section, I describe the learning theory and associated interpretive framework that guides analyses of student learning. Social constructivism offers an analytical lens for viewing and explaining the complexities of learning in the classroom (Rasmussen & Stephan, 2008). Therefore, the learning theory that was used to interpret this study's data was a version of social constructivism, called the emergent perspective (Cobb & Yackel, 1996; Stephan and Cobb, 2003). The label 'emergent' is used to stress the notion that classroom mathematical practices were not set a priori, but rather emerged from interactions among me and my students. Additionally, models were not given to students beforehand but students were encouraged to model their reasoning; therefore, models emerged from students (Doorman & Gravemeijer, 2009). The emergent perspective looks at mathematical activity through the lens of diverse social and psychological, or individual, perceptions (Cobb et al., 2001). As a result the emergent perspective goes beyond observing learning as an individual versus social process, and characterizes learning as both with neither taking primacy over the other (Stephan & Cobb, 2003).

The CTE also has a set of method/data collection that generally supports the goals of experiments (e.g., pre-post interviews, sometimes pre-post tests, video of small groups and whole class, student artifacts, etc.). This data helps determine the learning of the classroom. However with the learning theory of the emergent perspective there is a simultaneous analytical tool, or interpretive framework, that is used for analyzing data.

Interpretive Framework

The interpretative framework constructed by Cobb and Yackel (1996) was used for the analysis of the data collected in this study. When using the emergent perspective learning theory, "Cobb and Yackel (1996) constructed an interpretive framework useful for detailing the learning of a classroom community and its participants" (Stephan & Cobb, 2003, p.37). The interpretive framework poses two perspectives, social and individual (see Table 1). The social perspective relates to the normative ways in which students perform, reason, and argue in the classroom (Cobb et al., 2003), while the individual perspective targets individual students' participation within the classroom activities. Both perspectives were taken into consideration when analyzing students' learning in the classroom activities because without one the other would not be complete. The social perspective generates normative taken-as-shared ways of discourse and reasoning, while the individual perspective generates students' diverse ways of contributing in the taken-as-shared activities (Cobb et al., 2003).

Table 1: Interpretive framework

Social Perspective	Individual Perspective
Classroom Social Norms	beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical Norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical Conceptions

Social norms involve the roles in which the teachers and the students play within the classroom participation structure. Social norms, not specific to only mathematics classrooms, are when students are expected to clarify their solutions and their ways of thinking by explanation, justification, and argumentation (Yackel & Cobb, 1996). For that reason, if there are misunderstandings or uncertain reasoning being stated it is the responsibility of the teacher and the other students to question for complete comprehension. To establish social norms individuals must contribute while participating in classroom activities. When "making these contributions (social perspective), students reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity" (individual perspective) (Cobb et al., 2001, p.123).

The construct of sociomathematical norms is significant, clarifying students' development of mathematical beliefs and values and how they become independent in mathematics. Sociomathematical norms are "normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom" (Yackel & Cobb, 1996, p.461). The understanding of sociomathematical norms has been essential to the understanding of classroom interaction, students' mathematical activity and teachers' actions toward construction of teaching and learning opportunities. The teacher gains from the learning opportunities that arise when listening to their students' explanations. Their innovative selection of tasks and responses made to students' solutions illustrates their personal development of understanding the students' mathematical and conceptual development (Yackel & Cobb, 1996).

The distinction between social norms and sociomathematical norms is subtle. For instance, understanding that students are expected to explain their solutions is a social norm, whereas understanding what is adequate for a mathematics explanation is a sociomathematical norm (Yackel, 2001). There is a reflexive relationship between individual students' learning and the social context in which they learn.

Mathematical practices are considered mathematical interpretations that become normative through specific mathematic interactions. These derive from the social and sociomathematical norms participation structure of the classroom, depicting what is normative in terms of teacher and student discourse. Therefore the mathematical practices differ from social and sociomathematical norms by referring to more content specific activity (Stephan & Cobb, 2003). In other words, classroom mathematical practices refer to the mathematical content ideas that become taken-as-shared in the public discourse. Similar to social and sociomathematical norms, the classroom mathematical practices and activities also share a reflexive relationship. As mathematical practices were shaped from students' individual contributions, these contributions were allowed and limited by student participation in the mathematical practices (Yackel &

Cobb, 1996). As they participated in and contributed to the mathematical practices, students modified their individual ways of participating in these practices (i.e., their intellectual development).

For the purpose of this research classroom mathematical practices were defined as "localized to the classroom and are established jointly by the students and the teacher through classroom discussion; they emerge from the classroom rather than come in from the outside" (Stephan & Cobb, 2003, p.41-42). The interaction between social and individual environments incorporated students' contributions to the activities, and students' restructuring of knowledge as they participated in the classroom activities and discussion.

This study was conducted during the 2014-2015 school year, after the school year had started. Class lessons, prior to the study, implemented procedures for students to engage in classroom discourse and activities in order to establish norms and familiarity in the study. As new situations arose throughout the study's duration, negotiation of norms took place. This study utilized the elements of social constructivism and this interpretative framework in the data analysis to assess and refine conjectures made throughout the research process.

Instructional Sequence

As designed by Underwood Gregg and Yackel (2002), the setting of the instructional sequence was a candy shop owned by a fictional cartoon character family, The Simpsons. The instructional activities were designed so that each topic was presented in the context of a candy shop that is distributing candy to customers. The initial instructional task introduced the students to the ways in which the candies were packaged into rolls and individual pieces, depending on the flavor of the candies, in The Simpsons' Sweet Shop. This task incorporated students' prior knowledge of arithmetical operations and rules to find the total number of candies a customer received. The students were given a 'packing rule' and a visual of rolls or individual candies in order to figure out the total (see Fig.2). The intention of this task was for students to use the visual images to both explain and document transactions within the candy shop (Underwood Gregg & Yackel, 2002). In this particular problem students had to deal with missing pieces. The dotted circle, on the second problem, represented pieces that were missing from that specific roll. In this case, there were 4 missing individual pieces from the roll that has the dotted circle on it.

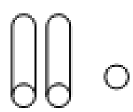
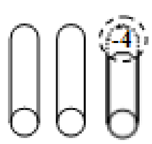
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.

Figure 2: Initial candy shop task. Students are to find the total quantity of candy that a customer receives using given rules.

As students continued through the instructional sequence they were presented with the scenario of customers wanting specific flavors and quantities of candy. Word problems, such as "Draw 3 different ways to have 95 pieces of candy packaged if the packing rule is 10 pieces per roll," allowed students to create their own visuals of how they interpreted or solved the problem. Such a task also encouraged students to draw on their arithmetical understanding of multiplication, the basis for which to generalize to *x* number of pieces in a roll.

One of the characters, Maggie, later created “Mystery Rolls” to promote more candy sales for the candy shop, which introduced students to letter variables representing an unknown quantity. Maggie’s cousin, Brainy Brian came into the scenario with a solution to help find the ‘mystery’ by using a balance. Again, students were initially given visuals and packing rules to discover the packing rule in order to find out the quantity each customer was receiving creating algebraic expressions (see Fig. 3).

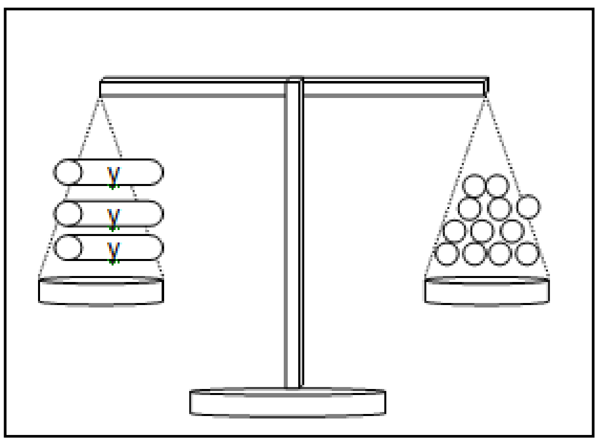


Figure 3: Cousin Brian’s Balance Strategy. This visual illustrated 3 of the mystery rolls “y” is balanced to 12 pieces of candy (3y=12).

As students actively engaged in *The Candy Shop* sequence using the visual tools (i.e. balance and candy rolls) consistently it was conjectured that they would learn how to create and solve for a variable in algebraic equations (See Fig 4.).

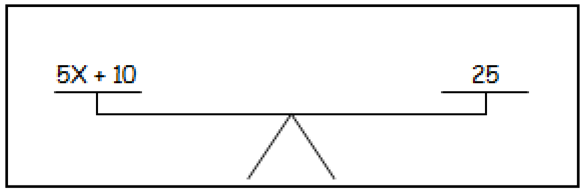


Figure 4: Quantity of candy rolls. Students used this illustration to create an algebraic equation and solving for X.

As the instructional tasks continued in the sequence students were asked to manipulate visual tools with the intention of building concepts of how to create, solve and simplify algebraic equations and expressions. Throughout the instructional sequence the given tools were supplied and later eliminated with the intention of giving the students an opportunity to create their own way of interpreting and solving the algebraic concepts.

Ethical Considerations

Before this instructional sequence was implemented in the classroom setting with these students, the study had to be approved through the university's Institutional Review Board (IRB). Every student that was asked to participate in the study had to have a parent or legal guardian fill out consent that gave them permission to participate in this study. To preserve anonymity, pseudonyms were used in the data analysis. The CTE took place in one of my five classes, with twenty students. Students were introduced to the ideas behind the study and the Institutional Review Board (IRB) process to get their parental consent before participating in the CTE. All consent forms were sent home with students for their parents or legal guardians to review and sign, with a given deadline of returning within three to four days in order for me to organize classes and start implementation of the study.

Participants and Setting

This site for this study to be conducted was a charter school located in an urban area in North Carolina. For the school year 2014-15 the school served approximately 1,204 students in a middle school and high school setting, compared to the state average of 441 for students in similar settings. The participants were in a unique setting as they were placed within the middle school, rather than elementary building. There were 20 fifth grade participants. Participants were both male and female students which a majority were considered “on grade level” in mathematics.

This study took place over sixteen class sessions during the 2014-15 school year. Because this school participates in a year round schedule, the study was planned to be conducted around breaks, so that they do not fall in the middle of the study.

A professor at UNC-Charlotte participated along side of me in some of the classroom teaching experiment lessons. The research team, which consisted of a university professor, which was also the dissertation committee chair, and me, as the researcher-teacher, were in contact weekly (face to face, via email or teleconference) to discuss the process of the study and make adjustments if needed and/or create further conjectures on how to best support the students’ conceptualizations of the mathematical content.

Research Question

The intent of this study was to analyze the development of students’ conceptualizations of algebraic equations and expressions through RME implementation through *The Candy Shop* instructional sequence. Therefore the research question was:

What are the classroom mathematical practices that emerge as students engage in *The Candy Shop* sequence?

Data Collection

One of the steps occurring continuously during this type of Design Research (Classroom Teaching Experiment) was the data collection. When conducting a CTE it is important to collect a variety of data, including video recording of whole class sessions, student interviews, field notes and student work (Cobb et. al., 2001). The data was collected over the duration of the sixteen lessons (approximately thirty 45 minute instructional sessions) in my fifth grade classroom of a middle school charter school setting. This data was used to analyze students' interactions and conceptions within the classroom setting. The instruction focused on students' cognitive development of creating, simplifying and solving algebraic expressions and equations using the Realistic Mathematics Education instructional sequence, *The Candy Shop*, designed by Diana Underwood Gregg and Erna Yackel (2002).

Data Collected

The data collected included video recordings of whole-class discussions and activities, individual student interviews, student work, field notes and notes of our research team meetings. The video camera was manipulated by a high school student assistant or one of the instructors to capture varying aspects of the classroom. Prior to this study, the student assistant was trained on how to operate the video camera and the procedures for videotaping needs put into place by the research team.

Five students were interviewed individually before and after the instructional sequence was implemented. The students were selected based on the diversity of their reasoning in mathematics, represented on a baseline assessment given at this point in the school year. Each interview was videotaped and lasted approximately 30-40 minutes. During the interview the students were asked to complete tasks and answer questions that illustrated their cognitive understanding of algebraic equations and expressions, as well as using variables. The interviews were conducted by one of the instructors and included questions that illustrated how students conceptualized the idea of what variables represent and how to manipulate algebraic expressions and equations (i.e., solving for a variable, simplifying). Students' work from the interview was collected but not used in this study's data analysis.

Student work that was collected throughout the duration of this study included class work, assessments, and scrap work. The data collected from the research team included field notes and reflective journals from classroom observations and discussions and research team discussions about implementation. The research team met weekly to discuss if the learning goals were being met and to plan accordingly for the remainder of the sessions.

Corresponding with these efforts, the research team analyzed the data and highlighted any means of support essential to achieve development, such as norms for argumentation, the role of the instructions and students in deliberately facilitated class discussions, and the use of the instructional tasks.

Data Analysis

The meetings during implementation did not consist of formal data analysis. Rather, the research team brought forward their interpretations of the status of the classroom mathematical practices by discussing student thinking from the class period.

Real time conjectures of student thinking and how that related to the HLT drove the decisions about instructional tasks used in subsequent class periods. Upon completion of the CTE, however, a more formal analysis of student learning was completed. The classroom mathematical practices were analyzed using a method designed by Rasmussen and Stephan (2008). This data analysis technique is described below.

Classroom Mathematical Practices Analysis

The collective activity of a mathematics classroom refers to the normative ways of reasoning within classroom community. It is a social phenomenon in which mathematical ideas become established in a classroom community through patterns of interaction. Documenting collective activity is pertinent in design research because “it offers insight into the quality of the students’ learning environment, an environment in which students are active agents in the construction of understandings” (Rasmussen & Stephan, 2008). When documenting collective activity it is important to analyze all three features of collective activity that include (a) a taken-as-shared purpose, which consists of what the teacher and students are doing together mathematically, (b) taken-as-shared ways of reasoning with tools and symbols, consisting of the ways tools and symbols are utilized and defined by the classroom community, and (c) taken-as-shared forms of mathematical argumentation, which consists of methods in which students explain and justify their solution and the process of their solution (Cobb et al., 2003). For this study the classroom mathematical discussion were analyzed to determine what became as taken-as-shared by the classroom community.

“Because collective activity refers to the negotiated normative ways of reasoning that evolve as learners engage in genuine argumentation, we will use argumentations that occur" (Rasmussen & Stephan, 2008, p. 197) in this study's classroom discourse as part of the analysis. The classroom discourse included discussions that could be heard by all participants within small and whole group activities. After all of the data was collected and recorded the Toulmin model was used as the analytical tool to analyze argumentation. Stephan & Cobb (2003) adapted Toulmin's model "to draw out the aspects to documenting mathematical practices" (p.43). For Toulmin (1969) an argument consists of three components, making up the basis of an argument: (1) the data, which is usually facts that supports the claim and lead to a conclusion, (2) the claim, which is a the statement, argument, or solution made to a mathematical problem, and (3) the warrant, which a challenge or justification needed for clarification of why the data lead to the conclusion. If a warrant is challenged a backing, the fourth component of Toulmin's model, comes into play. A backing is provided to justify why the warrant is accepted, therefore validating the entire mathematical argument (Rasmussen & Stephan, 2008; Stephan & Cobb, 2003).

For example, when solving a problem such as x + 2 = 5, what is *x*?, a claim would be that the answer in 3. Data for this problem could be that 3 + 2 =5. A student may not see how 3 + 2 = 5 relates to the answer that x = 3. The person must provide a warrant to explain how 3 + 2 = 5 leads to the conclusion that x = 3. A warrant that a student might provide is that they know that 2 + something equals 5 so that something is 3 since 2 + 3 is 5. Some students might understand how 2 + 3 = 5 leads to the answer of x =3, but may call in question the validity of that statement. What does 2 + 3 = 5 have to do with x at all? This students is calling for a backing, an explanation of why this addition sentence is a support for x = 3. A backing that could be given here is that when you have a letter in a statement, it stands for an unknown quantity that must be found. To find the unknown quantity, x, one must find the number in the addition statement that adds with 2 to achieve the sum of 5. The backing provides the mathematical authority for the argument (See Figure 5).

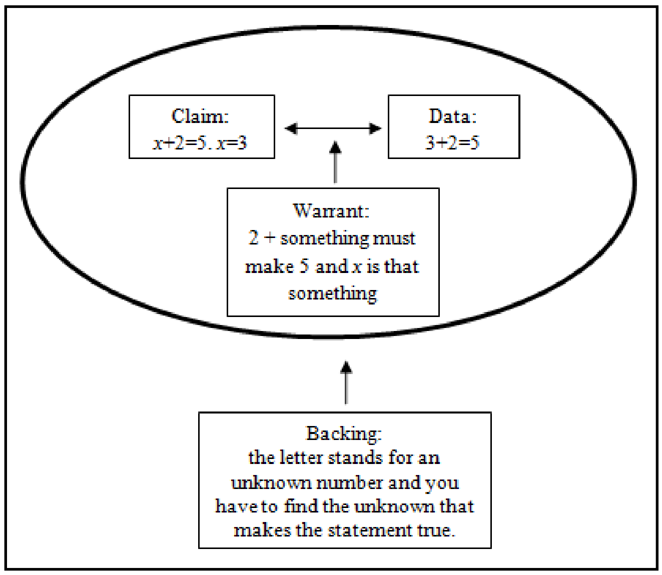


Figure 5: Toulmin's Argumentation Model

While Toulmin's argumentation scheme details the structure and function of a single argument, Rasmussen and Stephan (2008) have a three phase method for documenting collective activity that extends over several weeks of classroom discussions. This is the method that I used in my research study. In the first phase, transcripts are created of every whole-class discussion from all of the class periods that were considered. After the videos are transcribed, they are analyzed using Toulmin's argumentation model to create an argumentation scheme. For this study, the professor and I worked together to identify and classify the claims, data, warrants and backing for each transcript, which help me develop the argumentation log, ordering all whole-class discussions sequentially over the weeks (Rasmussen & Stephan, 2008).

Phase two "involves taking the argumentation log as data itself and looking across all the class sessions to see what mathematical ideas expressed arguments because part of the group's normative ways of reasoning" (Rasmussen & Stephan, 2008, p.199-200), or taken-as-shared. First, the dialogue has to be analyzed, looking for where there were no more warrants or backings being used. When the claims and data are no longer challenged, then the mathematical idea stands as self-evident. Second, we look to see if "any of the four parts of an argument (the data, warrant, claim, or backing) shifts position (i.e. function) within subsequent arguments and is unchallenged (or, if contested, challenges are rejected" (Rasmussen & Stephan, 2008, p. 200). If the warrant for an argument later becomes the data for a new mathematical idea, and is not challenged then it functions as if it were shared. Using the previous example, the warrant was '2 plus something must make 5 and x is that something'. If there was a future problem given asking what is x in the equation x + 5=7 and a student claims the answer is 2 because '5 plus something must make 7 and x is that something' with no challenge then the idea of x equaling the unknown number is taken-as-shared.

The final step of phase two, after analyzing the arguments across the whole unit, the taken-as-shared ideas will be used to create a mathematical ideas chart for each class day. These charts organize into three categories; taken-as-shared, future perspective taken-as-shared ideas to look for, and additional notes. The charts are compared to the previous days' charts to see which ideas move columns, consistent with Glaser and Strauss' (1967) constant comparison method, where ideas are analyzed, and more evidence is looked for in order to determine conjectures. For this study, I analyzed the dialogue to identify and categorize.

The third, and final, phase of Rasmussen and Stephan's (2003) model is to use the charts created in the last phase to create lists of mathematical ideas that became mathematical practices. The definition of mathematical practice that Rasmussen and Stephan (2003) use is "a collection of as-if-shared ideas that are integral to the development of a more general mathematical activity" (p.201). In other words, we characterized some as-if shared ides as parts of more than one practice, suggesting that the practices can overlap. These mathematical practices that are established through this phase can be used to improve the hypothetical learning trajectory for further research. I completed this final phrase by following the guidelines of Rasmussen and Stephan's (2003) model and identified mathematical practices that were established in this study.

Possible Limitations

The intent of this study was to develop a theory about the ways in which the implementation of RME and an instructional sequence can support students' learning of algebraic equations and expressions. Because of the qualitative nature of a CTE, only studying one class of 20 fifth grade students, the results cannot effectively be generalized to all beginning algebra students. Additionally, a majority of the students that will participate in this study were "on grade level" which may add another limitation to the data.

The participants in the study had not shared all of the same prior mathematics education. Therefore, the prior experiences that each participant had may or may not have been helpful in the development of their conceptualizations within the algebraic concepts.

The instructors that participated in this study came from diverse background experiences. The professor had 20+ years in the classroom and was very familiar with implementing inquiry lessons and using social and social norms. However, I brought less than 10 years experience as a teacher as I was new to implementing inquiry with the use of social and sociomathematical norms. Therefore, our prior experiences that we had as instructors may or may not have affected the implementation of the instructional sequence.

Conclusion

This chapter described the overview of the Design Research methodology used for the implementation of this study. Using the Design Research methodology, along with a Realistic Mathematics Education instructional sequence, the intent of this of study was to provide evidence of the development of students' conceptualizations of algebraic equations and expressions through the methods in which the social and individual environments worked together in a classroom. This was accomplished using a cyclical methodology to organize the affects the social portion had on individual students' mathematical understanding, as well as how each student contributed to the emergence of the classroom mathematical practices. The following two chapters will offer the results from this study.

CHAPTER FOUR: DEVELOPING SOCIAL AND SOCIOMATHEMATICAL NORMS

During the study the professor and I shared the responsibility of implementing the instructional sequence activities (See Table 2).

Table 2: Instructional Sequence Activities and Instructors

Implemented	Title	Instructor(s)	Page
Day 1	The Candy Shop Introduction	T	1
Days 1 & 2	Mrs. Simpson’s Packing Rules	T	2
Day 3	Krazy Kustomer Chaos	T	3
Day 4	Mischievous Maggie and the Mystery Rolls	T	4
Day 4	Mystery Solved!	T	5
Days 5	Brainy Brian’s Balances	T	6
Day 5	Balance Bonanza	T	7
Day 6	*Balance Bonanza Bonus	T	8
Day 6	Brainy Brian’s Balances Again	T	9
Days 6 & 7	Rambunctious Rolls!!!	T	10
Day 8	*Rambunctious Roll Review	T/P	11
Day 9	*Packing Problems	P	12
Days 9 & 10	*Positively Preposterous Peach, Pear, and Pork Chops	T	13
Day 10	Missing Pieces Miasma	T	14
Days 10 & 11	*Missing Pieces Murk	T	15
Day 12	Candy Scales I	T	16
Day 13	Candy Scales II	T/P	17
Day 14	*Quiz	T	18
Day 14	*Candy Scales III	T	19
Day 15	Candy Scales IV	T	20
Day 16	Candy Worker Test	T	21

T = Teacher, P = Professor

As mentioned previously, this was my first experience as a teacher and researcher working with social or sociomathematical norms as well as implementing this type of inquiry teaching without participating in professional development. As a result, as the social and sociomathematical norms were being analyzed, there was a noticeable difference between the approach of the teacher (me) and the professor in the instruction. This chapter highlights the social and sociomathematical norms that were established and how the professor's and my prior experiences resulted in diverse approaches of instruction. The chapter will conclude with a data-based critique of the implementation approaches for social and sociomathematical norms in a classroom teaching experiment with the presentation of common themes found.

Social Norms

Social norms are normative patterns in the ways in which the teacher and students expect each other to participate in classroom discussion. Yackel and Cobb (1996) have found the following norms are constituted in inquiry classrooms: students are expected to 1) explain and justify their thinking, 2) listen to and attempt to make sense of each other's interpretations, and 3) ask clarifying questions when a misunderstanding or disagreement occurs. In more traditional classrooms, the social norms that emerge would be different. Social norms regard the instructors' and students' roles within the classroom, not specific to subject. The instructor(s) and students simultaneously generate expectations of participation among the group, individual responsibilities, and collective negotiation of understanding (Bauersfeld, 1988; Dixon, Egendoerfer & Clements, 2009; Stephan & Cobb, 2003).

The social norms that were collectively constituted within this study included 1) explaining and justifying solutions, 2) interpreting other students' methods, and 3) asking clarifying questions wherever misunderstandings transpired. This study began in the last semester of the school year after these norms were somewhat established, as the students were already participating in discourse in their mathematics class in which they were prompted to explain, question, and interpret others' solutions. The professor that took part in some of the instruction in the study also facilitated student discourse to sustain the norms. To maintain the norms, the professor and I consistently reminded students of their expectations and held them accountable when they had violated a previously agreed upon norm. The social norms were renegotiated as the study progressed to ensure students focused on appropriate learning goals.

Explaining and Justifying

The expectation that students were required to explain and justify their ways of thinking is a social norm (Yackel & Cobb, 1996). This social norm, expecting students to explain and justify their thinking, was established prior to *The Candy Shop* sequence being initiated. In order for students to explain and justify their thinking they were expected to describe why or how they decided on what methods of solving they would utilize and the solution that they created.

During the study, students were given problem-solving scenarios that were intended to promote mathematical thinking. Independent and small group problem solving tasks were designed to engage and encourage students to think about how to clarify their thinking. As Dixon et. al. (2009) stated "research indicates that learning opportunities arise as students participate in whole-class social interactions. These interactions provide opportunities for students to reflect on their methods, justify solutions, and share their information with others.” (p. 1068).

Looking through the data, the research team saw that I was inconsistent with facilitation of student explanations and justifications. Throughout the duration of the study I sometimes gave the explanation for the students, prompted students to explain or directed the students to specific answers; yet, there was also inconsistency when I was making sure that there was a clear understanding of the class when an explanation is given. However, the professor consistently provoked student collaboration, having the students talk to each other and explain or help each other explain their interpretations. The following examples of dialogue will highlight the changes in approaches and how I used my observations of the professor’s approaches to better my instructional approach.

On the second day of *The Candy Shop* sequence, students were introduced to Mrs. Simpson’s Packing Rules consisting of a picture representing an unknown total of candy and various packing rules to use to interpret the visuals.

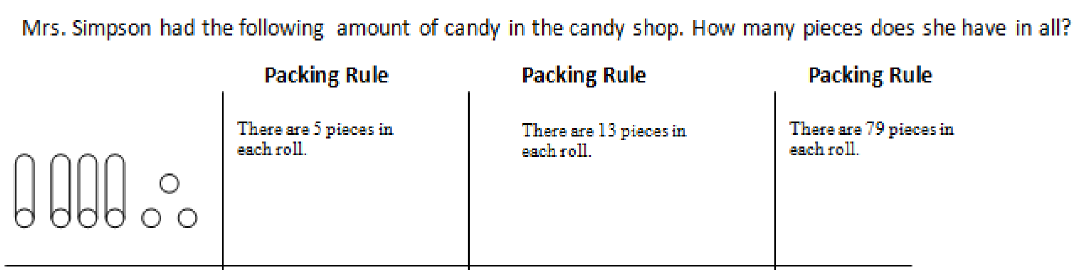


Figure 6: Mrs. Simpson’s Packing Rules, second packaging

The teacher began the whole class discussion by asking students to share their thinking while solving the problem (See Fig. 6). The following example illustrates how I prompted students to share their solutions with explanations, yet did not check other students' understanding of the explanations.

Teacher:	Okay let's go to the second picture down here, um, we're going to the second row but we're going to do the packing rule for the thirteen pieces. So we are looking at this right here, everyone with me? And Savannah what do you see on that picture?
Savannah:	Four rolls and three pieces left over.
Teacher:	Okay four rolls and three pieces left over she says. Anybody have any other way of looking at that? (*students respond no*). Okay, so I need to know how do I need to do this if there, if Mrs. Simpson gives me a packing rule for thirteen pieces in each roll, what are you going to do? London?
London:	You're gonna, um, it says she has thirteen pieces in each roll. There are four roll so you multiply 13 times 4. You have to multiply 13 times 4 and since there are 3 left over you have to add three and then you have to find your answer. 13 times 4 equals, oh wait, 13 times 4 equals 52, plus 3 equals and then you add 3 to that and that will equal 55, so 55 will be your answer for that.
Teacher:	Okay so come write your, um, number sentence on the board (*London writes his number sentence on the Smart Board*). **Okay** is there another way to do that? Renee?
Renee:	Um, can I come up and show?
Teacher:	Uh huh
Renee:	You can do repeated addition.
Teacher:	Repeated addition? Show me what you mean. Talk me through it.
Renee:	You do, you add 13 four times, (*Renee writes her number sentence on the board*) then you could add 3 and it gives you 55.
Teacher:	**Okay.** Renny?

Savannah clarified what was being represented in the visual for the class. I followed Savannah's statement by asking what we should do and London gives his solution with an explanation. I tell London to write his number sentence and ask for another way. There is no discussion from the students or myself about whether these explanations made sense to all or if they are even correct.

On Day 7 the students were providing explanations without prompting which suggests that the expectation to explain was taken-as-shared. However, the excerpt below shows that I was still providing parts of the explanations for the students rather than allowing them the change to complete their argument. Students had been working with Brainy Brain's Balance Strategy to solve for the packing rule in order to help figure out Maggie's Mystery Rolls. The students continued to work through multiple examples independently and then participated in whole class discussion to share their methods and solutions.

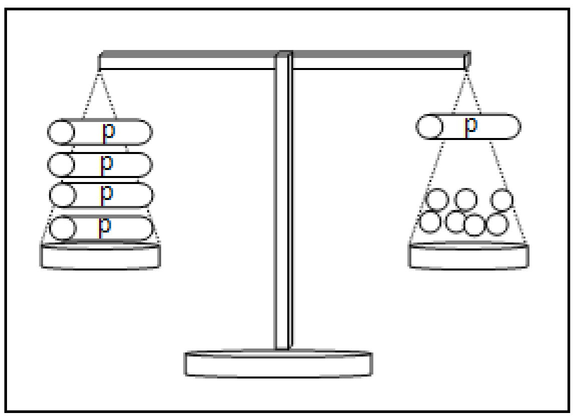


Figure 7: Rambunctious Rolls, #1

The following dialogue is a discussion about the first problem of the day (See Fig. 7), in which I provided parts of the explanations for the students instead of allowing them to explain.

Teacher: If you get confused, use your hand scales that you used yesterday, think about the scale. Let's look at the first one together, I know a lot of you haven't finished but let's look at the first one together. So I want you to think about you being the scale. Alright? So put your hands up, like this (*modeling both hands up like they are carrying trays*), like

	a scale, and you have 4*p* on one side and you have *p* and how many individuals?
Students:	Seven.
Teacher:	Seven. So you have one *p* equals 7 individual pieces over here (*incorrectly states problem*). What do we want to do? Figure out what?
Jordan:	How many pieces are in one *p*?
Teacher:	How many pieces are in one *p*? So we got to get rid of this extra *p* (*shows the side of the scale with the p and individual pieces*), or get rid of three of these *p*'s (*shows the other side of the scale*). We got to get it down to something, so what are we going to do? Anthony?
Anthony:	Take one roll off of each so they equal each other out.
Teacher:	Okay, he says we are going to take one out of each. So let's take one off of the side that has individual rolls, which tilts our scale a little bit, and then we're going to take one off of the other one so that we are back to where we are now, balanced. So, now how many *p*'s do you have on this side?

In this selection of dialogue, the students are prompted with questions on what to do. However, my discourse pattern resembled that classic funneling strategy (Voigt, 1995) in which the teacher's questions become so specific that the strategy is revealed in the teacher's questions rather than the students' explanations. As the excerpt shows, I gave them the strategy to use and then provided explanations to why their solution (the answer to the question that I gave them) worked or not. There was no student to student collaboration in the explanation and understanding of the procedures that were being made at this point on Day 7 of the instructional sequence.

On Day 8 of the instructional sequence the professor stepped in for me, as the instructor. Prior to the following dialogue the students had been working independently on a practice quiz that was designed to understand their knowledge of the instructional sequence concepts to this point. The professor explained to them that they would be sharing and discussing their thinking from the quiz with the whole class. The following dialogue is part of the first exchange with the students.

Professor: Okay. I did not get to talk to everybody (*during the monitoring time*) so if you wouldn't mind, if you're not finished with the quiz keep going but I just want to come around and ask you how you solved them so I can kind of get an idea of what we are going to talk about. Alright? So you guys talk to your partners about how you solved maybe one or two of them. Start with the first one, talk to your neighbors about it. Do you guys have shoulder partners or whatever partners at your groups?
Students: Yeah.
Professor: Okay. Talk to your partner or whoever that might be and compare and contrast your solutions.

In this exchange the professor highlighted the importance of peer collaboration and support for the students while working through the problems. The professor encouraged students to explain their solutions and solving methods by having them "compare and contrast" with their peers.

When small group sharing ended, the professor pointed out that most students understood the first problem, but still ignited a discussion by prompting students to explore the elements of the problem by asking questions. The following dialogue illustrates how the professor facilitated the classroom discourse.

Professor: Okay, we're going to get started and we're going to start with number 2 because it seems that most people got number one. What was the answer to number one? Just so everyone knows they got the same thing. What was the packing rule?
Students: y = 5.
Professor: So this yellow candy, whatever it is. What is a y candy?
Students: It's just a variable.
Professor: It's just a variable and so, sometimes people that I know that have done this before they like to name the candy after the letter that's used. So, we'll call it some kind of yellow candy.
London: Yellow banana.
Professor: Yellow banana, okay. So the yellow banana packing rule is how many pieces per roll?
Students: Five.

Professor: Five pieces every roll, so if you, if Bart opens up the candy what should he expect to see fall out?
Students: Five yellow pieces.
Professor: Five yellow banana pieces, you got it!

The professor did not allow the solution to be the end of the discussion. The professor used questions to encourage students to explain the idea of a variable and what the packing rule meant in this situation.

Throughout the remainder of the sequence it is apparent that both instructors and students initiated and/or prompted student explanations. My actions on Day 8 illustrated a turning point for me in my instructional practices. By observing the professor, I was able to see and, consequently, utilize alternative ways to encourage more student initiated explanations rather than giving or directing the explanations and confirmation of knowledge myself. I began to initiate elaboration when students presented simple answers or struggled to understand how to arrive at the solution. Yet, there were occasions when I did not have to request explanations, indicating that explaining and justifying was taken-as-shared early in the study.

Interpreting Others' Methods

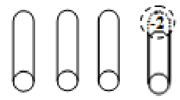
As students interpret each other's methods, they delve into other perspectives; therefore, grasping deeper understandings of the concepts. Prior to *The Candy Shop* sequence, students were familiar with having to provide statements of whether they agreed or disagreed with other students' methods. Therefore, interpreting others' methods was a social norm that was already in place before *The Candy Shop* started. Students used this strategy with the expectation that they were to understand others' thinking to compare and contrast their own thinking, as well as generate visual and/or verbal interpretations.

Throughout the study, I prompted students to interpret others' methods in whole class discussions. I used two techniques to hold students accountable for this particular norm by indicating whether they agreed or disagreed with the others' methods. The first technique involved students being expected to help others when misunderstandings emerged. I prompted students to help clarify the misunderstanding by participating in discussions with other students, analyzing and elaborating a comprehensive understanding of the concept. The second technique used involved students being expected to restate other students' actions or comments in order to confirm their thinking. There were times in the study, especially at the beginning, that I (again) did what I should have expected the students to be doing, such as giving the interpretations instead of allowing the students the opportunity to do so. However, after the professor and I discussed the need for the students' input, I began to refine my expectations for interpreting others' methods. With this clarification and reestablishment I was able to focus more on giving the responsibilities to the students.

As an example of my interactions with students early on, consider the excerpt below that occurred on the first day of the instructional sequence. The students were introduced to the scenario and characters of *The Candy Shop* and were discussing the representation of the candy in *The Candy Shop*. In the discussion below, the students were told previously to figure out how many pieces are there were of the given flavors with the given picture (See Fig. 8).

Representing Candy in the Candy Shop

Three rolls and a roll missing 2 pieces <u>or</u> Four rolls less 2 pieces



How many pieces are there altogether in each picture if...

Packing Rule Orange: 10 candies per roll
Packing Rule Rootbeer: 17 candies per roll
Packing Rule Grape: 53 candies per roll

Figure 8: Representing candy

Teacher: Alright, I'm going to stop you for a second. So, as I'm walking around, I am seeing that you are understanding that there's 10 per roll so you put the 10 in the roll and you saw the individual pieces. And they don't count as 10 because that's not a whole roll, correct?

Students: Correct. Yeah.

Teacher: That's what I heard from you guys. Does anyone want to add anything to that? (*No students reply*). What happened over here, Nathan?

Nathan: For the first one, I did 10 times 4 minus 2.

Teacher: And how do you know it's missing the 2?

Nathan: Because it says so.

Teacher: Because it has the drawing on there where it shows -2 which shows it's missing. Alright, well I want you guys to turn over to the next page.

This exchange began by me giving my interpretation of what I had observed the students discussing and doing in their groups. I then asked a student to add more but when he explains his act of subtraction by answering "because it says so" I, again, interpret his interpretation. Therefore the students did not have the opportunity to think through the process, or show their understanding.

This next excerpt illustrates how students built deeper understanding of a concept by interpreting others' methods. On day five, as the whole class revisited the scenario of Mischievous Maggie and the Mystery Rolls, I posed the question 'What is a variable?'

Brian shared his understanding and Chuck followed up by interpreting Chuck's and his understandings.

Brian:	A variable is what you use during like algebra, a type of thing like instead of showing a question mark after the equation; you would put five plus five equals *x*, something like that.
Teacher:	(*sees Chuck's hand raised*) Okay, you want to add to that?
Chuck:	**I disagree with Brian. A variable, well I agree and I disagree, because a variable can be that but it can also be part of the equation. Like five plus *x* equals ten.**

During this discussion, Chuck stated how his understanding of variables coincided, yet deviated from his interpretation of Brian's method. At this point Chuck is interpreting Brian's method without any prompting. He also provided other students' the opportunity to consider a more distinct definition of a variable by sharing his interpretation of Brian's original statement and his own understanding.

The expectation to assist others when misunderstandings occurred was revealed shortly after the recap of Mischievous Maggie and the Mystery Rolls. Students were introduced to a new character, Brainy Brian, and his balance strategy that he created to solve Maggie's Mystery Rolls. I asked the students to discuss what they read about this new character.

Teacher:	…we discussed somebody came up with a plan, who was he and what… go ahead Evan.
Evan:	Um, Brainy Brian.
Teacher:	Who's that?
Evan:	He's Maggie's nephew and he's in the sixth grade and a science brainiac. He came up with a code.
Teacher:	A code? What kind of code?
Evan:	Um, he came up with a code for each mystery roll.
Teacher:	Okay and what was that?
Evan:	Ummm…
Teacher:	**Anybody want to help him out**? Calvin?

Calvin:	He would put each roll on a balance scale with other pieces, loose pieces, and whenever they balanced out he would know that's how many pieces were in the roll.
Teacher:	Okay so you put it on a balance scale with an amount of pieces. **Can you give a little bit more detail there?** (*directing question to Evan*)
Evan:	He would do it each day and then he would keep going and see how many pieces equaled each roll.

As Evan contributed to the discussion with his synopsis about Brainy Brian and his method of solving Maggie's Mystery Rolls, he was not able to express the details of Brainy Brian's code that he referred to. For that reason, I prompted for the students to step in and assist him. At this moment I recognized that there was a misunderstanding and addressed this by asking others for help. Another student, Calvin, accepted the challenge and elaborated Brainy Brian's code. After Calvin helped, I reestablished dialogue with Evan prompting him to give a little more detail. By reestablishing the discourse with the student who was unable to elaborate an idea, the individual may be compelled to revise and clarify his understandings.

As the professor participated in instruction on Day 9 of the instructional sequence, I observed a way to hold the students accountable for making sense of other's solutions. Instead of asking someone to help another student out, the professor did not ask for volunteers. Instead, she asked a particular student who had not volunteered to re-explain what another student just said or did to solve a problem.

Prior to the following dialogue, the professor presented the students with a handout for them to solve given problems. The following dialogue occurred within the discussion of the following problem (See Figure 9) that was in the Packing Problems handout.

2. How much candy is in the collection below if the packing rule is g = 15? Draw a picture, if necessary.

$$3g + 5 + g$$

Figure 9: Packing problems, #2

Adam: So, first I did 15 times (*writes 15 x 3 on the board*) which is 45 (*writes the answer on the board*) and then I just said that g was 15 (*marks out the g in the problem 3g + 5 + g that was on the board*). So then I did 45 plus 5 plus 15 (*writes out 45 + 5 + 15 on the board*) which is 45 plus 15 is 60 and then I added 60 plus 5 which is 65, so the answer is 65.

Professor: Whew, a lot of stuff up there, but did you follow him?

Students: Yes.

Professor: You kind of had to follow him while he was talking. How many people raise your hand if you did it Adam's way. (*Students raise their hands*) Alright, so Adam's way, I'm going to put it up here, Adam's way was (*writes on the board while talking through it*) 45 plus 5 plus 15. Okay, so Adam if it's okay I'm going to erase this stuff. That was helpful for us to follow you but I think if we put it over there we are okay, we know how you did it. Alright, raise your hand if you did it a different way than Adam. (*Students raise their hands*) Nathan, I know you did I remember seeing your paper, and I think you did Luther. Come on up here. So, there's Adam's way, now Luther is going to show his way. So, if you did not do it Luther's way, see if you agree with him.

Luther: (*Luther goes up to the* board) Okay so I did 3*g* plus 1*g* equals 4*g* (*writes 3g + 1g = 4 g on the board*), so I had 4*g* plus 5 (*writes 4g + 5 on the board*). And so I did 4 times 15 is 60 (*writes 4 x 5 = 60 on the board*) and then I did 60 plus 5 equals 65 (*writes 60 + 5 = 65 on the board*).

Professor: Ok, so I would like, **Cindy, can you like summarize the difference between Adam's and Luther's (solutions)**

Cindy: Well, Adam, he did step by step, so he did 3*g* plus 5*g*, so 45 plus 5 plus 15. And Luther, he just took a short cut, and he added the two *g*'s together because its plus *g* and he just did 15 times 4, instead of 3 times 15 and adding the extra *g*. And then he just added the five.

Professor: Is that legal to do in here? Is it legal to add the 3*g* and the 1*g* first?

Students:	Legal (*majority of students say legal, some say illegal*)
Professor:	Wait. Wait. Wait. One person at a time. Debbie? Debbie's talking to you guys.
Debbie:	**It's basically the same thing except instead of actually using numbers, Luther used the *g* and Adam used the numbers that were with the letters.**
Professor:	**Did you hear what Debbie said? Okay, she said, and see if I got you right Debbie. Adam, kind of went in this order (*pointing to the expression on the board*) and he just used the numbers for g,** what was *g*? What does *g* stand for by the way?
Students:	15
Professor:	What does it stand for in the story?
Students:	The packing rule, 15 pieces per roll.
Professor:	**15 pieces per roll and so he just put 15 pieces per roll in the *g*'s in that order, 15 times 3 plus 5 plus 15 more. But Luther, according to you** (*speaking to Debbie*) did what?
Debbie:	He used the variables.
Professor:	**He kept the rolls, the pieces per roll as a variable and just combined them and got 4 rolls with *g* in each.** And that's okay to do you think? (*asking the class*) Debbie thinks so.

Within this discussion, as Adam and Luther explained their reasoning, two things happened. First, the professor asked Cindy, a student without her hand raised, to restate what the two students did to solve the problem. Cindy, indeed, verified her own understanding by interpreting the other students' methods of solving. She also showed that she had attempted to make sense of her classmates' thinking. By calling on Cindy, who had not asked to speak, the professor was illustrating to the rest of the class that she expected all students to try to make sense of the solutions being presented. If she only called on students who raised their hands, the other students would not feel accountable for listening to others.

As the discussion continued, the professor called on Debbie to explain why she thought it was "legal" to add numbers with variables with a single variable. As Debbie explained this, she added her interpretation of the two boys' solutions. The professor then restated Debbie's interpretation of Adam and Luther's solutions, and then asked the class if they thought Debbie's interpretation was correct. By this action, the professor was modeling how to restate others' methods of thinking. By posing the question at the end of the interpretation, the professor also promoted other students to generate interpretations of the two students' methods and how they may or may not apply to their own methods.

After observing how the professor held students accountable for interpreting others' methods, I decided to try this approach as well in order to better facilitate student-led discussions. However, I felt that I was still struggling not to give my interpretations. Then, as the discussion from above continued, the professor illustrated a strategy for highlighting a contribution that had the potential of supporting mathematical reasoning. The dialogue illustrates how the professor utilized restating to have the student think about her method.

Professor: What do you want to say? (*Speaking to Hannah*)
Hannah: I did it a different way.
Professor: You did it a different way? (*Hannah shakes her head yes*). Go on up (to the board) and use black, the only color we haven't used yet.
Hannah: I had, well it's a lot similar to what's in Adam's.
Professor: It's similar to Adam's, you think?
Hannah: But it's just a little bit different. So I did 3 times 15 (*writes 3* x *15 and then erases and writes 15 x 3 on the board*). And just to make it a little bit quicker I just did 45 plus 20 and that's 65 (*writes 45 + 20 = 65 on the board*)
Professor: Ah, it's a little easier that way, huh?
Hannah: Because if you had the 15
Professor: You could do it right here (*points to 5 + 15 on Adam's work of 45 + 5 + 15 written on the board*)
Hannah: Because if you add 15 plus 5 you get 20 (*writes 15 + 5 = 20 on the board*) and instead you just do 45 plus 20 (*points to her earlier work on the board*)
Professor: So you're saying that order doesn't matter where you add them (*Hannah shakes her head no*) and it's a little bit easier to chunk the 15 and 5 together. That's easy, that's 20. And then 20 is easier to add to 45. I like that.

As the professor observes that Hannah is noticing the commonalities between her and Adam's methods, the professor restated Hannah's method to help Hannah reflect and confirm her thoughts. She was also helping make Hannah's thoughts more comprehendible; while it was clear to Hannah that the addition order didn't matter (that was her criterion for her solution being different), her explanation did not explicitly bring that notion out. Therefore, in this exchange Hannah confirms Adam's method, and the idea that order in addition of numbers did not matter and that there are different methods of solving for faster or easier processes is validated.

In summary, both the professor and I encouraged students to restate what another student said or did, and how or why they said or did something most times. We believed that these restatements lead students to participate in and initiate more of the discussions, in which they would explain and inquire into others' methods of reasoning. The students in this study understood that it is acceptable to verbalize when one agrees or disagrees with others' reasoning, as long as they provided an explanation of their decision. The students also understood the purpose of whole group discourse was to assure that they and other students comprehend the concepts. Furthermore, they understood that it was responsibility as a classmate to help other students understand any misconceptions by sharing an explanation and justification of their own. Although there was very little need to reinitiate the social norm of interpreting others' methods of reasoning, there were some further negotiations that took place to improve the discourse. The students were persuaded to reflect more deeply upon the contributions of their peers. They were encouraged to take the steps to make sure that they understood others' methods before interpretations were made. These steps may include restating others' words and/or actions, or even questioning. Therefore it was understood at this point that interpreting others' methods included the use various strategies to understand others' thinking in order to compare and contrast one's own thinking was taken-as-shared.

Questioning

Unlike the other social norms, questioning students when they did not understand was not established prior to or even initially in the study. Students were expected to be able to analyze others' explanations and form questions to address the misunderstandings they might have in order to clarify the ideas. For the first half of the instructional sequence, there were a few students that asked questions but the majority of the questioning came from me, as a way to prompt students to process their thinking. Later in the sequence, the questioning norm was negotiated as the professor explicitly instructed students to ask questions about other students' actions and methods. This allowed me again to observe how to get the students involved in questioning and as the students were coached and learned how and when to ask questions; students' initiation took place more often, indicating that the norm became established.

In a previous section it was mentioned that students were introduced to the representation of candy rolls and pieces in *The Candy Shop* (See Fig. 8). Nathan gave his solution for the problem being discussed and when I requested an explanation he explained that he would have to take out two pieces because the roll was missing two pieces. At that point I began negotiating the norm of questioning when one does not understand. I asked Nathan "how do you know it's missing two?" to clarify that not only he understood what he was doing but that he understood the representation of the visual. I had identified an area of probable misconception and posed this specific question to challenge Nathan for clarification. As the whole class discussions continued I posed questions (i.e. “Does anyone agree or disagree? “Does that make sense?” “Anybody do anything different?”) that alluded to alternative solutions and methods, in order to encourage students to participate in further discussion.

The only questions that the students asked initially were very superficial questions, such as the example in the following excerpt.

Chuck:	Um, you could do repeated addition, instead of multiplication.
Teacher:	Okay, tell me what you would get for that.
Chuck:	**Uh, for that one?**
Teacher:	Uh huh (*yes*)
Chuck:	You would do ten plus ten plus ten plus ten minus two, which is 38.

When asked to solve a problem, Chuck was unsure about which problem to solve. Even though the question that Chuck initiated was not conducive to building conceptual understanding it did help redirect his focus. These types of general procedure questions are the only questioning that the students initiated for the first seven days of the sequence. Students, like Chuck, generated questions to find the location of a problem, to determine if they needed to come up to the board to share their solution, or presented answers hesitantly in the form of a question.

It was on the Day 8 of the sequence that the professor re-negotiated the questioning norm while facilitating instruction. Specific students were chosen to share their picture for the problem below (See Fig. 10). The professor urged the class to analyze Savannah’s drawing (See Fig. 11) and suggested Savannah may need their help with part of her drawing of the right side of the scale.

Solve for the Mystery Roll Packing Rule. The scales are there ONLY if you need to use them. Be sure to SHOW your thinking!

2. $12x + 7 = 18 + x$

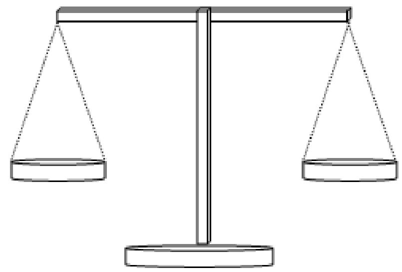


Figure 10: Rambunctious Roll Review, #2

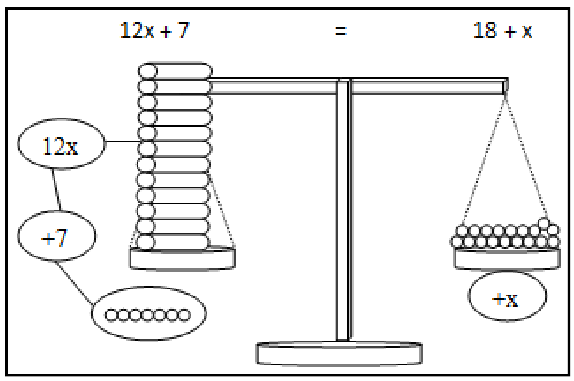


Figure 11: Savannah's interpretation (drawn on board)

Professor: Alright, who wants to ask, let's just work on the picture for right now, **who wants to ask Savannah a question about her picture?** Savannah, you ask, you call on whoever. You are the teacher right now. (*Savannah points to Debbie*).

Debbie: My OCD is bothering me, but **can you draw a full roll on the right side of the scale?**

Professor: So, Debbie, right (*questioning if she has the correct name*)?

Debbie: Yes.

Professor: I'm going to rephrase your question. You kind of told her what to do in your question. It really wasn't a question was it? You kind of don't know what to do so maybe ask her a question about her drawing, **like why did you…? Or why didn't you?** Or whatever. So some question about her thinking. What question could you ask Savannah? Not tell her what to do to match your picture, but ask her a question about hers. Go for it (*points to Cindy*).

When Debbie attempted to ask a question, she indirectly implied that Savannah's picture needed to match her own. The professor asked Debbie to rephrase her question to ask Savannah about her picture rather than telling her how to imitate her own drawing. At this point, the professor attempted to establish the norm of questioning by modeling to the class ways to question, such as "why did you…?" or "why didn't you..?"

Using one of the questioning formats the professor modeled, Cindy asked a question about Savannah's drawing.

Cindy:	**Um, why did you put 18 individual pieces instead of 18 rolls?**
Savannah:	Because it says pieces.
Professor:	Did you all hear her question? Great question. She asked 'why did you put 18 pieces on the right hand side instead of 18 rolls?' Who thinks they have an answer to Cindy's question?
	…
Professor:	Alright, go Savannah, she asked you so you explain it.
Savannah:	Because it didn't say 18*x*, it said 18 plus *x*.
Professor:	Repeat what she said (*speaking to Farrah*).
Farrah:	That it didn't say 18*x*, it said 18 plus *x*.

Not only did Cindy's question enable Savannah to justify her methods, but it also indicated a misunderstanding that Cindy had of the representation of rolls and pieces using variables. As a result, students' continued to negotiate Cindy's conception reinforcing the differences between the representation of rolls and pieces using variables. This allowed me to see which students comprehended the representation of the candy and candy rolls.

Subsequently, on Day 10 of implementing the sequence, students continued to struggle to ask questions when a conflict in interpretation arose.   While engaging in the daily review labeled Simpsons' Studies, students were instructed to write an expression for the given packing rule and drawing on the smart board, as seen in Figure12 below.

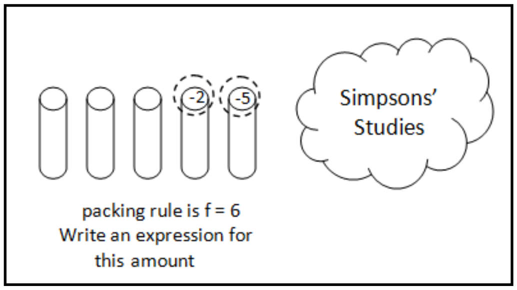


Figure 12: Simpsons' Studies, day 10 (drawn on board)

After one student, Chuck, presented five expressions as methods of solving the Simpsons' Studies (See Fig. 13), students were given time to look over and mark each expression with an emoji to show if they agreed, disagreed or did not understand. The students then participated in whole class discussion to share their understandings. As the first expression was discussed, the students unanimously agreed that it was correct. The students continued on to discuss the second expression ($5f - 2 - 5$) when Anthony, a student in the class, stated that he disagreed. The following dialogue shows Anthony's initial reason for disagreeing and how he eventually questions Chuck's method.

$5f - 7$

$5f - 2 - 5$

$5 \times 6 - 7$

$5 \times 6 - 2 - 5$

$3 \times 6 + 4 + 1$

Figure 13: Methods of solving day 10 Simpsons' Studies (written on board)

Teacher: So, with a show of hands, 5*f* minus 7, who agrees with, who gave a smiley face? (*students raise hands*) Frowny face? No Clue? Okay, so everyone agrees with that. Okay, 5*f* minus 2 minus 5. Smiley faces? (*a majority of the students raise their hands*). Frowny faces? (*Anthony raises his hand*) Anthony, why do you have a frowny face?

Anthony: I disagree with that.

Teacher: Why?

Anthony: Because it's all subtraction. So you do 2 minus 5 which would get you negative 3. And you wouldn't get negative 3 on this part.

Teacher: Chuck? Come on up here and you can explain.

Chuck: Actually you would put parentheses (*writes (5f-2)-5 on the board*) which would be 5*f* – 2 which would be 28 so then you subtract 5 and get 23.

Anthony: Now that makes sense but before…

Teacher: Do you have to do the parentheses?

Students: No.

Teacher: Because you would multiply, then you would subtract your 2 and then subtract your 5.

Anthony: I know, but I still disagree, **why wouldn't you just add them together? Add 5 plus 2 and do minus 7.**

Teacher: But is it the same thing?

Anthony: Yeah, but you're just over complicating it.

Teacher: Okay, over complicating things is different. Its' not over complicated, it's just a different way of thinking. Okay so we have negative, I mean minus 7 here and we have minus 2 minus 5 here which is the same thing. Now you guys got stuck on over complicating; it's just another way of thinking.

Chuck: You're not really over complicating because (*points to picture and how one package is missing 2 and one is missing 5 pieces*). You would be over complicating more if you used 7 because 7 you had to add it.

Teacher: Okay, so we can agree that both of these are the same, so let's go to the next one, 5 times 6 minus 7.

Here Chuck and I presented mathematical solutions referring to order of operations to help Anthony see that the method that he disagreed with would work. Then Anthony challenged Chuck's method of splitting the missing pieces of each of the two rolls instead of combining the values together to make it minus seven by questioning him. I, then, questioned Anthony's challenge to make him reanalyze his thinking. Through the questioning that took place, Anthony was able to see that this particular expression was just a different method that illustrated a different way of thinking and it did in fact work.

In this next segment of the same discussion, you will see how I tried to imitate the professor's modeling of prompting students to ask questions in order to sustain the questioning norm.

Teacher:	So, let's go to Kendall, come over here. She has *f*5 minus 7 equals *x*. **Any questions for her?** Brian?
Brian:	Why didn't you show the addition to get the 7? Because it never says anywhere that you have a 7. You have to add the numbers so you have to put that down on paper…
Teacher:	Well they got the 7 by the -2 and the -5, that's…
Kendall:	I just put it together to make it an easier problem to look at
Brian:	But don't you have to add them on the paper? Because it never really says you have seven.
Teacher:	Nope, just another way of thinking. That's what we were just talking about. Some people might have to add them. Some people can look at these two (*-2 and -5*) and know it's seven. Question?

In this dialogue I prompted students to ask Kendall questions about her solution, and Brian asks a question for her. However, I addressed Brian's questions by explaining and justifying Kendall's reasoning. At this point, I realized (by catching myself give the explanation) I was hindering the purpose of the questioning norm, which is to offer the students the opportunity to explain and justify their conceptions for deeper comprehension. Therefore, I stopped and Kendall started to answer the question that Brain had asked. As the dialogue continued, Brian initiated more questioning with justification and without teacher prompting.

As the sequence progressed, so did the students and my roles for this norm. On Day 11, I initially prompted and restated a student's response as Kathy presented her solution.

Teacher: Nine is her packing rule she says. **Anybody have any questions about her picture?**
Student: **I'm confused about where the 12 and the 9 came from.**
Teacher: **Where did the 12 and the 9 come from Kathy?**
Kathy: Um, the 12 came from right here (*points to the roll that is missing 12 pieces*) and the nine came from when you subtract 3, you get 9.

I restated a student's statement of confusion as a question, giving Kathy the opportunity to explain and justify. This resulted in highlighting how Kathy misinterpreted the visual. As the dialogue continued, Chuck presented his solution of x = 15 and, again using the professor's approach, I prompted student questioning. The strategy that I decided to focus on what throwing questions back at the students, whether it was to prompt or restate questions they were making, in order to sustain student collaboration and eliminate me giving them my explanations.

Teacher: **Anybody have any questions?** London?
London: I don't see how you got; I mean I see how you got that but…
Teacher: What did you get for your packing rule?
London: I got c = 9, I mean x = 9 because I did Kathy's way. Because I saw that there were 3 loose pieces and one package that said minus 12 so I, um, I tried to figure out what minus 12 equal 3 and 12 minus 9 equal 3 so that's my packing rule that I got.
Teacher: Anybody else get 9? (*No one raises their hand*). Anybody else get 15? (*A majority of students raise their hands*) **Can we help Kathy and London figure this out?**

When I utilized questioning to further London's thought process, London was given the opportunity to reevaluate his solution. I then posed a question to survey the class on which of the two solutions they felt was most likely correct. This verified the need for my next questioning, which promoted the class to help Kathy and London's resolve their misconceptions.

Shortly afterwards, in the same whole class discussion, there is a shift from teacher prompting to student initiation of this norm.

Teacher:	Calvin?
Calvin:	Aren't we supposed to… **So I have a question for you guys, aren't we supposed to find the original packing rule?** Because Chuck, you found the original packing rule for it. You guys (*speaking to London and Kathy*), you found another packing rule than the original.
Teacher:	**Explain why you think that.**
Calvin:	I think that because they subtracted 12 pieces, so it's equal to 3. So you have to add 12 and 3 together to get your original packing rule back to normal.
Teacher:	So, Debbie?
Debbie:	**And question… How would you get 9 pieces?**
London:	Oh, because um, I didn't do any adding. I just did, um, I thought what times (*meant minus*) equals 3. Because of the other side, and I knew 9 from 12 equals 3.

In this exchange, Calvin and Debbie asked questions without my direct prompting. Although Calvin posed a rhetorical question to the class about a prior learned procedure, this type of question persuaded students to reassess the problem and choose the best method of approach. I challenged Calvin to explain his interpretation, which he elaborated was the application of inverse operations. Consequently, more student initiated questioning was generated to decipher the misconceptions of all students. At this point, the students continued to utilize the questioning norm and without much prompting. Therefore, it was with this specific episode that the questioning norm was taken-as-shared.

Even though the explaining and justifying norm and the interpreting others' methods norm were both put into place before the study began, they had to be renegotiated in the context of *The Candy Shop* sequence. The students were able to build onto these norms by negotiating techniques to improve their explanations and interpretations in which they were already familiar. As the students proceeded through the instructional sequence they improved their questioning techniques, progressing from asking only superficial, procedural questions to asking questions that help them build conceptually to clarify their thinking and interpreting. I also felt that my role as the teacher had improved as I was able to alter some of my approaches to help the students by observing and using methods that the professor illustrated in her implementation of the classroom lessons. Through the negotiation and re-establishment of these social norms, the students created a basis to explore mathematical concepts through a more inclusive class discussion. As discussed earlier, social norms only concern the instructors' and students' roles in the classroom, not specific to subject. However the social norms help direct the sociomathematical norms, which are exclusive to mathematical content, regarding students' mathematical activity.

Sociomathematical Norms

According to Cobb and Yackel (1996), sociomathematical norms are social norms applied specifically in a mathematics content area, which in this study is the application of arithmetic to comprehend algebraic expressions and equations. Students' understanding the expectations to explain, justify, and interpret others' thinking when participating in class discussion are social norms; in a similar way, there are ways of participating in conversations that involve arguing mathematically; the criteria for the mathematical argumentation has been termed sociomathematical norms. Examples of sociomathematical norms include the criteria for what counts as a mathematically different, efficient, sophisticated and acceptable explanation (Stephan & Cobb, 2003;

Yackel; 2001; Cobb & Yackel, 1996; Cobb, Yackel, & Wood, 1991; Cobb, Yackel, & Wood, 1989).

The criteria for sociomathematical norms can be significantly different from classroom to classroom. Even though the instructor may have ideas of what specific norms they hope to produce in the instruction, the sociomathematical norms are not imposed in the class sessions. Just like the social norms, sociomathematical norms are normative understandings that are collectively negotiated and established by students and instructors (Cobb, Stephan, McClain, & Gravemeijer, 2001; Cobb & Yackel, 1996). This implies that all members of the classroom develop a consensus of taken-as-shared ways of participating in *mathematical* conversations (Zembat & Yasa, 2015; Yackel & Cobb, 1996) and negotiations (Dixon, Egendoerfer, & Clements, 2009; Bauersfeld, 1993).

Yackel and Cobb (1996) have suggested four different sociomathematical norms that can support productive mathematical discourse: the criterion for what counts as 1) a different mathematical solution, 2) a sophisticated mathematical solution, 3) an efficient solution and 4) an acceptable mathematical explanation. The sociomathematical norms established in this study consist of regulating what is considered as 1) an acceptable solution, 2) a different solution, and 3) an efficient solution.

Acceptable Solution

Throughout the duration of teacher-facilitated mathematical discussions in *The Candy Shop* sequence, the class negotiated the criterion for what constituted an *acceptable* solution. According to Herhkowitz and Schwarz (1999), a solution is considered *acceptable* when students are able to interpret their explanations and justifications describing the actions taken with mathematical objects. As it turns out in this classroom, what counted as an acceptable explanation was an explanation that focused on both the calculations as well as what the numbers in the calculations stood for in the picture (Thompson et al., 1994).

Prior to the study, the students were obligated to provide explanations and justifications for their mathematical methods in class. Consequently, when I prompted students to clarify their reasoning the students complied by attempting to describe the mathematical calculations in their reasoning. Recall from the previous section on social norms, in the initial discussions I provided many of the explanations and justifications for the students' solutions, or specifically directed the students to "explain" their solutions in the attempt to conform to what was *acceptable.* During subsequent conversations, I made a conscious effort to ensure that the students had opportunity to deliver the explanations and justifications themselves. As the students were challenged to try innovative ways to interpret their own and others' methods mathematically, the students' criterion of what constituted as an *acceptable* solution evolved. In the next few sections, I provide analysis that shows the establishment of a new criterion for what counts as an acceptable explanation.

During the first day of the sequence, students were asked to write number sentences to show their work on how they arrived at total number of candies found in each of Mrs. Simpson's Packing Rules. As students examined Mrs. Simpson's first packaging (see Fig. 14), they explained their thinking visually and verbally with and without teacher prompting. The following excerpt illustrates the negotiation between the students and me about what criterion was needed to constitute an explanation as acceptable.

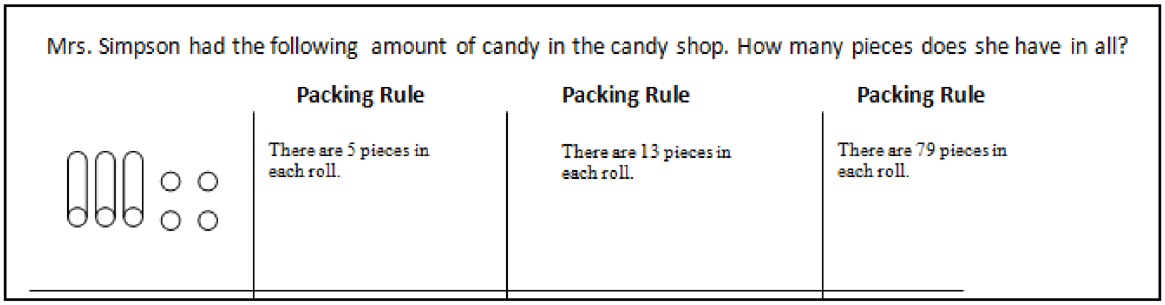


Figure 14: Mrs. Simpson's Packing Rules, first packaging

Chuck: You could do four rolls and then, uh, you could subtract one from it, because there are five pieces in each roll, if you wanted to do one thing. And then five minus four equals one.

… (*Hannah writes her number sentence, 5 x 3 + 4 =19, on the smart board at the teacher's request*)

Teacher: Okay, so she got 5 times 3 plus 4. **Explain.**

Hannah: Oh, I did 5 times 3 plus 4 because, um, you have 5 rolls of can… you have 3 rolls of candy and there's 5 pieces in each, and you have 4 pieces left over.

Teacher: Okay, so the 5 times 3 comes from?

Hannah: The um, there's 5 pieces of candy in each roll and there are 3 full rolls.

Teacher: And then the 4 is the?

Hannah: 4 extra pieces.

Teacher: And then Chuck you said there is something different you could do. You want to come show us?

(*Chuck writes his number sentence, (5* x *4) – 1, on the smart board*)

Chuck: Since there are 3 rolls and then there are 4 extra pieces and there are 4 pieces in each roll, you could do 5 times 4, which is, say you added one extra piece but then it's not actually a real piece so you do 5 times 4 minus 1 because there are 5 pieces in each roll and so 5 and 5 minus 4 equals 1 and so 5 times 4 minus 1 equals 19.

Chuck and Hannah both contributed to the negotiation of the sociomathematical norm of what constituted as acceptable explanation as they answered my prompts to share and explain. Chuck initially offered his solution without any mathematical reasoning. As I prompted Hannah through her explanation, the criterion for what constituted as an acceptable answer became more explicit. I had Hannah restate her explanation to emphasize her successive mathematical thinking, relating to the problem that she has solved. Here, I was encouraging the students to see that their explanations must focus on both the calculations as well as what the numbers in the calculations stand for in the picture. This is what Thompson, et. al (1994) called a conceptual orientation of math. As the discussion was redirected back to Chuck, he immediately presented his explanation and justification for his solution. However, he also had noted the criterion that was negotiated in the dialogue between Hannah and myself and was able to utilize it to make his explanation acceptable.

As the lesson continued, I generated opportunities for students to participate in the discussions by asking if they agreed, disagreed, or had anything to add.

Teacher:	Anybody have any other thinking for that one? Anybody agree, disagree, have anything to add?
Jordan:	I agree.
Teacher:	Why do you agree? Which one do you agree with?
Jordan:	I agree with both of them. I mean both of them are right, they are just done in different ways. Just different thinking.
Teacher:	Okay. Anybody else? Brian?
Brian:	I agree because they both get you to nineteen, and they both show the correct math. Like they both, um, both number sentences are correct, because they use the math from the picture and put it into sentences.
Teacher:	Okay.

Both students stated that they agreed with the two solutions presented, acknowledging the two students' mathematical thinking presented was different yet correct. This acceptance and acknowledgement of students in the class highlights that what constitutes as an *acceptable* answer was being negotiated by the students. At this point I accepted the answer with a simple “okay” but did not elaborate or have students elaborate more in the whole class discussion.

On Day 8 of the instructional sequence, the professor was present to observe my implementation of the instructional sequence. As revealed in the following selections of dialogue and discussion in this section, I was being a substantial contributor in the whole class discussions. Therefore, the research team felt it was necessary to reestablish the role of the instructor as more of a facilitator.

Prior to the following discussion, Nathan confirmed the following visual (See Fig. 15) presented a new concept unlike the previous visual examples. The students were now presented with a combination of candy pieces and rolls together on one side of the scale and pieces on the opposite side, rather than only pieces on one side and only rolls on the other. In this example, the roles of the instructors and students are renegotiated.

Figure 15: Rambunctious Rolls, #5

Cindy:	You have to do 46 minus 26 and you have to cancel the plus 26 on this (left) side (of the scale).
	…
Teacher:	… So then she had 5*x* and 20, so how many pieces did you have in the rolls?
Cindy:	Um, 4 since 20 divided by 5 equals 4.
Teacher:	**Okay, so you divided both sides by 5 or just one?**
Cindy:	Um, I did 20 divided by 5 since it's 5*x*.
Teacher:	**How did you get rid of your five on the other side?**

Cindy: Um, on that side… since there were 20 on that side because you did 26 minus, I mean 46 minus 26 and it equaled 20. So you are supposed to do 20 divided by 5 equals 4.

Teacher: **Okay, but I'm asking you… you had 5*x* over here (*left side of scale*). How did you get rid of your 5? You divided over here (*right side of scale*); what did you do over here (*left side of scale*)?**

Cindy: I divided the 20 on this (*right*) side by the 5 on that (*left*) side.

Teacher: **So you got *x* equals 20?**

Cindy: No, *x* equals 4.

Teacher: *x* equals 4, so how did you get the *x* by itself is what I'm asking you. You're telling me that you got 5*x* and 46 minus 26 and you got 20 over here (*right side of the scale; teacher writes 5x = 46-26 and then writes 20 under the 46-26*). And you still have 5x over here (*left side of the scale; teacher writes 5x = 20)*. Correct? (*Cindy nods yes*).

Teacher: **So you divided here (*right side of the scale*) and got 4 (*teacher writes 20/5 = 4*). How did you get rid of that 5 (*points to the 5x on the left side of the scale*)?**

Cindy: Oh, you divide it by 5 also.

Cindy explained her solution for the presented problem by explaining and justifying her mathematical reasoning. In this dialogue I initiated questioning to encourage Cindy's mathematical thinking, indirectly dictating Cindy to explain how she isolated the *x* variable with division. As Cindy was lead through the explanation by my prompting there was confusion that took place during this part of the discussion because Cindy did not see the traditional, calculational process that I knew. Therefore I was forcefully negotiating the need to know and share all of the procedural mathematical steps, without considering the ways that students would grasp or explain their thinking. The idea that I was negotiating here was that none of the students divided the 5*x* by 5, which at this point is not a logical thing to do as students are beginning to be introduced to algebraic concepts. This idea can only be understood when studying properties of equality, which is not the point here. This would be a better topic when students are in algebra.

As the students continued with the next task, the research team held an impromptu briefing to discuss the prior discussion. My use of specific questioning was suggested, by the professor, as limiting the diversity of the students' thinking and their input in the negotiation of what constituted as an *acceptable* explanation. In this briefing, we determined that it was necessary to allow students to provide their explanation of actions on mathematical objects in order to reinvent mathematical concepts. At this point, we attempted to reestablish the role of the instructor as one who facilitates explanations, rather than directing the route of students' explanations. Facilitation helps the student reach the goal of explaining their thinking within the means of the classes' negotiated criterion and definition. This particular model that I implemented was dictating because I forced the student to articulate my understanding and explanation rather than her own. To model this role of instructor, the professor taught the remainder of the class for the day.

The following activity (See Fig. 10) was used previously to highlight the reestablishment of the social norm, questioning. Prior to the following dialogue, Savannah presented her drawing (See Fig. 11) and the students were prompted to ask Savannah questions about her drawing.

Cindy:	Um, why did you put 18 individual pieces instead of 18 rolls?
Savannah:	Because it says pieces.
Professor:	Did you all hear her question? Great question. She asked why did you put 18 pieces on the right hand side, instead of 18 rolls. Who thinks they have an answer to Cindy's question? (*Two students raise their hands*). **Only two people think they can answer her question?** (*Savannah raises her hand*) I know you can Savannah. **Wow, that**

means there's some confusion. Do you think you could answer her question? (*Speaking to Anthony*)

Anthony: Well, because…

Professor: I'm not asking you to do it; I'm just asking if you can.

Anthony: Oh. Yeah.

Professor: … Alright, go Savannah, she asked you so you explain it.

Cindy challenged Savannah's representation of the number 18. The professor restated Cindy's question and prompted students to interpret the solution, in which they were hesitant or unable to do so. Regarding this as a possible existing misconception, the professor continued to *promote student participation* in the discussion. By prompting the students to continue through a misunderstanding, the professor established that an explanation would not be considered as *acceptable* if students were not able to interpret the actions of others' mathematical reasoning. Therefore, even after Savannah provided her answer, the professor continued to facilitate students' negotiation of Savannah's solution to elucidate what was constituted as an acceptable explanation.

Savannah: Because it doesn't say 18*x*, it said 18 + *x*.

Professor: Repeat what she said. (*Speaking to Farrah*)

Farrah: That it didn't say 18*x*, it said 18 + *x*.

Professor: What's the difference? Isn't that your question Cindy? What's the difference? How do you know what to put? (*Speaking to Savannah*) You want to call on someone or you want to take that question? How did you know to put 18 pieces?

…

Professor: How did you know to draw 18 little loose pieces instead of 18 rolls? What do you think? (*Speaking to Renee*) Talk to us Renee. Ya'll see what you think about what Renee says; see if you agree.

Renee: Um, yes?

Professor: Yes, what? What do you agree to here Renee?

Renee: The pieces.

Professor: The pieces. She agrees they should be pieces. **No one has given a reason why they should be pieces.** Who thinks they know?

The professor utilized multiple strategies to renegotiate the criterion for what counts as an acceptable explanation (i.e. restating, questioning, interpreting others' thinking, etc.) to encourage students' to explain the meaning behind the numbers and variables in the problem's expression. The professor had concluded that in spite of the explanations given, no one had linked the mathematical symbols to the rolls and candies. This conclusion led her to reestablish the criterion that mathematical explanations were only acceptable if the reasons for the calculations could be tied to the contextual situation. As the dialogue continued, the class renegotiated the criterion while establishing a mathematical concept.

Renny:	I know!
Professor:	You go for it (*Points to Renny*).
Renny:	Um, they can be pieces because it says 18 just by itself, and not 18 with an *x* by it, and *x* would be the packing rule.
Professor:	*x* is the packing rule for?
Renny:	For the amount of rolls they had.
Professor:	For the rolls.
Renny:	So, if it doesn't say *x* then it wouldn't be rolls.
Professor:	(*Circles the 18 in the visual on the board*) So, this (*18*) because it's all by itself, the little loose pieces. And this part (*Circles the x in 18 + x in the same visual*) *x* is the packing rule (*Points to Renny*), that Renny tells you is the *x* tells you about the packing rule. What do you think about that Cindy? Does that answer your question? Repeat what you heard.
Cindy:	So I heard that *x* is the packing rule and there's 18 loose pieces because 18 is all by itself and *x* is not beside of 18?
Professor:	Renny, does that, did she summarize the point that you were trying to get across?
Renny:	Yeah, basically.
Professor:	Basically, alright. Does that help you with your question that you had? (*Cindy nods yes*) Okay, thank you for asking your question.

Renny accepted the professor's challenge to attend to the need of mathematical reasoning for the impending questions about representation: 'What is the difference between 18x and 18 + x?' and 'How do you know to use loose pieces or rolls?' The professor acknowledged Renny's explanation by restating and emphasizing the components in the problem presented on the board. The professor then directed the discussion back to the Cindy, who originated the discussion, by prompting her to repeat what she heard.

Through this collaboration, the students and professor renegotiated the mathematical meaning for the representations used in the expressions of this sequence. At this point, it was taken-as-shared by the class that the variables represent varying values for packing rules (how many candies are in each roll) and the independent numbers represent the value of the number presented (loose pieces, not in rolls). In this discussion the idea that for a solution to be considered as *acceptable* it must include mathematical reasoning that ties back to the contextual situation. Also, the social norm that all members of the class must be aware of each others' interpretations for full understanding was also applied.

Given the opportunity to observe the professor use her voice to facilitate student-led discussions, I was able to figure out strategies that would help me give less and facilitate more in further whole class discussions. Throughout the remainder of the sequence the students and both instructors continued to negotiate and utilize what they had constituted as the criterion for what counts as an *acceptable* mathematical explanation to this point. Students continued to work on giving explanations that focused on how their calculations tied to the symbols and context within which they worked. I was able to attempt to be more of the facilitator of discussions, providing purposeful opportunities for the students to construct acceptable explanations. As the instructor, we both continued to support the students, encouraging the use of the established social norms (explaining, justifying, interpreting others' understandings, questioning and restating) as well as criterion that was negotiated along the way in order to create accurate explanations.

Different Solutions

The tasks in this instructional sequence were intended to generate opportunities for students to reinvent alternative representations so they would be able to build a deeper understanding through comparing and contrasting methods. Prior to the study, I had encouraged students to share different solutions in their mathematics class by asking them to raise their hands if they agreed or disagreed, having them explain why they came to their conclusion. Students were also familiar with comparing and contrasting solutions in whole class discussion. In the sections that follow, I illustrate how the sociomathematical norm for what counts as a different solution was negotiated within the discourse.

On Day Two while students discussed their solutions to Mrs. Simpson's Packing Rules using the rule of 13 (See Fig. 6) on the second packaging, I began the process of negotiating the criterion for what counts as mathematical difference in the candy shop activities. Prior to this discussion, students had established that the packing rule implied how many pieces of candy were in each roll and the individual candy counted as one when they discussed the representation of the candy for *The Candy Shop*. The following excerpt illustrates how I prompted the students to think of alternative ways to represent a solution for a problem.

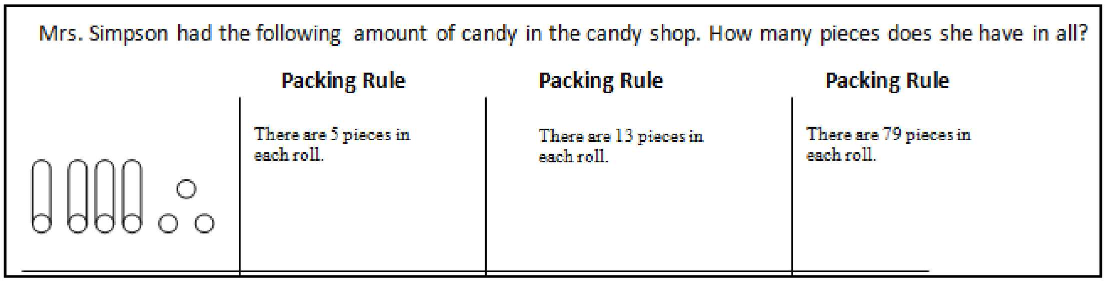


Figure 6: Mrs. Simpson's Packing Rules, second packaging

London: You're gonna, um, it says she has thirteen pieces in each roll. There are four rolls so you multiply 13 times 4. You have to multiply 13 times 4 and since there are 3 left over you have to add 3 and then you have to find your answer. 13 times 4 equals, oh wait, 13 times 4 equals 52, plus 3 equals and then you add 3 to that and that will equal 55, so 55 will be your answer for that.

Teacher: Okay, so come write your number sentence on the board. (London *writes 13* x *4 +3 = 55 on the smart board*). Okay, is there another way to do that? Renee.

Renee: You can do repeated addition.

Teacher: Repeated addition? Show me what you mean; talk me through it.

Renee: You do, you add 13 four times, (*Renee writes 13 + 13 + 13 + 13 + 3 on the board*) then you could add 3 and it gives you 55.

After London presented his solution, I urged students to consider alternative ways to solve the problem. Renee stated that the solution could be attained using repeated addition. When I prompted for an explanation, Renee elaborated on the calculations needed to reach the solution with repeated addition, justifying her method by reiterating the same solution as London. As the discussion continued, Renny offered another alternative solution method and the students concluded that there were different ways to solve a problem.

Teacher: Okay, Renny?

Renny: So you could put 13 and you could put times 5 minus ten because there are four full rolls and then there are 3 extra pieces. So you do 3 minus 13 (*writes 13-3*) equals 10. So you just subtract that and 13 times 5 is obviously 55. (*Students math works out correctly, as his verbalizing is incorrect*).

Farrah: I agree with all of them. The way I did it though was the same exact way as London, just without the parentheses.

…

Teacher: So let's look back over, we had three *different* ways for number two. Could we think of a third way to do the first one? Did you think of a fourth way? What did we learn from these two (*pointing to London and Renee's solutions*) right here?

Evan: There's multiple ways to do each problem.

Teacher: There's multiple ways to do each problem. Okay.

Even though Renny struggled to verbalize his explanation, he illustrated how his alternative method worked by writing it on the board. Farrah acknowledged all explanations that were presented by Renny, affirming the similarity between her and London's methods.

I concluded the discussion of this particular task by reflecting on the students' solutions and questioning the students targeting their negotiation of the sociomathematical norm of what counts as different solutions. Following my prompting, Evan confirmed the idea that *different* solutions included alternative methods (not just any alternative method; ones that combined the rolls and pieces differently) of attaining the solution to a problem was taken-as-shared in the classroom at this point.

On day three, students were introduced to a task that encouraged them to create different solutions. I reiterated the directions to draw three representations of solutions and then prompted students to share their different representations of the situation.

Teacher: So how many (total pieces of candy) are in that (visual) example right there?

Jordan: Um, sixteen.

Teacher: Sixteen.

…

Teacher:	It says draw three different ways that the candies can be packaged if the packing rule is six pieces of candy per roll, so what did you do?
Luther:	I did sixteen loose pieces.
Teacher:	Sixteen loose pieces. That’s one way. Did anybody else do that? Okay, some other ways.
Farrah:	I did two rolls with four extra pieces.
Teacher:	Two rolls with four extra pieces. Anybody agree with that?

The solutions that are presented in this activity were followed with procedural explanations. The students’ explanations consisted of stating how given representation of candy rolls and pieces were manipulated to generate alternative visuals that could be interpreted as equivalent to the total of the original visual. Therefore, the criterion for what counted as a different solution method was that students should manipulate the packages to produce alternative structures that maintained the quantity.

Throughout the study instructors, the professor and I, frequently prompted students by asking if they agreed or disagreed with presented solutions or if they had something different for their solutions. These prompts initiated student-led negotiations of what constituted a *different* solution method and with time the students started initiating differences without prompting.

The following dialogue, from Day 9 of the sequence, is an example of how a student used the criterion of manipulating the packages in an alternative way.

Professor:	Alright, let’s go ahead and discuss these. I don’t think we’re going to discuss this one because I think everyone around the room kind of got that one. What was the answer to that one?
Students:	39 pieces.
Renee:	I got 36.
Professor:	Ah, thank you for saying that, okay 36 (*writes 36 on the smart board beside the 39*). Alright, so Renee, talk about that. So Renee said she got 36 pieces so I bet we’re curious what happened. Go ahead Renee.

Renee:	I got 36 because I did 9 times 4 since the packing rule is 9 and there's 4 rolls (*writes 9 x 4 = 36 on the board*), but I didn't add those (individual pieces) because they weren't in packages.
Professor:	They weren't in the packages so she didn't add those three. Looks like you got lots of hands (*students in the class are raising their hands*), do you want to ask around who? (*Renee points at Hannah who has her hand raised*)
Hannah:	I kind of disagree because those three pieces are there for a reason. It's like there up there and they're there to be added. There not just up there.

This discussion features two significant ideas. First, Renee offered her solution as one that differed from that of the majority of students without any prompting from the professor. This is one of the few times that a student initiated a different interpretation without any prompting in the whole class discussion for the day. This strengthened the earlier conclusion that students felt obligated to state when they did not agree with someone's reasoning method. Second, Renee's presentation of her alternative answer played an important role in the students' negotiation of what constituted as a *different* solution. Renee's different interpretation was that she did not count the individual pieces when she determined the total amount of candy. Students' rejection of Renee's interpretation contributed to refinement of the criterion of what counts as a *different* solution; a different solution could include different manipulations but it must *include all quantities of the candy*. Both, the instructors and students, initiated future conversations regarding different solutions and solving methods, which supports this conclusion.

The difference between the professor and my approach to consider "different" solutions was the professor consistently facilitated student collaboration, taking the time to solicit students to produce how or why the solutions were different, while I focused on students presenting alternative solutions or solving methods. As the different approaches were used through the instructional sequence the students were able to determine that different methods of solving were considered different ways of thinking. By the end of the instructional sequence the whole class was still in the process of negotiating the importance to elaborate on the differences.

Efficient Solutions

As students presented alternative solution and methods of solving, the negotiation of what was considered to be an efficient solution materialized concisely in whole group discussions. This usually happened as a student shared a complex strategy and it was rebutted as “over complicated” by other students. The first occurrence of this was on Day 2, while the students were working on the following problem (See Fig. 14) for Mrs. Simpson’s Packing Rules.

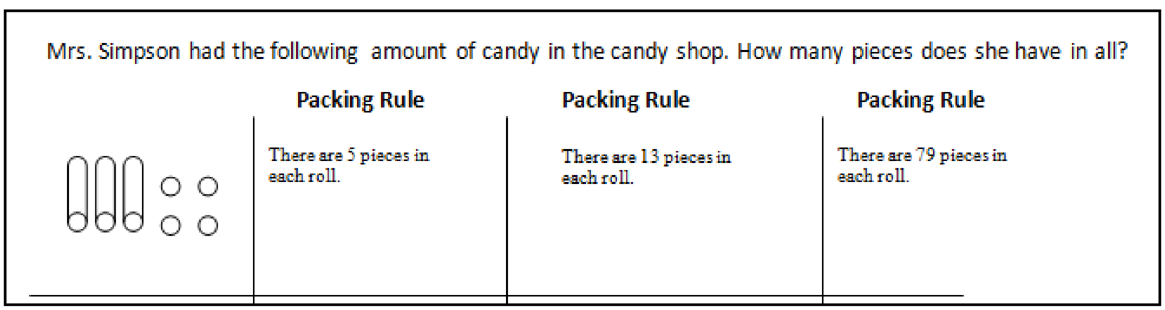


Figure 14: Mrs. Simpson’s Packing Rules, first packaging

The following dialogue illustrates a student sharing his complex solution for the third packing rule of 79.

Chuck: Um, I just came up with this you could 79 times 4 minus 83.
Teacher: 74?
Chuck: 79.
Teacher: Oh okay, sorry, 79 times 4?
Chuck: Yes, minus 83.

Teacher:	Minus 83 (*Teacher writes 79 x 4 – 83 on the board*). Explain that to us.
Chuck:	Uh, well um, you could add another imaginary roll and then you would do 79 times 4 which is I have no idea. Okay you would do 79 times 4 minus 83 because you would take the extra roll which is 79 and then take the extra missing four (pieces) and then add those together and then subtract those from the 74.
Teacher:	Anybody have any comments about that?
Jordan:	Um, I agree with that; it is right. But I think adding a whole new imaginary roll and just taking it away is **just too much**. It would be just easier to count the rolls you have and take away four.
Renny:	I agree Jordan. I think **it is unnecessary** to add a whole other roll, especially when the number that you're multiplying is 79. Um, I think you should just do 79 times 3 and subtract 4 since you get the same answer but it's shorter and it takes less time.
Teacher:	Okay.
Anthony:	I agree with Jordan and Renny because if you do like Chuck did and add a whole other roll **it's just over complicating it** which could get you very confused and not the right answer.
Teacher:	Okay, anybody else?

Chuck gave a solution method that involved adding imaginary rolls and then taking away the amount of pieces that would be in the imaginary roll. However, even though his answer matched other students' solutions, his method was rebutted by three of his peers. His classmates were negotiating what constituted as an *efficient* answer as they deemed his solving method as unnecessary, too much, and over complicated. I accepted the answers from the peers and did not delve further into elaboration of their ideas. Therefore, at this point the students had negotiated that a solution is not efficient if you added candies or packages that are not accounted for in the problem.

As the students continued through the sequence, the idea of adding "imaginary" pieces or rolls was brought up a couple more times where students were trying alternative methods of solving problems. However, the negotiation continued to be that this idea was NOT an *efficient* solution.

On Day 9, the students were engaged in a whole class discussion sharing their solutions and solving methods for number three on the "Packing Problems" handout that the professor had presented to them. The problem asked the students to answer the question "How could the Simpsons package 59 cherry candies if the packing rule is c = 10?" by drawing at least two different pictures. Prior to the following conversation, multiple students had shared their solving method, coming up with various solutions. Brian stated that he approached the problem using seven rolls to solve the problem. This excerpt illustrates the negotiation of the use of imaginary candies.

Professor:	… Describe what you did.
Brian:	Okay, so I had seven rolls. And what you do is, I took 10 away from one roll, which pretty much just gives you zero, but that's on way to do it. So, you have 7 rolls, take away 10, and then from the next one (roll) you take away 1 and then you get 59. Because 10 times 7 gives you 70 and 70 minus 10 equals 60, 60 minus 1 equals 59. And that's equivalent to 59 pieces.
	(*The professor has the students discuss Brian's solving method with their table groups*).
Professor:	Alright, let's come back together. I hear some Oh's and Ah's. What did you want to say Savannah? Listen up.
Savannah:	I agree with him but **he's like over complicating it.**
Professor:	**Over complicating it?** Okay, what do you want to say?
Jordan:	Yeah, he's just adding in another roll to get rid of it. That makes sense, why can't you have just 9 rolls in the first place. I mean it is right.
Professor:	It is right.
Jordan:	It is right, **it's just overcomplicating it.**
Professor:	Okay, so **I'm hearing, from at least this group, that it's right, but we have a little thing to say about it.** What does this group (consisting of Evan, Anthony, Hannah, and Kathy) think? Is it right?
Hannah:	It is right, but we think it's over complicating it.
Professor:	Over complicated. What about this group?

Luther: I was going to say, since he had seven rolls, I thought he would take one away from each roll or something.
Professor: Oh, that's a different picture isn't it?
Luther: On like, three of them you would take away 2 of them or four…
Professor: That would be really cool wouldn't it? There's all different kinds of pictures you could have for this isn't there? Did you hear what Luther said, have seven rolls and take one out of each roll and maybe have to adjust some things. But you could have a bunch of rolls with one missing piece as long as you put other pieces in there to compensate for it. So I think what they are saying to you is…
Brian: It's making it too hard, but I just…
Professor: **I don't, I kind of like it. If it makes sense.** Now what would this one look like in the store? In the Big Lots?
Brian: It would look like the wrapper that someone ate out of it and put it back and then you go to buy it and you're like what?
Professor: That would be really awful would it? It's just a wrapper in this case. Alright, last comment and then we're moving on.
Calvin: It's not hard; we're just saying that you're over complicating it.
Professor: Oh yeah, we all agree. Okay, so we're not going to mince words there, we all agree this (Brian's Interpretation) works and it's kind of cool that you can add a bunch of empty rolls and it doesn't change anything in this case. It's a little over complication but it's really good thinking, I think.

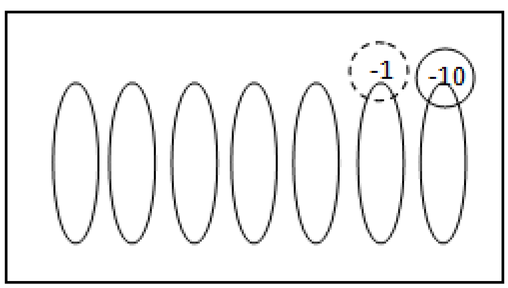


Figure 16: Professor's drawing of Brian's interpretation (on the board)

As Brian offered his solution and the professor drew the visual on the board (See Fig. 16), the students could see that his method did work. However, the students were stuck on the idea that adding more "imaginary" candies was "over complicating" the solving, negotiating that this was NOT an efficient method of solving. At this point professor continued the negotiation of what constituted as an efficient solution, guiding the students to see that even though the solution or method may be complex, it could still illustrate the mathematical processes that took place to find the solution.

The next day, on Day 10 of the sequence, the students were discussing the following Simpsons' Studies problem of the day when the rebuttal of "over complicated" arose again.

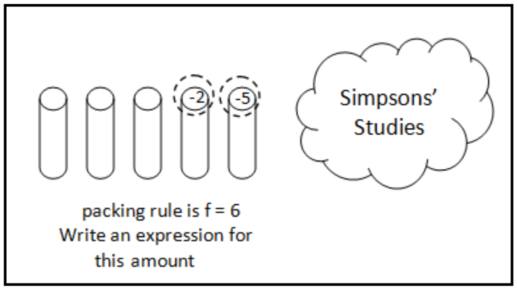


Figure 12: Simpsons' Studies, day 10 (drawn on board)

Chuck presented the multiple methods, shown below (See Fig.13), that he came up with to solve the Simpsons Studies (See Fig.12). Prior to the following dialogue below, the students were discussing his various answers and asking questions about how or why he did some of the methods that he came up with. There were some students that were confused with different expressions so I had the students write down the expressions and code the problems with emojis to show if they agreed, disagreed, or had no clue or didn't understand.

$$
\begin{gathered}
5f - 7 \\
5f - 2 - 5 \\
5 \times 6 - 7 \\
5 \times 6 - 2 - 5 \\
3 \times 6 + 4 + 1
\end{gathered}
$$

Figure 13: Methods of solving day 10 Simpsons' Studies (written on board)

In the following dialogue, Anthony, a student from the class, disagrees with Chuck's second expression (5*f* – 2 – 5) and begins a renegotiation of the idea of "over complicated" solving methods.

Anthony:	I disagree with that.
Teacher:	Why?
Anthony:	Because it's all subtraction. So you do 2 minus 5 which would get you negative 3, and you wouldn't get negative 3 on this part.
Teacher:	Chuck? Come on up here and you can explain.
Chuck:	Actually, you would put parentheses here (*writes (5f-2)-5*) which would be 28 so then you subtract 5 and get 23.
Anthony:	Now that makes sense, but before.
Chuck:	Still you would have to do that.
Teacher:	Do you have to do the parentheses?
Students:	No.
Teacher:	Because you would multiply, then you would subtract your 2 and then subtract the 5.
Anthony:	I know, but I still disagree. Why wouldn't you just add them together; 5 plus 2 and do minus 7?
Teacher:	But is that the same?
Anthony:	Yeah, but **you're over complicating it.**
Teacher:	Okay, **over complicating things is different. It's not over complicated; it's just a different way of thinking.** Okay, so we have negative, I mean minus 7 here and we have minus 2 minus 5 here, which is the same thing. Now you guys got stuck on the over complicating; it's just another way of thinking.

As Chuck presented multiple methods, Anthony rebutted his solutions with the argument the second expression would not work in this situation because it was all subtraction, which would give them a negative number. As Chuck explained using parentheses Anthony stated that it did make sense. However, Anthony stated that it was "over complicated." I intervened here and reiterated the professor's comment, from a previous class discussion, that just because an idea was more complex didn't necessarily mean that it was "over complicated" and it could dismissed. I continued to explain how the first and second expressions were the same. The class established that complex solutions that presented alternative solutions and methods of solving were considered different ways of thinking. Therefore, the idea of what constituted as an *efficient* solution was re-established. The simplicity or complexity of the solving method was insignificant for what constituted as an *efficient* solution. The criterion for what was considered to be an efficient solution was that there had to be a comprehensible link between the mathematical reasoning and the contextual situation that was presented.

Discussion

According to Yackel and Cobb (1996), sociomathematical norms are extensions of general classroom social norms that are essential to a classroom's microculture. Establishment of sociomathematical norms maintains a classroom environment that encourages problem solving which generates means to clarify, analyze, and discuss classroom member's mathematical expressions. Socomathematical norms comprise modes of clarifying the properties of mathematical solutions that determines it to be *acceptable*, *different*, *sophisticated*, and *efficient.*

The notion of what constituted as a sophisticated solution was never the focus of instruction and consequently, never established by the class in this study. The table below summarizes the norms that were established in this study.

Table 3: Social and sociomathematical norms of *The Candy Shop*

Social Norms	Sociomathematical Norms
Students were expected to explain and justify their thinking.	The criterion for what counts as an acceptable mathematical explanation was that the explanation had to be focused on both the calculations as well as what the calculations stood for.
Students were expected to interpret others' methods	The criterion for what counts as a different mathematical explanation was that the explanation could present alternative methods and manipulations, but MUST include all quantities of the candy.
Students were expected to use questioning when they did not understand	The criterion for what counts as an efficient mathematical explanation was that the explanations could include different solving methods with the same answer.

The norms of *explaining and justifying*, *interpreting others' methods*, and *determining what constituted as a different solution* were established before *The Candy Shop* instructional sequence began. Though these norms were established they continued to be sustained and negotiated by the instructors and students throughout the sequence.

The *explaining and justifying* norm was established as describing how and why solutions were attained. It appeared that this norm was taken-as-shared early on in the study. Even though the norm of interpreting others' methods was established prior to the study, negotiations of strategies helped the class renegotiate this norm. Before the sequence, the students were familiar with stating and expressing why they agreed or disagreed with another person's methods. However, during the instructional sequence the students learned the significance of interpretation through two techniques: 1) assisting other with misconceptions and 2) restating. These strategies allowed renegotiation of the *interpreting others' methods* norm by giving responsibility to students to provide their input, having students help with misunderstandings, and having students restating others' interpretations to ensure accountability.

The social and sociomathematical norms that were established, with ongoing negotiation, throughout the sequence were questioning, acceptable solutions, and efficient solutions. These three norms became established throughout the instructional sequence. After becoming established, the questioning norm became a taken-as-shared idea in this classroom. The students progressed from asking superficial questions to initiating questions without prompting that helped them build conceptual thinking. The instructors helped with this progress by leading renegotiation of the format of questioning. The norm of what constituted as an acceptable solution and an efficient solution were both established, and continued to be negotiated throughout the remainder of the sequence.

These norms helped define the roles of both the students and teachers through the sequence. These norms also guided students to successfully partake in the following classroom mathematical practice.

Data Based Critique

Prior to this particular study, the professor had 20 years of teaching experience in which inquiry, social and sociomathematical norms are the focus of participating in mathematics classroom. Therefore, through these prior experiences the professor knew when to intervene in discussions in order to support the establishment of these inquiry norms. This was my first time attempting an inquiry method in front of an expert inquiry teacher. Although I know that inquiry mathematics involved providing contextual problems, engaging students in mathematical discourse, and providing students with opportunities to solve problems in their own way, the constructs of social and sociomathematical norms were new to me. At the beginning of this dissertation process I expected most of my challenges to be in the role of the researcher. However, there was a definite learning curve for me as the teacher and it is in this section that I use the analysis above to reflect on my growth as a teacher.

As noted earlier, the professor and I had different understandings of social and sociomathematical norms and the way in which mathematical discussions would take place. In Fig M, I note four distinct differences in the ways in which the teacher and professor negotiated mathematical meaning with the students in class discussions.

Table 4: Differences in instructional techniques of instructors

Norms	Teacher Technique	Professor Technique
Social Norm: Explaining and Justifying	Provided explanation for the students; Directed students to specific answers; did not elaborate for whole class comprehension.	**Asked students to explain** and then restated it
Social Norm: Interpreting Other Students' Methods	Gave interpretations for the students	Asked specific students (who did not volunteer) to restate students' words or actions; restated students' questioning to generate student interpretations; **held students accountable for interpretations**
Social Norm: Asking Questions when you don't understand	Asked questions of the students	**Asked students to ask questions** about others' methods and actions; coached, guided and acknowledged student questioning
Sociomathematical Norm: What counts as acceptable mathematical explanations	Prompted students to "explain;" prompted students to agree or disagree; directed students to targeted explanations about the calculations through specific questioning which was not	**Facilitated student-led discussions**; restated solutions prompting students interpretations; promoted student participation; used social norms to encourage math

Table 4 (continued)

	acceptable	reasoning; redirected student explanations to focus on problem, not just calculation
Sociomathematical Norm: What counts as a different solution	Prompted students to use alternative ways to solve; reflected on student solutions; focused on	Facilitated student collaboration to explore how or why a solution was different; **prompted**
Sociomathematical Norm: What counts as an efficient norm	Accepted student rebuttals without elaboration	Guided students through elaboration to **rethink rebuttal of complicated ideas**

These four distinct differences were 1) Student collaboration, 2) Student responsibility, 3) student accountability, and 4) Instructor facilitation of whole class discussions. As mentioned previously, having limited experience implementing inquiry lessons, I was not familiar with all four of these techniques at the beginning of the instructional sequence. I wanted to take on the responsibility of making sure that the students understood the concepts (the way I did); therefore, I was the main participant in the discussions giving the students no accountability for gaining knowledge about the concepts at hand. As I was able to see how the professor was able to give the responsibility to the students I tried some of her techniques to facilitate stronger opportunities. The bold print in Table 3 highlight the techniques I tried to implement throughout the remainder of the instructional sequence.

Conclusion

The results presented in this chapter lead us to two different sets of conclusions. The first set concerned the nature of the instructors' techniques when implementing the instructional sequence, and the second set concerned the students' negotiation and establishment of social and sociomathematical norms. Both of these conclusions brought about the implications of a presence of educator experience in this type of classroom experiment and with the social and sociomathematical norms.

The fact that the students were already working together before the study, influenced the establishment of the consistent whole class discussions. However, the presence of the professor, which was an expertise with facilitating the students to use and establish the norms, also played a significant role. The students and I, as the teacher, were able to build off the prompts of the professor in the discussions to negotiate and establish the norms efficiently.

CHAPTER 5: CLASSROOM MATHEMATICAL PRACTICES

Classroom mathematical practices are identified as taken-as-shared ways of reasoning, arguing and representing established though discussion of particular mathematical ideas (Cobb, et.al., 2001). With the focus of study being targeted on students' conceptions of algebraic expressions and equations, the analysis of mathematical practices was concerned with specific arguments and ways of reasoning about the use of variables and how to solve for unknown values.

There were four mathematical practices established over the duration of the sixteen class periods for this study (See Table 1). These mathematical practices were established using Rasmussen and Stephan's (2008) three-phase method for documenting collective activity. For Phase 1, all of the whole-class discussions were transcribed and coded with Toulmin's system, using data, claims, warrants and backings to develop an argumentation log. In Phase 2, the argumentation log was used as data to determine occurrences that signaled mathematical ideas becoming taken-as-shared. And finally, in Phase 3, the list of taken-as-shared mathematical ideas, established by the class, were organized into mathematical practices.

The following paragraphs depict the mathematical ideas that became taken-as-shared in the classroom and then were organized by the research team into classroom mathematical practices. The constructs *data, claim, warrant* and *backing* are used to analyze a selection of the discourse; however, this was reported in moderation to keep the reader informed of, yet not exhausted, how argumentation assists in verifying ideas as taken-as-shared.

Table 5.
Classroom Mathematical Practices in *The Candy Shop*

Classroom Mathematical Practices
Practice 1: Evaluating pictures of candies given a different value for x (packing rule). • With given packing rules, a total number of candies can be found by multiplication (number of rolls times the packing rule) and addition of individual pieces • Repeated addition can be used as an alternative of multiplication • Multiple expressions and illustrations can be used to show the value of x. • Subtraction of individual pieces from rolls could be explained as “stolen” pieces
Practice 2: Interpreting the meaning of variables • The variable in *The Candy Shop* represents a whole number quantity, defined by a packing rule (how many individual pieces are in each roll) which may or may not be given. • Variables do not always represent fixed values. • “Like terms” such as $4x$ and $2x$ represent multiple units of a fixed quantity ($x+x+x+x$ and $x+x$) and therefore can be combined • Quantities such as $4x$ and 2 are not multiple units of a particular quantity, and therefore the sum of these units can only be represented as $4x+2$
Practice 3: Interpreting Algebraic Expressions • A number beside a variable (i.e., $6x$) represents multiplication • Expressions represent the amount of candy rolls and pieces in *The Candy Shop*.
Practice 4: Reinvention of Methods for Finding the Value of x (Packing Rule) • Division (pieces/rolls) can be used to find the packing rule • Numbers are “canceled out” by applying the inverse operation to isolate rolls and pieces to opposite sides of the balance (= sign) • When solving for x, an “imaginary roll” can be added in the expression • A “collection” of candy should be the sum of both the rolls and pieces.

Practice 1: Evaluating pictures of candies given a different value for x (packing rule).

As students solved for the total amount of candy with given visuals and packing rules, there were four mathematical ideas that became taken-as-shared:

- With given packing rules, a total number of candies can be found by multiplication (number of rolls times the packing rule) and addition of individual pieces; and
- Repeated addition can be used as an alternative of multiplication.
- Multiple expressions and illustrations can be used to show the value of a collection of candies while maintaining equality.
- Subtraction of individual pieces from rolls could be explained as “stolen” pieces

The instructional sequence began with an experientially real context of an infamous fictitious family, The Simpsons™. The teacher asked the students to read about the scenario to find out that they are going to be a part of The Simpsons™ taking over Uncle Wiz’s candy store, and opening it as the Simpsons’ Sweet Shop.

> *Now you guys are going to be part of the Simpsons’ Sweet Shop. You are actually part of this, so you are going to join them in their “drama drama’ as it says. And as you mentioned their drama is they are arguing over how many pieces of candy should go into each roll for each flavor. So I want you to look at the* ***representing candy at the candy shop*** *at the very bottom of the page and I want you to talk about what is represented and how it is represented with visuals that they give you.*

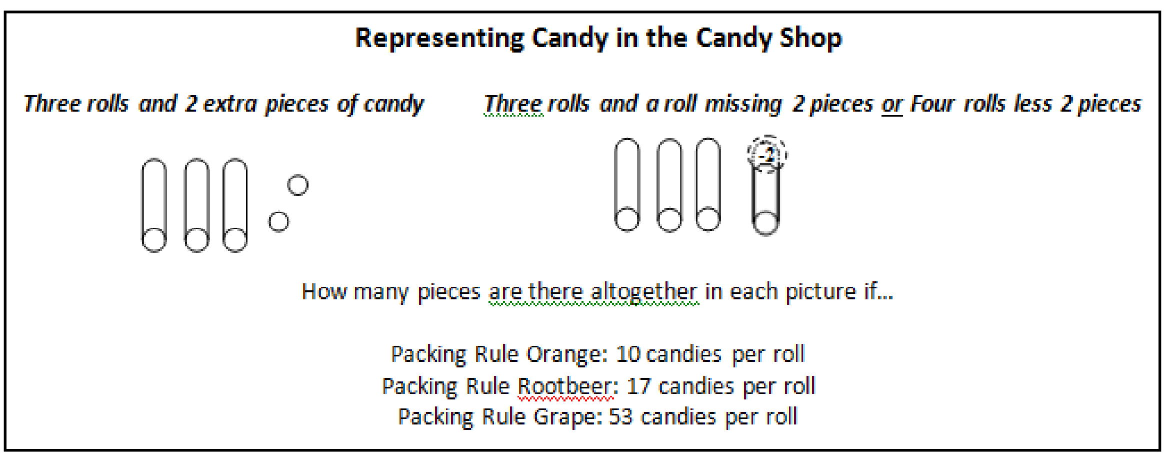


Figure 17: Representing candy in *The Candy Shop*

Students suggested that if there was a roll that was missing pieces the individual pieces could be used to put into the roll to create a full roll. Students argued that the flavor of the individual pieces was unknown; therefore, they questioned being able to add them back to the roll, since all the rolls were the same flavor. Students also suggested finding the amount of candies in the roll(s) using the operation of multiplication or repeated addition; since all the rolls have the same flavor then their packing rule would be the same.

The teacher then presented the question “how many pieces are there altogether in each picture if…?” directing students to the three different scenarios of flavors and packing rules presented in ‘Representing Candy in the Candy Shop’ (See Fig.17). The students were encouraged to explore the three scenarios with the given visuals and try to figure out the amount of candy for each of the given packing rules. To make meaning of these three scenarios the students had to be able to interpret the visuals of the rolls, individual pieces, and missing pieces. The candy *rolls*, represented by a cylinder, were packages that contained a certain (unknown) number of candies. The candy *pieces*, represented by a circle, were individual pieces of candy out of the package. If a roll was missing pieces from its package it was represented with a negative number within a dotted lined circle on the roll. The negative number was how many pieces were missing from that one roll. In order for the students to later be able to use variables, the students must first understand how the visuals along with the given packing rules create a total of one order of candy.

On Day 1 of the class sessions, the students were given visuals of Mrs. Simpson’s fictitious amounts of candy with given packing rules of 5,13, and 79 and asked to help find how many pieces she had in all. The teacher asked the students to show their thinking by coming up with a number sentence. The students showed no difficulty working out how many total pieces of candy Mrs. Simpson had in her first visual (See Fig. 18) with the packing rule of 5, as they all calculated 19 pieces. Below is a discussion in which the claim of 19 is defended by multiple students:

Mrs. Simpson had the following amount of candy in the candy shop. How many pieces does she have in all?

Brian
5 x 3 + 4 = 19

Renny
5 x 4 = 20 – 1 = 19

Anthony
5 + 5 + 5 + 4 = 19

Kendall
5 x 3 = 15 + 4 = 19

Figure 18: Equations of four students for Mrs. Simpson's Packing Rules

Teacher:	Alright, let's talk about the first one. So, we have three rolls and four individual pieces and there are five pieces in each roll. So I want you to tell me a number sentence that you can up with. Brian what did you come up with?
Brian:	5 times 3 plus 4 equals 19.
Teacher:	(*Writes Brian's number sentence on the board*) Okay, any other ones? Anthony?
Anthony:	5 plus 5 plus 5 plus 4 equals 19.
Teacher:	(*Writes Anthony's number sentence on the board*) Anything else?
Renny:	5 times 4 equals 20 minus 1 equals 19.
Teacher:	Okay (*Writes Renny's number sentence on the board*). Kendall?
Kendall:	5 times 3 equals 15 plus 4 equals 19.
Teacher:	So, let's talk about these. Tell me what you think.
Cindy:	Um, the first two are correct but the last two are not correct.
Teacher:	Explain to me what you are talking about.
Cindy:	Because the last two; it has an equal sign in the middle of the problem. So it's two different problems. It's five times four equals twenty, and then twenty minus one and that's two different problems.
Teacher:	Okay so you are seeing two equations there and two equations here? (*pointing to Renny and Kendall's number*

	sentences). Agree? Disagree? You have thumbs up Brian; tell me what you are thinking.
Brian:	I think because if you do five times four is twenty then you should um, have just that. But if you are trying to make the entire number sentence you just do five times four minus one.
Teacher:	So you are saying that I can change this one (*pointing to Renny's number sentence*) to this (*points to Brian's number sentence*)?
Brian:	Yep.
Teacher:	Okay.
Farrah:	The last one you could just change five times three and you can put that in parentheses that way you would know to get fifteen and then do plus four… which would give you nineteen.
	…
Hannah:	Or you could just do the first one, it's almost the same (as what Farrah just said)… you just take out the parentheses because if you start from the beginning of it and do five times three plus four which would be nineteen.

With given packing rules, a total number of candies can be found by multiplication (number of rolls times the packing rule) and addition of individual pieces.

This is the idea that appears to become taken-as-shared in the discussion above. In this discussion, Brian made the claim that Mrs. Simpson's total candy amount is 19. At the request of the teacher, Brian provided data by sharing his number sentence in which he multiplied and subtracted to show his representation of the problem. Being aware that there were multiple ways to solve this problem, the teacher asked the students if there were any other representations. Anthony provided more data to support Brian's claim, by sharing that he created a number sentence using repeated addition. Renny and Kendall also offered alternative solutions that were challenged by Cindy. Both students did not maintain equality in their number sentence. Cindy explained that they needed to show their work as two number sentences, because with the equal sign in the middle their number sentences were untrue. After discussing the procedural problems with these two solutions Renny and Kendall’s solutions were rejected, as they all agreed that these two solutions should have be written like Brian and Anthony’s number sentences.

On Day 2 of the classroom sessions, the class continued Mrs. Simpson’s Packing Rules sheet from Day 1. Prior to the following discussion, Savannah stated that the picture that is being discussed (See Fig. 19) had four rolls and three pieces left over. Below is the continuation of the whole class discussion of students defending their claims for Mrs. Simpson’s packing rule of 13:

Teacher:	Okay, four rolls and three pieces left over she says. Anybody have any other way of look at that? (*students respond no*). Okay, so I need to know how do I need to do this if there, if Mrs. Simpson gives me a packing rule for thirteen pieces in each roll, what are you going to do? London?
London:	You’re gonna, um, it says she has thirteen pieces in each roll. There are four rolls, so you multiply 13 times 4. You have to multiply 13 times 4 and since there are 3 left over you have to add 3 and then you have to find your answer. 13 times 4 equals, oh wait, 13 times 4 equals fifty-two, plus 3 equals and then if you add 3 to that, that will equal 55 so 55 will be your answer for that.
Teacher:	Okay, so come write your, um, number sentence on the board (*London writes his number sentence on the Smart Board; Figure H*). Okay is there another way to do that? Renee?
Renee:	You can do repeated addition
Teacher:	Repeated addition? Show me what you mean. Talk me through it.
Renee:	You do, you add 13 four times, (*Renee writes her number sentence on the board; see Figure H*) then you could add 3 and it gives you 55.

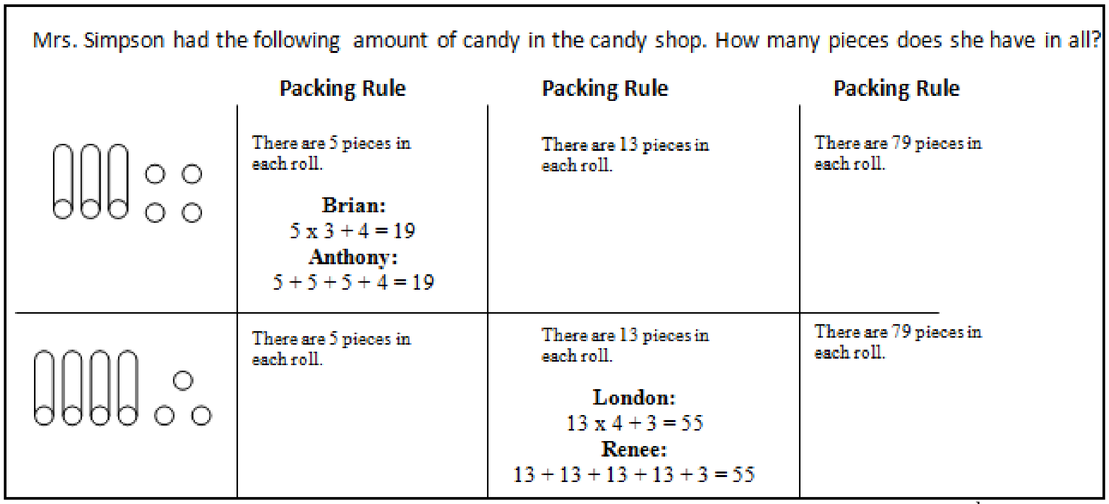


Figure 19: Students' solutions to Mrs. Simpson's Packing Rules, 1st & 2nd packaging

Repeated addition can be used as an alternative of multiplication.

This idea appears to be taken-as-shared in the above discussion. In this discussion, London made a claim that the total number of pieces for Mrs. Simpson's packing rule of 13 is 55. London provided data with his number sentence, explaining his use of the multiplication and addition operations to solve the problem. Furthermore, he provided backing to his claim by justifying his use of the operations with the visuals that were given. Renee followed by making a claim that the same packing rule could be solved using repeated addition instead of multiplication. The discussion continued as other students confirmed both solutions as correct. Observing the students' different number sentences on the board, the students also confirmed that there were multiple ways to approach solving for the total number of candies.

This is the second instance in which the use of multiplication and/or repeated addition to solve the packing rules occurred in the whole class discussion. By the end of the activity, evidence from students' arguments signified that the use of multiplication and/or repeated addition to find the total number of (unknown) candies in the rolls was taken-as-shared. Students used this as their data for the following activity in Day 3 of the instructional sequence which implies that it had become taken-as-shared. As the students engaged in the activity, the focus was students' building onto their arithmetic reasoning by finding arithmetical patterns while repeatedly adding the number of candies in each roll or multiplying the number of rolls times the packing rule. This later formalized into for the amount of candies in rolls using a variable. This verifies Philipp's (1999) assertion that one way to understand algebraic equations is as generalized arithmetic.

Multiple expressions and illustrations can be used to show the value of a collection of candies while maintaining equality. For example, you can write a quantity of 40 candies as 4 rolls of 10 or 4 rolls with 1 missing in each and 4 loose pieces. The amount of candies is the same but they are organized in different packing arrangements.

The Candy Shop instructional sequence tasks were intentionally designed so that students would start with given numbers for the packing rules and then move into the unknown, introducing variables. As the students moved through the activities they utilized the ideas that were taken-as-shared with Mrs. Simpson's Packing Rules activity. On Day 3, students engaged in another activity, Krazy Kustomer Chaos, where they had to create multiple visuals to show a specific amount of candy. One of the questions asked the students, "What are 3 other ways to have 95 pieces of candy packaged, if the packing rule is 5 pieces per roll? Draw them below." Some students did not fully understand the meaning of packing rule and struggled with drawing different representations due to being one of the initial activities they were engaged in. However, as they discussed within their groups they were able to see that the visualizations could differ yet have the same value. They used this conception as they moved through other activities and became more familiar with the meaning of packing rules represented the amount of candies in each roll. By the end of Day 3 the idea that *multiple illustrations can be used to show the value of a collection of candies while maintaining equality* was taken-as-shared.

As they continued, the idea of using a variable to represent an unknown quantity was introduced with Mischievous Maggie and the Mystery Rolls on Day 4, and Brainy Brian's Balance Scales on Day 5, which presented another method for students to represent and understand expressions. Students replace the value of the packing rule with variables. The students were challenged to solve for the value of the variable, using Brainy Brian's scales.

As the instructional sequence progressed, so did the students' challenges. The students initially worked with iconic illustrations for rolls and pieces as well as algebraic expressions. Over the duration of the study, the idea that both *multiple expressions and illustrations can be used to show the value of a collection of candies while maintaining equality* was taken-as-shared.

Subtraction of individual pieces from rolls could be explained as "stolen" pieces

When the students were introduced to Mrs. Simpson's packing rules, they were also introduced to different visuals to interpret. One in particular was a visual that showed a roll with a negative number in a circle on the top of the package. The students had quick instinct that this represented the pieces that were missing out of the package that the circle was on. In fact the first day that this visual was shown London interpreted a visual (See Fig. N) stating "So I saw in this picture that there were two rolls and there was three missing pieces. So, and then I looked over here and saw that there were three extra pieces." Students continued to do as London did in this situation throughout the instructional sequence calling these pieces 'missing' until Day 9 when the professor offered a back-story. As the students presented and discussed different representations of 59 total pieces of candy for the Simpsons, one of the students came up with a representation that had 'missing pieces.'

Professor: Can I tell you a real live story?
Students: Sure.
Professor: That relates to this. I promise you my friend went to Big Lots the other day and bought a bunch of food and stuff in there, in Big Lots. Have you guys ever been to Big Lots?
Students: Yeah.
Professor: And when she went home, she had a box of graham crackers and she started to open them and she noticed it was already opened. And when she opened it there were graham crackers missing. I promise. Somebody had, in the store, had opened it up and eaten a few graham crackers and put it right back in there, as if…
Debbie: As if nothing happened.
Professor: As if nothing happened. So that kind of reminded me of what you guys are doing in this class. It's kind of like that Big Lots person, they, what did they do? (*draws an example problem on the board, See Fig. 20*)
Students: Ate it, Stole it, someone took your stuff.
Professor: What would you say, was the Big Lots person doing here?
Calvin: Pretty much stealing a piece.
Professor: Yeah, then they are repackaging it, putting it back on the shelf, and so, one of the rolls has a piece missing, like our Big Lots thief.

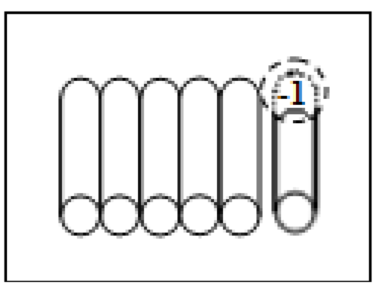


Figure 20: Stolen piece example

In this exchange the professor made the claim that the visuals that individual pieces subtracted from the rolls could be explained by thinking about pieces of candy being stolen in the store. For data, the professor shared a real-world story of a friend having missing (or subtracted pieces) from her graham crackers to help the students connect with the idea. Here the idea of the individual pieces subtracted from the rolls is being negotiated as a possible stolen piece of candy in the store. In the same discussion, Brian shared his representation of 59 total pieces of candy using an imaginary roll (See Fig. 20) when the professor prompted Brian to think of what his interpretation would like in the store.

Professor:	I don't. I kind of like it, if it makes sense. Now what would this one look like in the store? In the Big Lots?
Brian:	It would look like the wrapper that someone ate out of it and put it back and then go buy it and you're like what?
Professor:	That would be really awful wouldn't it? It's just a wrapper in this case…

In this part of the dialogue, Brian used the idea that the subtracted roll illustrated a package that was empty, which warranted his own claim and data. Within this conversation the argumentation shifted from a claim to a warrant and was never questioned. Thus, the idea that *subtraction of individual pieces from rolls could be explained as "stolen" pieces* is taken-as-shared.

Practice 2: Interpreting the meaning of variables

In traditional instruction, students that are beginning to learn about variables and algebra expressions are presented with formal definitions that provide little quantitative significance to them (Chalouh & Herscovics, 1988). Being that this is one of the most fundamental (Philipp, 1992), and most misconstrued (Pollack, 2012), concepts in mathematics, it is imperative for students to have the opportunities to construct multiple definitions of variables through mathematical activities.

Throughout *The Candy Shop* instructional sequence, activities were designed for students to build their own definitions and understanding of variables in a quantitatively significant way rather than handing them conventional mathematical symbolism to be unpacked. For example, as students participated in the activities 'Mischievous Maggie and the Mystery Roll' and 'Brainy Brian's Balance Strategy' the students identified with the concept of a variable as being a "name" for an unknown value. This understanding of variable was used in many of the students' claims and data throughout the instructional sequence. As students moved through the activities and discussions, the idea of variables became more familiar. With this familiarity, the students identified multiple interpretations of variables which established the following four taken-as-shared ideas:

- The variable in *The Candy Shop* represents a whole number quantity, defined by a packing rule (how many individual pieces are in each roll) which may or may not be given.
- Variables do not always represent fixed values.
- "Like terms" such as $4x$ and $2x$ represent multiple units of a fixed quantity ($x + x + x + x$ and $x + x$) and therefore can be combined
- Quantities such as $4x$ and 2 are not multiple units of a particular quantity, and therefore the sum of these units can only be represented as $4x + 2$

The variable in The Candy Shop represents a whole number quantity, defined by a packing rule (how many individual pieces are in each roll) which may or may not be given. On Day 4 the students are introduced to Brainy Brian's Balance Strategy which is a tool to help students find the packing rules for Maggie's Mystery Rolls. The first time that a variable was defined, in the instructional sequence, as representation for the packing rule was on Day 4 when Jordan is explaining Brainy Brian's Balance Strategy:

Jordan: Um, well, in here it says he put the roll of *x* pieces on one side of the scale and seven pieces on the other side of the scale and um, it can help us figure out what the packing rule is because the separate pieces, if it's the same as the pieces in the roll, it will always weigh really close to the same. The only thing that weighs different is the paper rolling them up but that won't make a difference. But the little (individual) pieces, if you put the pieces on (the scale) and it's exactly the same weight as the roll those pieces will be the same amount of pieces in the roll.

Jordan insinuated, but did not use the word "variable" as he explained that *x* is the number of pieces in a roll. During this activity, the class never pursues a direct discussion about the variable after Jordan's interpretation is given. However, the students continued to use this particular interpretation while engaged in the next activities.

On Day 7 the class delved into the meaning of variable when students have different interpretations of the concept. The dialogue below goes along with the following problem in the 'Rambunctious Rolls' activity (Fig. 21):

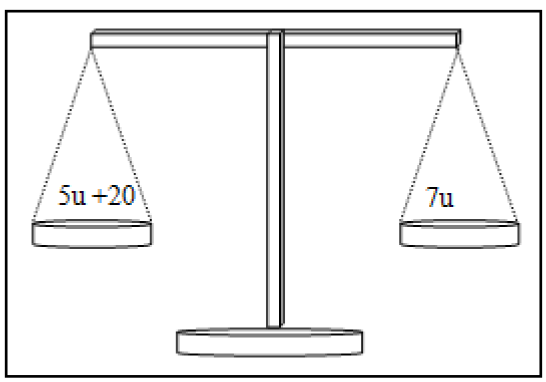


Figure 21: Rambunctious Rolls, #3

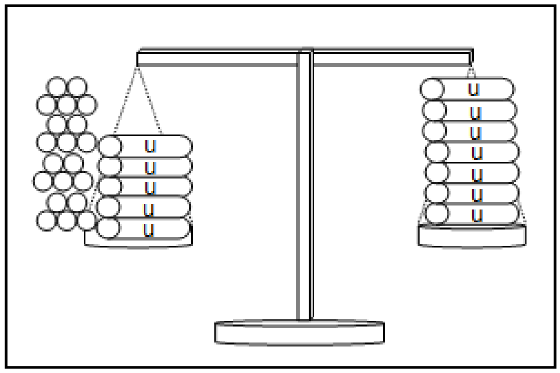


Figure 22: Chuck's interpretation of Rambunctious Rolls, #3 (drawn on board)

Chuck:	So we have 5*u* +20 on one side and then we have 7*u* on the other side of the balance.
Teacher:	Okay, and what did you do? Can you go up to the board and show us?
Chuck:	(*draws a balance on the board* [Fig.22]) So, we have 5*u* (*draws five elongated ovals and labels each with the letter u inside on the left side of the scale*)
Teacher:	And what does the *u* stand for?
Chuck:	Uh, the rolls?
Teacher:	Okay.

As Chuck described the scale and begins to draw his own visual, using rolls and pieces, the teacher prompted him to define the variable *u* in the problem. Chuck hesitantly made claim that the variable *u* represented the rolls. Chuck continued to present his visual, informing the class of his solving method. The teacher concluded Chuck's presentation by restating his methods and prompting the class for additional thoughts, which initiated a challenge of Chuck's interpretation.

Teacher:	So basically what I am seeing is we are balancing out by taking from one side and then taking from the other the exact same thing. Does anybody have anything to add to that?
Calvin:	He said that *u* equals the rolls, but wouldn't *u* equal the packing rule?
Teacher:	Hmm. Good question, what do you guys think?
Savannah:	Yeah.

Teacher:	Yeah? What do you mean ‘yeah’ Savannah? Which one do you agree with?
Savannah:	I agree with Calvin.
Teacher:	You agree that *u* is the packing rule?
Savannah:	Yeah.
Teacher:	Okay, explain what the packing rule is though, Calvin. What does the packing rule tell you?
Calvin:	The packing rule tells you how many pieces are in the roll. So 5*u*, *u* equals how many pieces are going to be in the rolls.
Teacher:	Okay.
Chuck:	I mean five means how many rolls there are and *u* meant the packing rule.
Teacher:	Okay, so *u* is the individual pieces inside of the package which gives us the packing rule, is that what you are saying?
Calvin:	Yeah.
Teacher:	And Chuck?
Chuck:	Yeah.
Teacher:	Okay, so *u* is not the rolls, the number of rolls is in front of the *u*.

Many students respond incorrectly to the representation of variables, due to what is known as the “reversal error,” where students treat variables as literal symbols instead of quantities (Philipp, 1992). This error is more than likely caused by students’ learning about variables in arithmetic, such as *m* or *c*, representing *meters* or *cents* rather than the *number of meters* or the *number of cents*, or not understanding that variables have different functions. Even in algebra there are variables that are easy confused by being read as a statement rather than a relationship of quantities, such as $a = l \text{ x } w$ that is used to find area (Booth, 1988). Calvin challenged Chuck’s claim of the representation of the variable *u* by questioning an alternative representation of the variable as a packing rule. At the teacher’s request, Calvin defined what a packing rule was along with providing an explanation of how the definition fit this particular problem. While correcting his initial claim, Chuck defined the number in front of the variable as the number of rolls and clarified the variable as the packing rule. At this point, the teacher concluded by restating the definition of the variable's representation that the students agreed upon.

The notion that the variable represented the packing rule came from students; therefore, it wasn't until Day 8 that this idea because taken-as-shared. Students were discussing a drawing that Savannah presented on the board (See Fig. 11) when Cindy, a student from the class, asked Savannah the question "Why did you put 18 pieces instead of 18 rolls?"

Professor:	Did you all hear her question? Great question. She asked why did you put 18 pieces on the right hand side, instead of 18 rolls. Who thinks they have an answer to Cindy's question? (*two students raise their hands*) Only two people think they can answer her question? (*Savannah raises her hand*) I know you can Savannah. Wow, that means there's some confusion. Do you think you could answer the question? (*speaking to Anthony*)
Anthony:	Well, because
Professor:	I'm not asking you to do it; I'm just asking if you can.
Anthony:	Oh, yeah.
Professor:	Where's your hands? Can you answer it? Alright, go Savannah, she asked you to explain it.
Savannah:	Because it doesn't say 18*x*, it said 18 + *x*.
Professor:	(*speaking to Farrah*) Repeat what she said.
Farrah:	That it didn't say 18*x*, it said 18 + *x*.
Professor:	So what's the difference? Isn't that the question, Cindy? What's the difference? How do you know what to put? You (Savannah) want to call on someone or you want to take that question? How did you know to put 18 pieces?

Confusion of many of the classroom members emerged with Cindy's question. There was little response when the teacher requested the class to assist Savannah in her explanation. The professor kept the discussion on track, encouraging the students to explain why Savannah chose to draw pieces instead of rolls for her representation of the number eighteen in the problem:

Professor:	… How did you know to draw 18 little loose pieces instead of 18 rolls? What do you think (*Renee*)? Talk to use Renee. Ya'll see what you think about what Renee says, see if you agree.
Renee:	Um, yes?
Professor:	Yes, what? What do you agree to here Renee?
Renee:	The pieces.
Professor:	The pieces. She agrees they should be pieces. No one has given a reason why they should be pieces. Who thinks they know?
Renny:	I know!
Professor:	You go for it (Renny).
Renny:	Um, they can be pieces because it says 18 just by itself, and not 18 with an *x* by it. And *x* would be the packing rule.
Professor:	*x* is the packing rule for the…
Renny:	For the amount of rolls they had.
Professor:	For the rolls.
Renny:	So, if it doesn't say *x* then it wouldn't be rolls.
Professor:	(*draws a circle around the 18 on the board*) So, this (*18*) because it's all by itself the little loose pieces and this part (*circles the x in 18 + x on the board*) *x* is the packing rule that Renny tells you, the *x* tells you about the packing rule. What do you know think about that Cindy? Does that answer your question? (*no response from Cindy*). Repeat what you just heard.
Cindy:	So, I heard that *x* is the packing rule and there's 18 loose pieces because 18 is all by itself and *x* is not beside of the 18.
Professor:	Renny, does that, did she summarize the point that you were trying to get across?
Renny:	Yeah, basically.
Professor:	Basically, alright. Does that help you with your question that you had (Cindy)? (*Cindy nods head yes*). Okay, thank you for asking your question.

This discussion occurred from one student's misunderstanding and questioning of the claim made about the representation of a variable. Renny provided support to Savannah's visual, making the claim that the 18 should be drawn as individual pieces. For his data, he stated that the variable (in this case the variable x) that was placed with the number represented the packing rule for the rolls. The teacher warranted Renny's claim by using the visual on the board to emphasize what he stated. In this instance, indicating the variable in *The Candy Shop* as representation for the packing rule (how many individual pieces are in each roll) was established in the data of the student's argument. There were no more challenges made towards Renny's claim; therefore, indicating variables in *The Candy Shop* signified for the numerical amount for the packing rule (how many individual pieces are in each roll) was taken-as-shared at this moment in the discussion. Additional argumentations made in subsequent discussions confirmed this conclusion.

Variables do not always represent fixed values. As students are first introduced to variables and begin to interpret them, they tend to connect variables with fixed exclusive values (Booth, 1988). Many students think that once the variable a has a value of 10, then a is always going to be equivalent to 10. When the idea of variables was introduced in *The Candy Shop* sequence, there were multiple student generated interpretations that surfaced during the whole class discussions. In the whole class discussion about the first two scales in the 'Brainy Brian's Balance' activity (See Fig. 23) students made claims and provided data to support their conclusion that $x = 10$ and $c = 5$. At the request of the teacher, one of the students in the class, Chuck, made an alternative claim that included the variables from both scales.

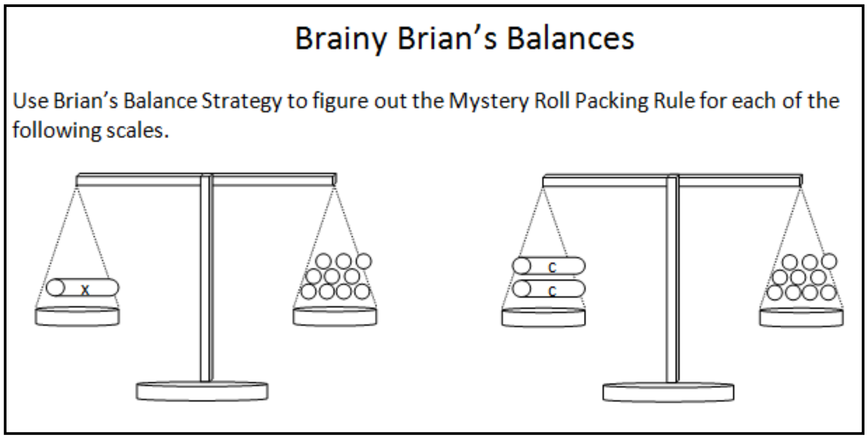


Figure 23: Brainy Brian's Balances, scales 1 & 2

Chuck:	Well, um, I did basically the same thing, except since I saw that one roll of *x* equals 10 pieces and two rolls of *c* equals ten pieces, I did *x* equals 2*c*, because one roll of *x* equals two rolls of *c*.
Teacher:	Can you explain that to me?
Chuck:	Well, since one roll of *x* equals 10 pieces and two rolls of *c* has 5 pieces in each, it's 10 pieces. 1*x*, or just *x*, could 2*c*.
Teacher:	So you used these two (scales) to put together a number sentence, or an expression? (*Chuck nods yes*) Does everyone understand what he is talking about? (*Students answer yes*) Do you really or are you just saying yes?
Students:	Yes.
Teacher:	Go ahead (*Speaking to Anthony, whose hand is raised*).
Anthony:	I disagree with Chuck, because *x* could stand for anything. If you just showed this (second) scaled to a person but they didn't see the first one they wouldn't know what *x* was, so it wouldn't make sense at all.
Teacher:	Alright.
Hannah:	I agree with Anthony, because still if they just look at the second one then he said *x*, that could mean anything. That could be 15 pieces; that could be 185 pieces; it could be any amount unless you stated that.
Teacher:	Alright.
Jordan:	I disagree, and I don't know if I'm right, but what I'm hearing is that you think people would get them mixed up if you don't have the first one (scale) cause then you don't know that you're just counting the one roll. If that's right, then I'll continue. Um, I don't think you would need to show this (the first scale) to show that (the expression). I think they could still understand it because the *x* could be any number.

Teacher: Debbie.
Debbie: I have something to add onto that because if *c* is 5 pieces each, then *x* equals 10. And if this person knows how to add, whoever it is, and they don't show the first one (*referring to the scale*), then they could still get it because *c* is 5 and they probably can figure out the *x* is, will be, 10 because 2*c* equals 5 plus 5.

Chuck's made the claim that the two differing variables, *x* (from the first scale) and 2*c* (from the second scale), could be equivalent. He used the values of the individual variables, accomplished by solving the individual scales to provide data to support his claim. Anthony and Hannah rebutted Chuck's claim with their collective claim that the variable *x* could represent any value. Hannah introduced the idea that *variables do not always represent the same value*, as she supported her and Anthony's rebuttal providing random values that the variable *x* could possibly represent.

The discussion lead into other aspects of the problems that were being discussed, but the teacher later reverted back to the topic of the expression $x = 2c$.

Teacher: Okay, and then Chuck, you brought up the point that $x = 2c$ and Anthony disagreed. So I want to discuss.
Farrah: Going back to what we just looked at the flavors have different packing rules, *x* and *c* could very well be different flavors because they are on different days. So, if they are different flavors they might have more pieces or they might have fewer pieces, or they might weigh more than the other.
Renny: I agree with Farrah and I'm going to add a comment that if you want to figure out the packing rule. Let's just say that root beer was 11 and grape was 2, then you could figure out what kind is in the package.
Teacher: I guess my question is can *x* still equal 2*c*?
Students: Yes.

At this point in the exchange the students had agreed that the value of *x* could equal the value of 2*c*. The discussion continued as students debated Farrah's claim of the packing rule changing and if the flavor of the candies was or was not a significant factor in figuring out if there were times when $x = 2c$ would not work. As the students discussed and debated different factors, Brian claimed that $x = 2c$ will always work because, in this sequence, the variables will always represent the same flavor. Brian's claim was challenged by another student, Cindy:

> Cindy: Um, I disagree with Brian because they (the variables) won't always equal each other, because *x* and *c* are variables and variables can equal any number. So *x* could change and *c* could change (values). So they might not equal each other all the time.
>
> Teacher: Hmmm, interesting.
>
> Farrah: Well, I agree with Cindy because she's right that they're just variables, so *x* and *c* won't always equal each other because if you change *x* to equal another number, or even just one of those variables change, then it would be thrown off.

This was the initial discussion of the concept of variables not always having the same value in all situations. Later, on Day 8 of the instructional sequence the concept is brought up again by the professor. Prior to the following dialogue, the students were given the task: "How could the Simpsons package 59 candies if the packing rule is c = 10? Draw at least two different pictures." The students discussed multiple ways to illustrate this problem on the board. Luther suggested an illustration, using two expressions together on the scale, which the professor later alluded to in the discussion.

> Luther: Uh, I didn't think of this until London started to do his. I thought he was going to do this, but he did it in a different way. So where he could put one picture on, this would be $5c + 9$ (*writes 5c + 9 on the left side of the scale; draws 5 rolls and 9 pieces above the expression*). And then, you would put another picture over here, like $6c - 1$ (*writes 6c-1 on the right side of the scale; draws 6 rolls with one missing one piece*).

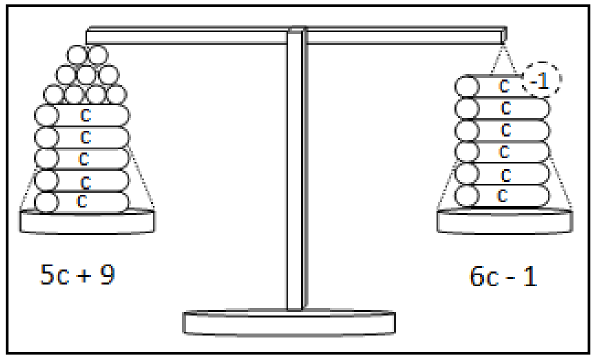


Figure 24: Luther's interpretation of 59 candies

The professor used Luther's interpretation on the board (See Fig. 24) to pose a question to the students to think about variables having different values.

Professor:	… before we move on, one quick question. You guys said that these two pictures on this scale (*referring to Luther's illustration* [Fig.24]) were equivalent that it would balance like that, is that always true?
Students:	No.
Professor:	In this case is it true?
Students:	Yeah.
Professor:	What would make this not true? Talk to your partners about it. What would make this not equivalent? Same picture. Hey guys, we're not going to add or take away anything in the pictures. The picture stays the same.
Debbie:	Well, then it's not going to equal a different amount.
Professor:	What are you talking about? Say it out loud.
Debbie:	Well it might not, because there's 5*c* and 6*c* – 1, I mean 5*c* plus nine is still 59 and then 60 is (6*c*) minus 1 but it might not be the same because it's two different rolls and the packing rule is 10 and it's just like, it might now, I'm just going in circles here.
Professor:	(Laughing) Going around in circles? Help her out.
Calvin:	I think it could be different because *c* might not be equal to the same amount, the packing rule might be different.
Professor:	Did you hear what he just said?
Student:	Yeah, but it says the packing rule is…
Professor:	It does say the packing rule is, in this case, that *c* is equal to 10, but
Jordan:	And that messes up the entire thing, no matter what *c* is, the packing rule is always going to be 10, throughout the whole question.
Professor:	Beautiful, you're right.

Jordan: It's not going to change.
Professor: You're right, but I did ask you, I challenged you is there a time where it wouldn't be the same and that's a case (*speaking to Calvin*) where it wouldn't be the same. If *c*, f the packing rule was not 10 do you think it would change; that equivalence?
Students: Yeah.
Professor: Yeah, it sure would. Nathan? Last word.
Nathan: Because I remember every week she would change the packing rule.
Professor: Oh, I know.
Nathan: So, it could be a week old package with a non-equal one.

When the professor made the announcement that none of the variables or numbers would change, Debbie made the first claim stating that if that was the case, then the values wouldn't change. As Debbie presented her data, restating that if the packing rule and expressions were to stay the same the sides would still equal 59, she started to second guess her own thinking by stating that it may or may not equal 59. Calvin stepped in and made the second claim, that there could be a difference in the value of both sides of the scale. For his data, he explained that the packing rule could change; therefore, the variable *c* would not be the same value. Nathan reminded the class that Maggie, the fictitious character from the instructional sequence, would change the packing rule weekly. As a result, Nathan warranted Calvin's claim with the hypothetical scenario that one of the rolls could be from a different week when *c* represented a different packing rule. At this point, the idea that *variables do not always represent fixed values* is taken-as-shared.

"Like terms" such as 4x and 2x represent multiple units of a fixed quantity (x+x+x+x and x+x) and therefore can be combined. Quantities such as 4x and 2 are not multiple units of a particular quantity, and therefore the sum of these units can only be represented as 4x+2. According to Linchevski and Herscovics (1996), there are multiple

studies (Carpenter et al., 1981; Kuchemann, 1981) showing from a cognitive prospective the idea of joining like terms in algebraic expressions as a significant problem for students. On Day 9 of the instructional sequence, the professor asked the students to complete four problems on the 'Packing Problems' worksheet that she handed out. After the students had time to look them over and work them out with their partners, the professor prompted the students to discuss their answers. On problem number two the students had the question "How much candy is in the collection below if the packing rule is g = 15? Draw a picture, if necessary" with the problem "3g + 5 + g" under the question. Initially Adam made the claim that the solution was 65.

> Adam: So, first I did 15 times 3 which is 45 (*writes 15 x 3= 45 on the board*) and then I just said that *g* was 15 (*marks out the g in the problem 3g + 5 + g that was on the board*). So then I did 45 plus 5 plus 15 (*writes on board*), which is 45 plus 15 is 60 and then I added 60 plus 5 which is 65.

For his data, Adam stated the process in which he substituted the variable g for the given packing rule value of 15 to achieve this solution. At the teacher's request, Luther went up to the board to share his alternative solving method.

> Luther: Okay, so I did 3*g* plus 1*g* (*writes 3G +1G = 4G on the board*), so I had 4*g* plus 5 (*writes 4G = 5*). And so I did 4 times 15 is 60 (*writes 4 x 15 =60*) and then I did 60 plus 5 equals 65 (*writes 60 + 5 = 65*)

According to Toulmin's (1958) argumentation scheme Luther was interpreted as having the same claim as Adam, stating that the solution was 65. However, Luther stated that he combined the 3g and 1g (two like terms) together with the operation of addition, creating 4g, and then substituted the g with the given packing rule value of 15 to solve the problem. With his statement Luther provided additional data that supported both students' shared claim, and also offered an alternative perspective to solving. The professor kept the discussion going when prompting students to raise their hand if they could see the difference between the two students' solutions. Not all the students raised their hands, so the professor had them talk within their table groups to discuss the possible differences.

Professor:	Okay, so I would like, Cindy, can you like summarize the difference between Adam's and Luther's?
Cindy:	Well, Adam, he did step by step, so he did 3*g* plus 5 plus *g*, so 45 plus 5 plus 15. And Luther, he just took a short cut, and he added the two *g*'s together because it's plus *g* and he just did 15 times 4, instead of 3 times 15 and adding the extra *g*. And then he just added the five.
Professor:	Is that legal to do in here? Is it legal to add the 3*g* and the one *g* first? (*Some students say legal and some say illegal*). Wait, wait, wait, one person at a time, Debbie? Debbie's talking to you guys.
Debbie:	It's basically the same thing except instead of actually using the numbers, Luther used the *g* and Adam used the numbers that were the letters.
Professor:	Did you hear what Debbie said? Okay, she said and see if I got you right Debbie, Adam, kind of went in this order (*pointing from left to right on the original problem* $3g + 5 + g$) and he just used the numbers for *g*, what was *g*? What does *g* stand for by the way?
Students:	15.
Professor:	What does it stand for in the story?
Students:	The packing rule, 15 pieces per roll.
Professor:	15 pieces per roll and so he just put 15 per roll in the *g*'s in that order, 15 times 3 plus 5 plus 15 more. But Luther, according to you (Debbie), did what?
Debbie:	He used the variables.
Professor:	He kept the rolls, the pieces per roll as a variable and just combined them and got 4 rolls with *g* in each. And that's okay to do you think (class)? Debbie thinks so. What do you want to say (Hannah)?
Hannah:	I did it a different way.

Cindy summarized the differences between Adam and Luther's interpretations of the given problem stating that their approach to solving was the difference. By providing this link from the students' individualized data to the shared claim, Cindy's summary was decoded as the warrant. From this summary, the professor developed the question, "Is it legal to add the 3*g* and the one *g* first?" which was the initial introduction to the ideas that *"like terms" such as 4x and 2x represent multiple units of a fixed quantity (x+x+x+x and x+x) and therefore can be combined and quantities such as 4x and 2 are not multiple units of a particular quantity, and therefore the sum of these units can only be represented as 4x+2.* As the discussion continued the professor provided backing for the Cindy's warrant by restating Debbie's statement and continued to prompt for understanding of like terms. Unfortunately, this discussion veered away from that particular concept, but would arise again.

During the instructional sequence, on Day 13, the professor proposed a challenge for the students (See Fig. 25). The students talked with their partners and brought their ideas back to a whole group discussion. The professor asked the students to give their answers first, so that she could record them on the board and then discuss. Jordan, one of the students, made the claim that 9 was the packing rule in which all of the students collectively agreed.

Professor:	No way, come on that was too hard. Okay, success, that means I have to come up with something harder, but who can explain this one? How about you… Alright here we go. Ya'll listen up, see if she matched what you did. Come on up to the front. Show your classmates what you did, show them what you did, did you have to draw a picture?
Kathy:	Yes.
Professor:	She drew a picture, excellent.
Kathy:	First I did 4*x* plus 2*x* and got 6*x* (*writes 4x + 2x = 6x; draws 6 rolls with x's in each roll*); and then I took out the 20 right here (*puts a mark through the 20 on the original problem*); and then I had to take out 20 here (*writes -20 behind the 74 on the right side of the scale*) minus 20 equals 54 (*writes =54 on right side of the scale*). And then, then you divide 54 divided by 6 equals 9. (See Fig. U)

Professor: Alright, who has questions for her? Repeat what she did. Nathan, repeat what she did.

Nathan: So, what she did was she added the 4 and 2 getting 6, then she took the 20 out and then she took the 20 from 74 getting 54. Then 54 divided by 6 equals 9.

Professor: (*speaking to Kathy*) Did he say what you did right? (*Kathy nods yes*).

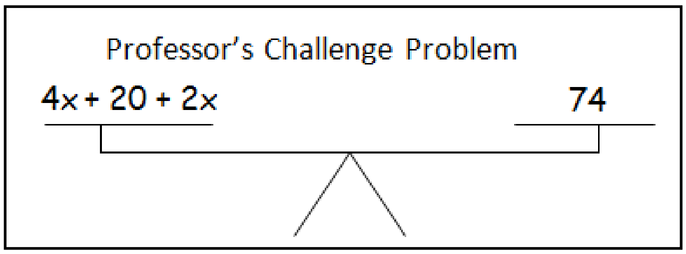


Figure 25: Professor's challenge

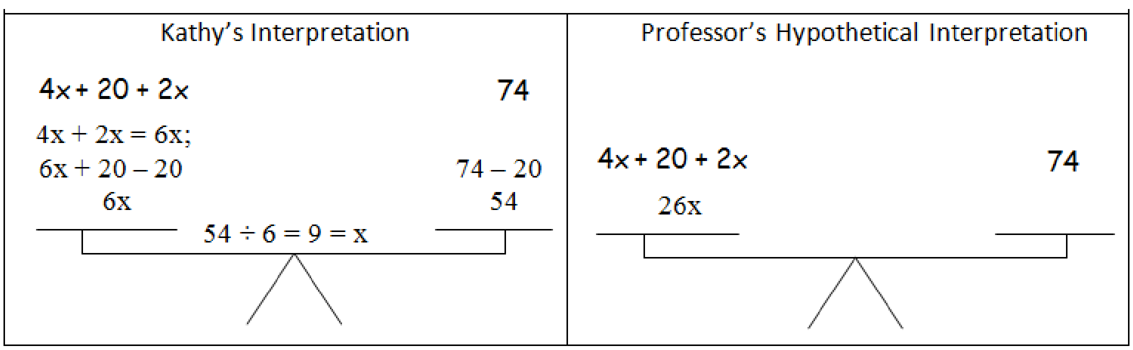


Figure 26: Two interpretations of professor's challenge (drawn on the board)

Kathy offered data that supported the collective claim of the students, stating that she added the two numbers that had variable *x* as part of them, 4*x* and 2*x*. By having another student repeat Kathy's data, the warrant for this given claim and data is implied. To confirm that the students had a comprehensive understanding of combining like terms, the professor offered a hypothetical solution (See Fig. 26) for the students to decipher.

Professor: I thought I was going to find somebody doing this. They said this (left) side is 26*x* and this (right) side is 74. How do you think they got that? Everybody think for like five seconds. Alright, turn to your neighbor and tell them where do you thing that (26*x*) came from. (*Students turn and share*).

Professor: Okay, so we got a volunteer to explain it, go ahead.

Renny: Okay, so basically what they did, they just added 4 plus 20 equals 24 plus 2*x* equals 26 and they thought they were all rolls.

Professor: Well, that makes sense to me, doesn't it to you?

Students: No.

Professor: Why not?

Jordan: Because the 20 there doesn't have a variable.

Professor: So what, it's got a number.

Jordan: But if there's an *x*, the *x* signifies that it's a roll so 2*x* means 2 rolls, but since there's just a 20 there, it's 20 separate pieces because there's no *x* there to signify 20 rolls.

Professor: Wow, did you hear what he said?

Savannah: No.

Professor: He's right there behind you, repeat what he said, or say it in your own words. He says you can't add these two things (*points to the numbers with variables and the independent numbers*) together. Do you agree with him?

Savannah: Yeah, because 20 is just the pieces that you're adding on.

Professor: But, there's pieces here (*points to the 2x on left side of scale*).

Savannah: That's a roll.

Professor: That's a roll, but aren't there pieces in there?

Savannah: Yeah.

Professor: Well, pieces plus pieces. Help me understand this. Go for it (Calvin).

Calvin: So what he's saying is so you can't add 20 plus 2 or 4 because 20 is not in a roll or a package. But 2*x* and 4*x* are in a roll, with the pieces but 20 is not in a roll. It's just loose pieces.

Professor: Okay, that's really nice, help me out (Farrah).

Farrah: So, even though it's 2, the value is, it means 2 rolls so the value is not 2. It's however many pieces times two.

Professor: Oh, did you hear that? She said, I'm going to say what you just said. Farrah says, now make sure that I got you right, Farrah. This 2 (*writes 2x*) does not stand for 2 pieces. (*Draws two circles*) That's not the picture for that. What would the picture look like?

Farrah: It would be two rolls.

Professor: Two rolls with how many pieces then?

Farrah: Nine.

Professor: Nine pieces? Well, if we didn't know that, how many pieces.

Farrah: x.

In this exchange, the professor made the claim that some would get 26x on the left side of the scale and prompted the students to think how they would have accomplished this. Renny's supportive statement, that they added all of the terms, with and without variables together, was interpreted as data for the hypothetical solution. However, when asked if that made sense the class challenged this solution, stating that it did not make sense. Jordan clarified the students' rebuttal, divulging the difference between the representation of rolls (coefficients or numbers with variables) and pieces (independent numbers). The professor continued to be the "devil's advocate" in this situation questioning the two different representations being added together, using Jordan's data and her own claim to present a warrant. Calvin's reiteration of the idea of two different representations for rolls and pieces was interpreted as a warrant, linking Jordan's data and the students' rebuttal that combining the two did not make sense. Farrah supported Calvin's warrant explaining that when the number was combined with the variable, the number was not representing how many pieces there were. The number represented the amount of rolls to multiply to the variable (the unknown value), to find the total value. This was the second instance that this concept that like terms could be combined with each other but not with independent numbers came up in whole class discussion. Students continued to use this concept as data in their arguments on problems afterward, indicating the ideas had become taken-as-shared.

Practice 3: Interpreting Algebraic Expressions.

According to Seng (2010) many studies found that students illustrate significant struggles when learning basic algebraic concepts (Herscovics & Linchevski, 1995; Kuchemann, 1981), among those being the interpretation of algebraic expressions.

As mentioned in the previous section, students struggle with the idea of variables, wanting to initially interpret them as a fixed value (Booth, 1981; Kuchemann, 1981) which can cause errors when students are presented with various representations of variables, such as composite units (Underwood Gregg and Yackel, 2002). Students also tend to use their knowledge of previously learned arithmetic to apply to algebra, such as the concept of "=" being an action to solve for an answer (Thompson, 1999). As explained by Tirosh, Even and Robinson (1998) the students struggle to accept the "lack of closure" solution; students tend to want to make the expressions "equal" a value or have a definite answer instead of leaving the open expression alone.

As the students worked through the instructional sequence, they were introduced to the idea of algebraic expressions through activities designed to lead them to make their own interpretations. Students were presented activities that would lead them into situation where they may face these common errors, mentioned above. These activities allowed students to build their knowledge of algebraic expressions, which led to the following two taken-as-shared ideas:

- A number beside a variable (i.e., 6*x*) represents multiplication
- Expressions represent the amount of candy rolls and pieces in *The Candy Shop.*

Independent number beside variable (i.e., 6x) represents multiplication.

One error that students face when first introduced to interpreting algebraic expressions is the confusion of the "place value" aspect of conjoining (Booth, 1988). This is when students are given a value for a variable (i.e. $y = 4$) and a number with a variable beside it (i.e., 5*y*) and they create a number (54) instead of understanding that these two values should be multiplied together (5x 4 =20). However, when the concept of solving for the variables (packing rules) was presented to this class the students seemed to have intuitive understanding to apply appropriate methods when solving, especially when there was a visual for them to utilize. Initially, some students were able to imply or express the use of multiplication within their claims, justifications, etc. as they solved for the packing rules. As the sequence progressed it became obvious that some of the students had prior experience with some of the algebra concepts that they brought with them to work through the activities and elaborate on in the discussions held in class. Being that this concept isn't introduced until 6th or 7th grade these students show that they developmentally ahead in their mathematical abilities. Not all students were able to instantaneously imply the operation of multiplication, until after the students with prior experience contributed to the discussions.

The first implication that the placement of a variable beside a number represented multiplication was on Day 4 the student were discussing Brainy Brian's scale, shown below in Figure 25.

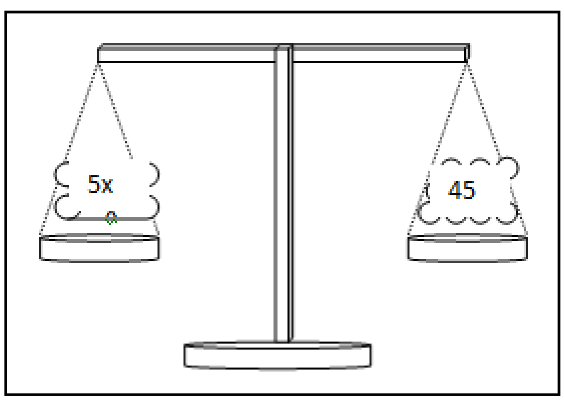


Figure 27: Brainy Brian's Balances, #5

The dialogue that follows illustrates how one student, Hannah, intuitively implied the use of the multiplication operation when interpreting 5*x*.

Hannah: What I did for that is, um, because it was 5 times and then it was 45 and I know that 5 times 9 is 45, so I got nine.

As the students continued to discuss this problem, Calvin offered his solution and explanation for the same problem, implying the same thing as Hannah but with the use of division.

Calvin: Oh, the 5*x*. I did 45 divided by 5 for that one.
Teacher: Why did you do that?
Calvin: Because if there is 45 pieces and there's five rolls and you just do 45 divided it would just tell you your answer for that.

In both selections of this dialogue the students use mathematical terms, such as "times" and "divided" to imply that they know that to get the value of the number and the variable that are placed together they need to use multiplication. However, it is not until later in the sequence that the students directly discuss what the number and the variable represent and why they are using multiplication to figure out value.

As the students discussed their Simpsons' Studies problem on Day 8, the class established the meaning of when a number and variable were placed together. The students had just finished discussing different methods they used to get the packing rule *x*, in the given equation $8x + 5 = 48$. Using their hands to represent a scale the class discovered that after subtracting the 5 from both sides, they would have $8x = 43$, which Nathan, a student in the class, just wrote on the board and explained.

Teacher: … So now we have 8*x* and 43. We want to get *x* by itself that was our goal yesterday right? (*Students nod yes*) So, 8*x* plus 43 (*teacher means to say equals instead of plus).* When we put the 8 and the *x* together what did we say we were doing yesterday?

Students: Multiplying.
Teacher: Multiplying, so how do we get the *x* by itself? What do you have to do to your balance?
Brain: You got to take away the 8 to get the *x* by itself.
Teacher: Take it away, as in what operation are you using?
Brian: Um, you would be using division.

In this discussion the teacher prompted the students to focus on the operation that was represented when putting a number and variable together. The teacher walks the students through the procedural steps as she refers to "yesterday," directing the students through to the prior conversations that they had discussed using "times" such as in the dialogue mentioned previously for Day 4. The students used the data that 8*x* represented multiplication here to explain how to get the variable by itself with division. The students had implied that a *number beside a variable (i.e., 6x) represented multiplication* through their claims up to this point, but now that it was used as data for a new claim, it was interpreted as taken-as-shared. The students' claims and data continued to support that this concept was taken-as-shared as the instructional sequence continued.

Expressions represent the amount of candy rolls and pieces in The Candy Shop.

As students worked through *The Candy Shop* instructional sequence they were able to use other taken-as-shared ideas that the class established to discover what the algebraic expressions for this sequence represented.

For example, as mentioned previously in this chapter the students were introduced to variables on Day 4 within the activity of Maggie's Mystery Rolls. Jordan made the claim early on that the variables that were being used in this sequence would represent the amount of candy in the rolls and that Maggie would change these amounts (with or without changing the variable itself) daily. Then on the following Day 5 the students used this same information as data to establish that the variable represented the pieces of candies in the package as taken-as-shared. The students further negotiated this idea later to reestablish that variables, or the number of candies in each roll, were known as the packing rule in this sequence. These taken-as-shared ideas about variables were significant in aiding further connections to be made as the class began working with Brainy Brian's Balance strategy, comprehending the way the numbers, symbols and variables worked together to create expressions. An example that was used multiple times in this chapter is on Day 7 when Chuck and Calvin negotiate the meaning of the variable and the number that is placed in front of it. They come to a consensus that the variable is a representation of the number of candies in the rolls (Calvin) and the packing rule, while the number represents how many rolls that are present. However, as these claims are used as data by students the following day, the idea that expressions represent the amount of candy rolls in *The Candy Shop* would be considered taken-as-shared. The students continued to refer to these components, the variable and number that were attached to the variable as the number of rolls and packing rule throughout the remainder of the sequence which supports this.

Earlier in this chapter the following problem (See Fig. 10) was used as an example to illustrate the questioning norm. In the discussion for this same problem the class established that *expressions represent the amount of candy rolls and pieces in The Candy Shop*.

Solve for the Mystery Roll Packing Rule. The scales are there ONLY if you need to use them. Be sure to SHOW your thinking!

2. $12x + 7 = 18 + x$

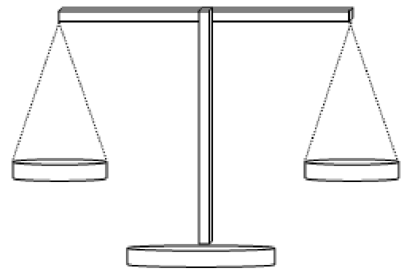


Figure 10: Rambunctious Roll Review, #2

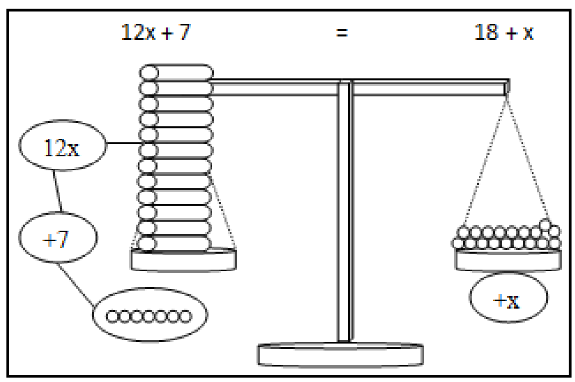


Figure 11: Savannah's interpretation, drawn on board

First, here is the recap of what has happened in the whole class discussion so far. The professor asked a student, Savannah, to draw her interpretation on the board (See Fig. 11) and prompted the other students to ask Savannah questions about her drawing. Cindy, another student in the class, questioned Savannah about her representation of the 18 on the right side of the scale as pieces instead of rolls. Savannah explained that she drew pieces "because it didn't say 18*x*, it said 18 plus *x*." The professor continued to encourage the students to provide reasoning of why the representation should be pieces.

Renny claims that he can provide reasoning, and the following excerpt shows the dialogue including his explanation.

Renny:	Um, they can be pieces because it says 18 just by itself and not 18 with an *x* by it, and *x* would be the packing rule.
Professor:	*x* is the packing rule for the…
Renny:	For the amount of rolls they had.
Professor:	For the rolls.
Renny:	So if it doesn't say *x* then it wouldn't be rolls.
Professor:	(*draws a circle around the 18 on the board*) So, this (*speaking of the 18 that was circled*0 because it's all by itself, the little loose pieces; and this part (*circles the x in the 18 + x on the board*) *x* is the packing rule that Renny tells you the *x* tells you about the packing rule. What do you think about that Cindy? (*Cindy does not respond*). Does that answer your question? (*Cindy does not verbally respond*). Repeat what you just heard.
Cindy:	So, I hear that x is the packing rule and there's 18 loose pieces because 18 is all by itself and x is not beside of 18.
Professor :	Renny, does that… did she summarize the point that you were trying to get across?
Renny:	Yeah, basically.

Prior to this discussion students had implied and made claims that the variables represented the packing rule, or the number of pieces of candy in each roll, that the number in front of a variable represented the number of rolls, and that the individual numbers represented individual pieces of candy in *The Candy Shop*. Renny took all of these claims and used them as data to explain why Savannah's interpretation was correct. At this point, the idea that *expressions represent the amount of candy rolls and pieces in The Candy Shop* was a taken-as-shared idea among the class. Student claims and data used in activities beyond this point serve as support for this data.

The activities that were designed for this instructional sequence helped students build knowledge of algebraic concepts that much research had reported were areas of concern for students beginning to learn algebra. The students did learn that variables did not always have the fixed values, as they understood that the variables were used for packing rules in this sequence and the variable values changed with the packing situation. The students could see that when given an expression, such as 5*c* +3, that the composite unit had a back story and could continue to grow by adding pieces of candy, or could lessen by losing pieces of candy. As students learned and used the balance strategy, some were confused with the difference between an expression and an equation. Some students continued to want the equal sign as part of the expression, just as the research had reported. All of the taken-as-shared ideas that derived from this practice create rules for the students to use as they work through the algebraic expressions.

Practice 4: Reinvention of Methods for Finding the Value of x (Packing Rule)

The tasks in *The Candy Shop* instructional sequence encouraged students to solve for the packing rule starting with visuals and moving to standard notation. Even though some continued to create visuals to solve, the whole class was able to establish that the variables used in the expressions represented the unknown packing rule (Herscovics, 1989). With this understanding, the students were able to invent methods for solving for the value of the variables, which lead the following taken-as-shared ideas (the variable *x* is used in the following section as a generic variable, as variables did vary):

- To find the packing rule, the rolls (i.e. 4*c*) and pieces (i.e., 20) must be isolated on opposite sides of the balance.
- Inverse operations are used to solve for the packing rule.
- Division (pieces/rolls) can be used to find the packing rule
- When solving for x, an “imaginary roll” can be added in the expression

To find the packing rule, the rolls (i.e. 4c) and pieces (i.e., 20) must be isolated on opposite sides of the balance.

As mentioned in the previous section, the students quickly understood that the scale had to be balanced; therefore the amounts on the two sides were equivalent. As they manipulated Brainy Brian's balances to figure out various packing rules, they discovered that to solve these problems operations had to be performed to find the packing rule. On Day 7, while the students were working through a balance (See Fig. 21), this concept arose as a student supported a claim made (Fig.22). Prior to the following dialogue, Chuck made the claim that $u = 10$.

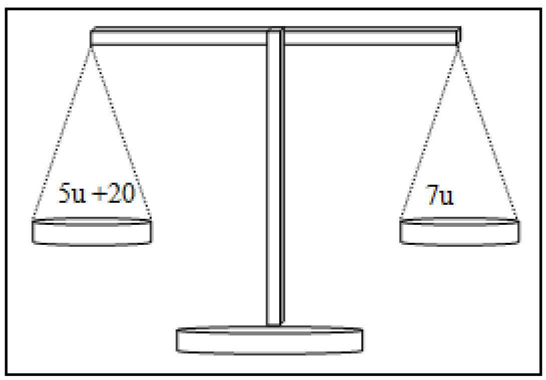


Figure 21: Rambunctious Rolls, #3

Chuck:	… So, what I did was since there were 7*u* on this (right) (Fig. 22) side and 5*u* on this (left) side, I took 5*u* off of each side, because I wanted to isolate the remainder of the pieces and the remainder of the *u*'s on this (right) side of the balance (*he erases 5u from both sides of the balance*). So we have 20 pieces left (*points to the left side*) and then there are 2 *u*'s left (*points to the right side*). So, 20 divided by 2 equals 10 (*writes 20 ÷2 = 10 on the board*). So, *u* must equal (*writes u = 10 on the board*).
Teacher:	Everybody understand that?
Students:	Yeah.
Teacher:	So, basically what we are seeing is we are balancing out by taking from one side and then taking from the other side the exact same thing…

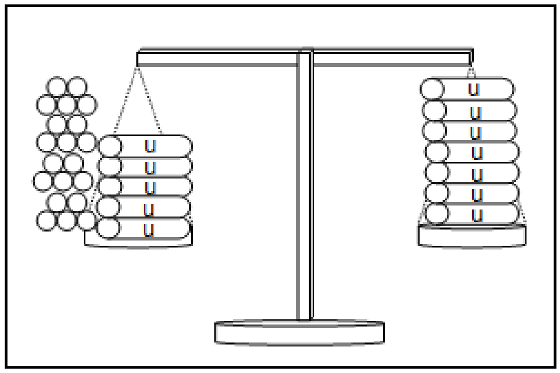


Figure 22: Chuck's interpretation of Rambunctious Rolls, #3 (drawn on board)

For his data, Chuck stated that he found this answer by subtracting 5*u* from both sides and then divided 20 by 2. He then warranted the linked his claim and data with his reasoning of wanting to isolate the pieces and rolls on opposite sides of the balance. The teacher then reiterated what Chuck is doing, providing a backing for Chuck. This is the initial occurrence of the concept of isolation of the pieces and rolls on differing sides of the balance. However, as the students continued through the next day's activities this concept is sustained. On Day 8, the whole class was discussing how to solve their morning activity, the Simpson's Studies, which was $8x + 5 = 48$. The students shared their answers, and a misconception arose as a student got confused with what to do with the 5 on the left side of the balance. Therefore the teacher and other students walked through the problem, coming to the conclusion through Nathan's claim that the five has to be subtracted from both sides.

Teacher: Okay, so we're going to subtract it (*the 5*) from this (left) side, and what does that do?
Nathan: Cancels it out.
Teacher: And our balance is kind of leaning now so we're going to…
Nathan: Take a five out of that (*referring to the right side of the balance*) which is 43 (*teacher writes 48-5, then marks out and writes 43*).

Teacher: Which is 43. So now we have 8*x* and 43 (on opposite sides of the balance). We want to get x by itself that was our goal yesterday right? (*Students nod yes*). So, 8x plus 43 (teacher means equals instead of plus)…

In this exchange, Nathan supported his claim with the data that the five was cancelled out and that in order for the balance to be balanced the five would have to be subtracted from the opposite side as well. The teacher reminded the students that they had discussed isolating the *x* yesterday, which is interpreted as a warrant tying Nathan's claim and data together. Being the second occurrence of the concept as warrants that tie together students' data and claims it is interpreted that at this point t*o find the packing rule, the rolls (i.e. 4c) and pieces (i.e., 20) must be isolated on opposite sides of the balance* was constituted as a taken-as-shared concept. There is further data that supported this interpretation as students continued to utilize this concept to support their thinking. This concept also complies with the taken-as-shared idea from the previous practice, that the balance represented equivalence. In order to find the value of a variable (which represented the packing rule) the rolls and pieces had to be isolated first before being able to isolate the variable by itself. In order to isolate the rolls and pieces on each side of the balance the balance had to show equality between the amount of rolls (with their individual pieces in each of them) and the individual pieces.

Inverse operations are used to solve for the packing rule

The idea of using inverse operations to solve for the packing rule arose early in the instructional sequence as students were introduced to Brainy Brian's balance strategy. Students were able to find the packing rule of $2c = 10$ (See Fig. J) using the inverse operation of division to split the multiplication of the 2*c*. However, there was not a clear understanding that the combination of a number and variable (i.e., 2*c*) represented multiplication yet. On Day 6, as the students progressed through solving various balances with visuals, the teacher challenged them to solve for the packing rule on one of Brainy Brian's balances that had numbers and variables only.

Teacher:	So, we have $5x + 6$ on one side of the balance and we have, on the other side 46. So if we have this expression, tell me how we are going to balance out and how we're going to figure out the packing rule.
Debbie:	Since the plus 6 is the extra pieces then you take that away.
Teacher:	So, you are talking about $5x + 6 - 6$? (*writes on the board*)
Debbie:	Yeah, because yeah and you take that away. Then the left side will be heavier.
Teacher:	So, these cancel out right? (*Teacher marks out + 6 -6 on the board*)
Debbie:	Yeah.
Teacher:	So we have $5x$ equals 46 (*Teacher writes on the board*)
Debbie:	No, you have to um, you have to take away the 6 from 46 to get 40.
Teacher:	So what you are saying is if you do something to one side of the equation you have to do it to the other?
Debbie:	Yes.
Teacher:	Are you sure?
Debbie:	Yeah.
Teacher:	So we took away (6) from 46. So now we have $5x = ?$
Debbie:	40.
Teacher:	40.
Debbie:	And then you divide 40 by 5 which is 8, so x equals 8.

Debbie made the claim that the variable x equaled eight. For her data, she used taking away (or subtraction) from each side of the balance and then division. Again, it is implied, and at this point being negotiated, that inverse operations are used to solve for the packing rule. The students continued to use inverse operations without explaining their claims or using the concept in their data, until later in the instructional sequence. On Day 13 of the class sessions, the teacher prompted a discussion about one of the Simpsons' Study for the day, $6x – 8 = 34$. Prior to the following dialogue, the teacher noted that Kathy and Evan had different solving methods and asked them to explain.

Kathy made the claim $x = 7$ using guess and check with multiplication as her data. The teacher then prompted Evan to share what he did.

Evan:	Um, so I did 34 plus 8 equals 42. And then I did 42 divided by 6 equals 7. And then 7 times 6 is 42.
Teacher:	So you took the 34 from here (the right side of the balance) and added the 8. Why did you do that?
Evan:	Because it says 6*x* minus 8 equals 34 and you could use the inverse operation.
Teacher:	Okay, so he used the inverse operation…

This is the occurrence in which the idea that *inverse operations are used to solve for the packing rule* was considered taken-as-shared. Evan made the claim that you could add eight to the opposite side of the scale. For his data, he used this concept of inverse operations to explain how he moved the numbers to solve for the variable (or the packing rule).

When solving for the total amount of candy, an "imaginary roll" can be added in the expression.

On Day 2 of the sequence, while the class was discussing different ways to solve Mrs. Simpson's Packing Rule #6 (See Fig. 28), a new idea was presented. Prior to the following dialogue the teacher had prompted students to share their solutions using the packing rule of 79 pieces.

Chuck:	Um, I just came up with this; you could do 79 times 4 minus 83.
Teacher:	Seventy-four?
Chuck:	Seventy-nine.
Teacher:	Oh okay, sorry, 79 times 4?
Chuck:	Yes, minus 83.
Teacher:	Minus 83; explain that to us. (*Writes 79 x 4 – 83 on the board*)
Chuck:	Uh, well um, you could add another imaginary roll and then you would do 79 times 4 which is, I have no idea. Okay you would do 79 times 4 minus 83 because would take the

extra roll which is 79 and then take the extra missing 4 and then add those together and then subtract those from the 74.

Teacher: Anybody have any comments about that?

Jordan: Um, I agree with that; it is right. But, I think adding a whole new imaginary roll and just taking it away is just too much. It would be just easier to count the rolls you have and take away four.

Renny: I agree Jordan, I think it is unnecessary to add a whole other roll, especially when the number that you're multiplying is 79. Um, I think you should do 79 times 3 and subtract 4 since you get the same answer, but it's shorter and it takes less time.

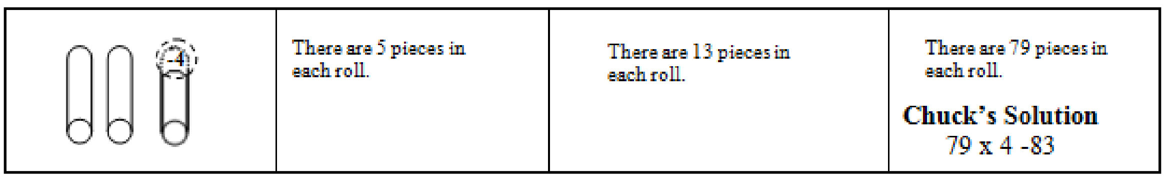


Figure 28: Mrs. Simpson's Packing Rules with student solutions, #6

In this exchange, Chuck made the claim that an imaginary roll could be added to the expression to show how many pieces Mrs. Simpson had with the packing rule of 79. For his data, Chuck explained that he would multiply the packing rule times the four rolls (which included his imaginary roll) and then subtract one whole roll plus the four individual four pieces to get the same answer. Jordan and Renny warranted that Chuck's solution is correct; however, they rebutted his solving method, stating that it was unnecessary. The idea of an imaginary roll being used was being negotiated by the students at this point.

Later in the instructional sequence, Brian used Chuck's idea of using an imaginary roll. On Day 9, the class was presented with the problem "How could the Simpsons package 59 cherry candies if the packing rule is c = 10? Draw at least two different ways.” The students had presented multiple ways they had solved and discussed as a whole class if the solutions were correct or not. Then Brian presented an idea.

Brian:	… I actually did seven rolls.
Professor:	Nice. Seven rolls? Oh, I remember yours. Yours was neat. Alright, everybody okay with this picture? (*pointing to what another student had presented*).
Students:	Yeah
Professor:	Alright, I’m going to leave this (Nathan’s interpretation) here and I’m going to put yours (Brian’s interpretation), you had 7 rolls. I’m going to make these last two rolls. Describe what you did Brian.
Brian:	Okay, so I had 7 rolls. And what you do is, I took 10 away from one roll, which pretty much just gives you zero but that’s one way to do it. So you have 7 rolls, take away 10, and then from the next one (roll) you take away 1 and then you get 59. Because 10 times 7 gives you 70 and 70 minus 10 equals 60 , 60 minus 1 equals 59. And that’s equivalent to 59 pieces. (*Professor draws Fig. 29 on the board*)

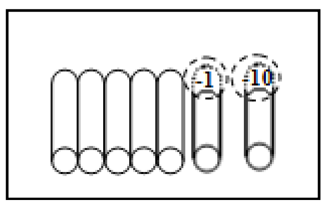


Figure 29: Brian’s interpretation with imaginary roll

Brian made the claim that to show the total of fifty-nine pieces, a seventh “imaginary roll” could be added to the visual and the expression. For his data he explained that procedural steps, getting the total of fifty-nine. As the students continued to make sense of Brian’s visual, they agreed that it was correct but that it was making it harder than necessary. Then Luther added another way to look at Brian’s idea.

Luther:	I was going to say, since he had seven rolls, I thought he would take one (piece) away from each roll or something.
Professor:	Oh, that’s a different picture, isn’t it?

Luther:	On like three of them, you would take away 2 of them or four…
Professor:	That would be really cool wouldn't it? There's all different kinds of pictures you could for this isn't there? Did you hear what Luther said? Have seven rolls and take one (piece) out of each roll and maybe have to adjust some things. But you could have a bunch of rolls with one missing pieces as long as you put other pieces in there to compensate for it. So I think what they are saying to you is…
Brian:	It's making it too hard, but I just…
Professor:	I don't. I kind of like it, if it makes sense. Now what would this one look like in the store? In the Big Lots?
Brian:	It would look like the wrapper that someone ate out of it and put it back and then go buy it and you're like what?

The professor brought Brian's attention to what the rolls from his interpretation look like in the store; tying his claim and data together they warrant this idea. Luther's statement that multiple rolls could each be missing pieces qualifies this idea as well, as it indicated the strength of the warrant and data. At this point, the concept that *when solving for the total amount of candy, an "imaginary roll" can be added in the expression* was taken-as-shared. This taken-as-shared idea is significant to this to instructional sequence because it allowed students to see that they could use complex thinking, including adding more rolls that were not there, to help them find the packing rule. These complex thoughts and ideas may not work for all students, but it allowed students that had diverse conceptual thinking to do so.

Division (pieces/rolls) can be used to find the packing rule.

The students intuitively used the operation of division from the beginning of the instructional sequence when they were presented multiple rolls on the balances. The students knew procedurally that they needed to divide the individual pieces by the number of rolls, but it was unclear if they had conceptual understanding of what they were doing or why. On Day 5, the idea that division could be used to find the packing rule was brought about.

Teacher: … We're going to go down to the bottom here and we're going to look at this balance right here (*points to the balance with 33 pieces on the left side and 3 rolls with w pieces in each on the right side*). We are just going to focus on this balance right here for a second. So, tell me how you interpreted that Savannah?
Savannah: Okay, so there are 33 pieces, so I knew to do 33 divided by 3 because there are 3 rolls on this (right) side, and I got 11. So *w* equals 11.
London: I agree.
Teacher: (*speaking to the class*) You agree? (*Students shake their head to agree*).

While Savannah explained how see achieved her claim of $w = 11$, she provided data stating that since there are 3 rolls she has to divide the 33 individual pieces by the 3. Even though this concept was not questioned and the class agreed with her data, being the initial introduction it is interpreted as being negotiated.

As the discussion continued onto solving the next similar balance problem presented, the same concept arose again. Prior to this dialogue, Hannah made the claim that $x = 9$. For her data, she explained that just by looking at the numbers, 5 and 45, she knew to use multiplication. As Calvin attempted to help another student's misunderstanding, the teacher asked him what he did with the $5x$ (See Fig.27).

Teacher: … What did you do with the $5x$?
Calvin: Oh, the $5x$, I divided 45 by 5 for that one.
Teacher: Why did you do that?
Calvin: Because if there is 45 pieces and there's 5 rolls and you just do 45 divided by 5 it would, it would tell you your answer.

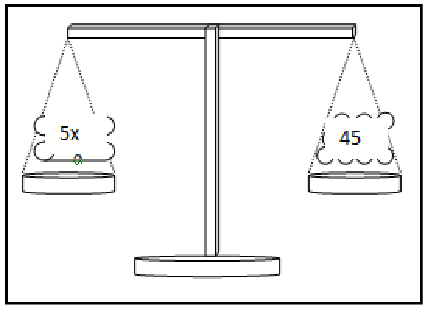


Figure 27: Brainy Brian's Balance, #5

When Calvin was asked what he did with the 5*x*, he explained that he simply divided 45 by 5. This statement was interpreted as additional data that supported Hannah's original claim. But as Calvin continued to explain why he did the division, using the numbers of individual pieces and rolls of candy, he tied his data back to the original claim which is interpreted as a warrant. Since the concept that *division (pieces/rolls) could be used to find the packing rule* had shifted from a claim to warrant, it is understood to be taken-as-shared.

Conclusion

This chapter has presented the taken-as-shared ideas that were established throughout the instructional sequence for *The Candy Shop*, which are summarized in both of Table 1 and Table 2 within the chapter.

Practices were chosen to illustrate how the social and psychological environments impacted students' learning of algebraic concepts. Data analyses lead to the discovery of how the individual participants and the classroom community contributed to these practices. The individual participants contributed to the class by 1) establishing new concepts and 2) providing additional evidence, both in the form of data, warrants, backings, or challenges. The classroom community influenced the individuals' mathematical knowledge by 1) identifying misunderstandings of concepts presented and 2) modifying their thinking to recognize the concepts correctly.

The classroom practices that emerged from *The Candy Shop* instructional sequence are associated with the modifications of the CHLT that took place over the duration of the sixteen class lessons. These modifications can be used for implications of future research to better the education of algebraic ideas to students.

CHAPTER SIX: CONCLUSION

This study was a sixteen session long classroom teaching experiment that focused on the ways in which a Realistic Mathematics Education (RME) instructional sequence could support students' conceptualizations of algebraic expressions and equations. The results from this study focused on *The Candy Shop*, the RME instructional sequence used in which students engaged in activities to build knowledge of these algebraic concepts. This chapter features a synopsis of the results as well as implications for further research in this content area.

Prior research of students' common mistakes and processes of learning algebraic concepts were used to create the initial hypothetical learning trajectory (HLT). This HLT included ideas from the original instructional sequence by Diana Underwood Gregg and Erna Yackel (2002). The sequence was designed so that students could work through realistic problems in order to build understanding of algebraic concepts, such as equations, expressions and variables.

The analyses from chapters four and five revealed the different approaches taken by the two instructors to establish the social and sociomathematical norms (Cobb et al., 2000); as well as classroom mathematical practices were taken-as-shared by the students in the classroom throughout the duration of the instructional sequence. The three-phrase approach of Rasmussen and Stephan (2008) was used to guide the documentation of collective activity to identify the taken-as-shared ideas. The Toulmin (1969) model was used to identify data, warrants, and backings provided by individuals' contributions in whole class discussions.

As the students engaged in the instructional activities they worked independently and in small groups, as well as participate in whole group discussions. Both instructors (i.e., the professor and me as the teacher/researcher) participated in implementing the instructional sequence. The professor brought 20 years of experience teaching, including working with inquiry and social and sociomathematical norms; I had just begun to integrate an inquiry approach into my teaching practice. To implement the instructional sequence, both instructors facilitated the activities within the sequence and whole class discussions for the opportunity for all students to have input in the students' thinking that was being shared.

During the whole class discussions, there were social and sociomathematical norms that became established, as well as noticeable differences in the instructors' approaches that became apparent. The desired norms, and the different actions used by the professor and by me to establish the norms, are presented in the table below.

Table 6: Social & sociomathematical norms with instructor techniques

Social & Sociomathematical Norms	Teacher Technique	Professor Technique
Social Norm: Students were expected to explain and justify their thinking	Provided explanation for the students; Directed students to specific answers; did not elaborate for whole class comprehension.	Asked students to explain and then restated it
Social Norm: Students were expected to interpret others' methods	Gave interpretations for the students	Asked specific students (who did not volunteer) to restate students' words or actions; restated students' questioning to generate student interpretations; held students accountable for interpretations
Social Norm: Students were expected to use questioning when they did not understand	Asked questions of the students	Asked students to ask questions about others' methods and actions; coached, guided and acknowledged student questioning
Sociomathematical Norm: The criterion for what counts as an acceptable mathematical explanation was the explanation had to be focused on both the calculations as well as what the calculations stood for.	Prompted students to "explain;" prompted students to agree or disagree; directed students to targeted explanations about the calculations through specific questioning which was not acceptable	Facilitated student-led discussions; restated solutions prompting students interpretations; promoted student participation; used social norms to encourage math reasoning; redirected student explanations to focus on problem, not just calculation
Sociomathematical Norm: The criterion for what counts as a different mathematical explanation was that the explanation could present alternative methods and manipulations, but MUST include all quantities of the candy.	Prompted students to use alternative ways to solve; reflected on student solutions; focused on presentation of different solutions or solving methods	Facilitated student collaboration to explore how or why a solution was different; prompted agree/disagree and different solutions; initiated conversations
Sociomathematical Norm: The criterion for what counts as efficient mathematical explanations was that explanations can include different methods but with the same answer, classifying as different ways of thinking.	Accepted student rebuttals without elaboration	Guided students through elaboration to rethink rebuttal of complicated ideas

As Table 6 illustrates, I played an authoritative role with the students as we negotiated the social norms by providing many of the responses and reactions I wanted the students to have. However, the intention of an inquiry perspective is that the students are expected to actively contribute to the classroom discourse. By watching the professor's strategies that incorporated more student autonomy and discourse, I was able to shift my role to less of an authority and more of a facilitator that guided students' contributions and held students accountable for participation.

In the analysis process there were multiple classroom mathematical practices that were established in the whole class discussions as well. As the argumentation log was analyzed, there were taken-as-shared ideas identified. These taken-as-shared ideas were put together and identified through themes that became the classroom mathematical practices. The chart below summarizes the classroom practices and their taken-as-shared ideas.

Table 7: Classroom mathematical practices & taken-as-shared ideas

Classroom Mathematical Practices	Taken-as-Shared Ideas
Practice 1: Evaluating pictures of candies given a different value for x (packing rule).	• With given packing rules, a total number of candies can be found by multiplication (number of rolls times the packing rule) and addition of individual pieces • Repeated addition can be used as an alternative of multiplication • Multiple expressions and illustrations can be used to show the value of x. • Subtraction of individual pieces from rolls could be explained as "stolen" pieces
Practice 2: Interpreting the meaning of variables	• Variables do not always represent fixed values • The variable in *The Candy Shop*

Table 7 (continued)

	• represents the packing rule (how many individual pieces are in each roll) • Only like terms (i.e. 4x and 2x) can be combined together mathematically.
Practice 3: Interpreting Algebraic Expressions	• A number beside a variable (i.e., 6*x*) represents multiplication • Expressions represent the amount of candy rolls and pieces in *The Candy Shop.*
Practice 4: Inventing Methods for Finding the Value of x (Packing Rule)	• Division (pieces/rolls) can be used to find the packing rule • Numbers are “canceled out” by applying the inverse operation to isolate rolls and pieces to opposite sides of the balance (= sign) • When solving for x, an “imaginary roll” can be added in the expression • A “collection” of candy should be the sum of both the rolls and pieces.

Some of the taken-as-shared ideas were established before the study, as the rest were negotiated and/or established during the study. Rasmussen and Stephan (2008) gave criteria for ideas to be taken-as-shared. To be taken-as-shared, the ideas had to shift in function or they were not questioned. Therefore the taken-as-shared ideas that are presented on the table above fell into at least one of these criteria.

Instructional Sequence

A classroom hypothetical learning trajectory (CHLT) was designed so that students could work through activities with realistic problems in order to discover and build understanding of algebraic concepts, such as equations, expressions, and variables. The following discussion shows the revisions that were made to the instructional sequence during the study and implications for the future uses of this sequence.

For the original CHLT there were four segments planned out. The first segment included activities that guided students to determine how many candies were packaged in collections, using their arithmetical skills to build understanding of algebraic concepts by manipulating the visual representations. The activities were designed for students to explore the concepts of finding the number of pieces of candy, writing arithmetic expressions, discovering equivalence of varying expressions (i.e. 3 x 13 – 4 vs. 2 x 13 + 9) and quantities, and packing loose pieces inside a roll that was missing pieces to make calculations easier. The second segment guided students to describing the candy quantities in terms of unknowns packing rules, which would move into learning and using variables. The activities for this segment were designed for students to explore generalizing numbers into unknowns, defining *x* (variables), writing expressions, and determining equivalent expressions. The third segment continued the use of variables to guide the students to create algebraic expressions from a picture or a picture from an algebraic expression. The activities for this segment were designed to have students simplifying expressions and interpreting a negative sign when there were no rolls in front of it (i.e., 10x – 4). The fourth segment guided students to discover how many candies were there when various amounts of specials were sold to customers. The activities of this segment included exploring the distributive property.

Key concepts were purposefully introduced and arranged throughout the instructional sequence to aide students in their understanding of the overall algebraic concepts. For example, as the idea of The Simpsons ™ taking over the candy shop was purposefully presented to the students at the beginning to introduce the students to the visual representations of the candy pieces and rolls. These visual representations, later in the sequence, were to turn into variable representations. Therefore the students could visualize what the variable was representing and build deeper understanding when defining a variable. However, as the instructional sequence progressed some of the initial CHLT concepts, along with the instructional sequence activities (See Appendix D) were revised. Some activities were deleted or changed, while new ones were created and/or added. For example, there was not enough time to do the activities for simplifying so the team had to delete these activities

Revisions

The research team met frequently during the duration of this study to discuss where the students were in learning the concepts that were being presented, which activities were or were not guiding students to understand, and to guide the instructor and direction of instruction. The students moved through the Segment One pretty smoothly without any necessary change needed. The students displayed their comprehension of determining how many candies were packaged in a collection. As indicated in the Classroom Mathematical Practice (CMP) analysis, it had became taken-as-shared that with any given packing rule, a total number of candies can be found by multiplication or repeated addition, which is compatible with the intentions of the designers.

As the students worked through Segment Two activities the research team started making some revisions. On Day 5 of the instructional sequence the students were introduced to a strategy in the activities, “Brainy Brain’s Balances,” and the “Balance Bonanza” activities (See Appendix D). The original taken-as-shared interest of this segment was for the students to be able to describe different candy quantities in terms of unknown packing rules. However, the research team made a significant revision for this segment. They felt that the students could deal more meaningfully with variables if the unknown quantity could be found, as opposed to starting with adding and subtracting expressions.

As the students participated in discussions they noted that the variable represented the packing rules and that they had to manipulate the candies on the balances to figure out the unknown packing rule. The students had a three day holiday so the team decided that the students needed to start back with concept before moving on to the next. This concept was imperative for the students to build onto, especially when they were trying to establish social and sociomathematical norms for the class. Therefore the research team created the “Balance Bonanza Bonus,” (See Appendix E) which continued the focus on finding the packing rules on new balances, whether it was through using the given numbers and variables or creating a drawing to work through. The students worked independently to come up with solutions. They then shared their solutions and solving methods within small and whole class group discussions, as well as negotiated norms. As the students worked through this activity the research team noted that the students negotiated ways to interpret the variable as the packing rule and reinvention of ways to solve for the variable, which later were identified as classroom practices that emerged from this study.

At this point the students knew of how to figure out the packing rule using balances that presented pictures of only candy rolls on one side and only candy pieces on the opposite side of the balance, and balances that presented only numbers and variables without pictures. Therefore, the instruction continued with the next activity from the original CHLT, “Brainy Brian’s Balances Again.” This activity flipped the students’ thinking back to solving for the packing rules using balances with pictures; however, the idea of candy rolls and pieces being together on one side with one or the other on the opposite side was introduced. This activity continued the negotiation of interpreting the meaning of variables as the students discussed the connection between the different representations they were presented through the activities.

Segment Three is where most of the instructional sequence revisions took place. The activities in this segment delve deeper into algebraic expressions. The segment started off with the students continuing to work with Brainy Brian's strategy through the next original activity, "Rambunctious Rolls," to find packing rule by creating equations from given pictures. As the students participated in discussion the instructors noted that a majority of the students seemed to get the correct answers, yet they questioned the students' understanding of their solution methods and establishment of the norms as explanations were limited. This activity also offered simplistic problems, with only one side of the balance having both candy pieces and candy rolls. Therefore, the team decided to create the "Rambunctious Roll Review" activity for the following day to give the students more time to negotiate the norms through discussion of how they were finding the packing rule. This activity presented students with 5 problems; one was similar to the simplistic form they did on the previous activity, three problems challenged them with having both candy pieces and candy rolls on both sides of the balance, and then the last problem was a written explanation of how they solved one of the challenging problems. This activity gave students the opportunity to reflect on their solving methods (i.e. creating an expression, drawing a picture of the candy rolls and pieces) and recognize if they independently understood how to determine what the variables and numbers were representing. Through the discussion of small and whole class groups, the students were able to compare and contrast with their peers to get deeper understanding of the concepts as well as negotiation of the norms.

As the instructional sequence approached Day 8, the research met and discussed concerns about my teaching techniques and the students' misunderstandings that were noted during instruction. The team decided that the whole class may not have a comprehensive understanding on the prior concepts presented due to the specific questioning that I was using to lead them to explicit answers. For example, as one student was solving a problem, she had $5x = 20$ and she said the answer was 4. This was the correct answer; however, I wanted her to explain how she got the x by itself. So I asked her how she got $x = 4$ and she explained that 20 divided by 5 is 4, which again was correct. I continued to push her to explain until she explained that she divided $5x$ by 5, because both sides you do the same thing. This explanation did not fit this particular students' problem solving methods. Therefore the team felt that it was necessary to create a new activity that would serve as a checking point of where students were on their understanding, as well to renegotiate some norms. The activity, "Packing Problems," was created and offered students four activities to revisit their understanding of the concepts that had been discussed up to this point. The students had learned how to find the total number of candies in a collection with a given packing rule and a drawing or expression, as well as create a drawing or expression given the amount of candy and the packing rule. The students were given a chance to share and explain their thinking to these activities within small and whole group discussions. The discussions had also changed to students asking questions and explaining more so that the students could interpret each others' methods.

The research team also questioned if the students fully comprehended that the same variable could represent the same unknown quantity in one problem but stand for a different unknown quantity in another problem situation. Therefore, the activity "Positively Preposterous Peach, Pear and Pork Chops" was created. As the students worked on finding the packing rules for the balances of three different flavors, the students discussed how all three flavors used the same variable *p*; however, the variable *p* had three different values in this one activity. The discussion in this activity allowed students to negotiate how a variable, even though it is the same letter, could be interpreted differently in different expressions. This page was not in the original instructional sequence; however, it was added because of a discussion that students had earlier. One student had solved a problem with the answer $c = 5$, and as the students moved on to solve and discussion the next answer $x = 10$, this student said $2c = x$. The discussion led to a debate of if this was true or not. In this case, the two could be used, if the same problems were presented. However the variables *c* and *x* do not always have the same value. Therefore, the research team thought that the P activity would allow the students to see this concept. All three problems were on the same page, and all used the variable *p*, but the value of *p* was not the same.

The students continued with Segment Three to the next activity called "Missing Pieces Miasma" which introduced the idea of candy rolls on the balances missing individual pieces. As the students participated in this activity some students worked effortlessly through the problems, while other did not. During this point of the study, there was also an unplanned break in the implementation of the instructional sequence due to school activities and schedules. Therefore, the research team decided to create the activity “Missing Pieces Murk,” which would extend the concept of missing pieces through additional balances presenting candy rolls that were missing pieces of candy. The team felt that both of the activities were imperative to students’ comprehension of solving algebraic expressions, because they presented subtraction as part of the problem. The students were building knowledge of how to represent a picture of a roll of candy that had a negative number on it, as subtraction in the expression. This activity also helped students find the packing rules and create expressions for the next activities, “Candy Scales I” and “Candy Scales II.” These two activities presented students with balances that only had numbers and variables on them; there were no more pictures given. The problems presented within these two activities were designed for students to create expressions that had adding and subtracting operations in them. The expressions that they created would help them find the packing rule, or how many candies were in each roll.

The “Candy Scales II” activity led the instruction into Segment Four of the CHLT. However, as the students worked on these two Candy Scales activities the research team observed inconsistencies within the classroom when utilizing prior knowledge of skills and norms from earlier activities in the sequence. For that reason, all of the concepts that the students had learned from the beginning of the sequence until this point were mixed up into one activity named “Quiz,” for the students to work through. The research team felt it was necessary to assess the students’ understanding to detect any possible patterns of gaps in the concepts, and it would help guide the direction of further instruction of the sequence.

From the results of the quiz, the research team noted that most of the students were more successful solving balances that presented more simplistic problems, such as only one side having a combination of the rolls and pieces while the other side had only one or the other. Conversely, the students struggled when navigating through more challenging problems (i.e. a balance that had both candy pieces and rolls on both sides). When the research team recognized this concern, they decided to delete the next two activities, “Candy Scales III” and “Candy Scales IV” from the original instructional sequence as both of these activities focused on simpler expressions. The original “Candy Scales III” and “Candy Scales IV” activities target both addition and subtraction problems, but most of the problems had only pieces on one side of the balance. So the team created a new “Candy Scales III” activity, which meshed up all different types of expressions, simple and complex. This allowed students the opportunity to delve into their peers’ and their own solutions with more the more challenging problems. The research team went on to rename the original “Candy Scales V” activity as “Candy Scales IV” to help sequential order in the instructional sequence.

The final revision was the use of another page as an assessing tool. The instructor decided to use the “Candy Worker Test” to assess the knowledge that the students learned throughout the duration of the study to solve chosen expressions.

There were many revisions made to the original CHLT. The research team revised instructional activities, concepts, and organization of each throughout the study to help the students’ conceptual navigation through the instructional sequence be successful. Below, Table 8 shows the revised CHLT that was utilized in this study. The individual pages for the activities can be found in Appendix E.

Table 8: Revised CHLT

Segment	Tools	Imagery	TAS Interest	Possible Discourse	Pages
One	Candy Drawings Smarties© Candy	(un)packing and packing rolls of varying amounts of candy	Determining how many candies are packaged in a collection	• How to find the number of pieces • Writing arithmetic expressions and/or equations • Equivalence of varying expressions (3x13-4 VS. 2x13+9) and quantities • Packing loose pieces inside a roll that's missing pieces to make calculations easier	1-3
Two	Candy Drawings Smarties© candies Brainy's Balances	Unknown pieces of candy in a roll; balances with candy pieces on one side and rolls on the opposite	Describing candy quantities in terms of unknown packing rules	• Does *x* describe any unknown amount or an unknown amount of pieces per roll • Writing equations • Solving equations for unknown	4-9
Three	Candy Drawings Brainy's Balances with and without pictures	*X* is a quantity that can be organized in different ways; symbols that represent quantities of candies; negative number on roll	Determining the algebraic expression and/or equation from a picture and vice versa (Missing Pieces)	• Writing equations • How do you find the packing rule • How do you interpret a negative sign number on a candy roll • Does a variable always represent the same object and value • Distributive property	10-15
Four	Candy Drawings Balances without pictures	Expressions on each side of the balance; mixture of type of problems (simple or complex)	Determining the value of the packing rule from various balances	• What if we have both candy rolls and pieces on both sides of the balance • Compare/Contrast solutions with peers	16-21

As mentioned previously the fourth segment of the original CHLT, which focused on the concept of simplifying expressions with the use of the distributive property was deleted. The revised CHLT's fourth segment focused on a mixing up simple (i.e., $ax + b = c$) and complex (i.e., $ax + b = cx + d$ and/or $ax - b = cx + d$) problems, so that the students could delve deeper into finding the packing rule. The revisions that were made lead to the implications discussed in the following section.

Implications

The analysis of the classroom mathematical practices indicated that some topics needed more time and that there was a need for solving problems that indicated multiple concepts. These revisions are thought to be unique for this group of students, as it may or may not be for other groups in which this instructional sequence may be used. Future research studies will need to address the need for students grasping the concepts before moving on to the next activity; therefore the periodic assessments are thought to be necessary additions for direction of instruction.

Instructor Implementation

As mentioned in Chapter 4, while analyzing the data there were many differences noted between the instructors' instructional practices. These differences were also analyzed to see how the professor and I, as the teacher, approached initiating and guiding the construction of the social and sociomathematical norms in the classroom. Overall, the analysis showed that the professor's experiences allowed her to approach the norms as facilitator of student collaboration in small and whole group interactions, while I (at first) took a more authoritative role in the negotiation process. The following sections will discuss the revisions that took place with the instruction during the study and the implications for instruction in future studies.

Revisions

Initially I provided the students with explanations, justifications, interpretations, and funneled them to the ways of solving that I valued. I never expected elaboration of any answers or solutions given. On Day 8, while the students worked on an activity that I had given to them, the professor pulled me to the side to hold an impromptu meeting. The professor and I discussed asking specific questions to the students to guide them to a specific answer. The professor explained that this action limited students solving the problems the same way I had, which she did not believe they would be able to do on their own at this point. It was clear that my instructional practices needed to be revised.

The professor instructed some of the lessons to model how I could interact in ways that would open up the classroom discourse. As the professor took turns in instructing the lessons, I was able to note new techniques for me to use. Throughout the rest of the instructional sequence I was able to make the transitions noted in the chart below.

Table 9: Evolution of teacher's techniques

Initial Technique(s)	New Technique(s) Tried
Provided explanations for the students; Directed students to specific answers; did not elaborate for whole class comprehension.	Asked students to explain
Gave interpretations for the students	Asked specific students to restate students' words or actions; restated students' questioning
Asked questions of the students	Asked students to ask questions about others' methods and actions
Prompted students to "explain;" prompted	Facilitated student-led discussions;

Table 9 (continued)

students to agree or disagree; directed students to targeted explanations about the calculations through specific questioning which was not acceptable	restated solutions
Prompted students to use alternative ways to solve; reflected on student solutions; focused on presentation of different solutions or solving methods	Facilitated student collaboration; prompted agree/disagree and different solutions; initiated conversations
Accepted student rebuttals without elaboration	Asked students to rethink rebuttal of complicated ideas

The new techniques that I learned from the professor and tried using as the instructional sequence continued allowed me to see the value of experience. I was able to give students the opportunity to take responsibility for their learning, instead of giving them the lesson and the methods of solving. I was able to give the students opportunities to build understanding by collaborating with their peers. These opportunities are what make this type of experiment one of inquiry and allow the students to establish the norms needed.

Implications

As it was noted that the techniques of the professor and myself are diverse, the rationale for this difference was experience. Therefore, for further research purposes there should be someone with experience in this type of inquiry lessons as the teacher/researcher, as well as someone who understands how to negotiate social and sociomathematical norms in ways that preserve students' autonomy. Since I had no experience with negotiating norms that support student autonomy, I had a different understanding of my role. As I worked through the analysis, with the professor's guidance, I was able to understand the impact of our different techniques on the implementation of the sequence.

Conclusion

The algebraic concepts that were presented in this instructional sequence are typically addressed in 7th and 8th grade algebra courses, and some 6th grade. This study had 5th grade participants that evolved in their understanding of this higher grade level concepts. Imagine the possibilities for students' conceptual growth if there were more opportunities, such as this instructional sequence, in elementary mathematics classes. This finding supports Cai and Knuth's (2011) idea of algebraization in elementary grades, instead of deferring these concepts until the students reach algebra class.

The findings from this study add to the research of upper-elementary students' development of algebraic expressions and variables through a classroom teaching experiment using an instructional sequence designed under the principles of RME. The RME instructional sequence supported the students' learning of some of these algebraic concepts. Throughout the duration of this study the class was able to develop ideas such as defining variables, adding numbers with variables, balancing expressions by adding and subtracting, representing candy with numbers (rolls and pieces), representing packing rules with variables, finding the value of a packing rule (variable) with missing pieces, creating visuals that represent algebraic expressions and/or equations, and creating algebraic expressions and equations.

The RME sequence added contextual meaning of the algebraic concepts for the students, through the fictitious scenario and situations that were presented in the activities. The instructional sequence's activities directed the students to explore and build conceptual understanding of expressions, equations, and variables. For example, the students were able to build their understanding finding the value of the unknown with the use of the fictitious character, Brainy Brian's balance strategy. The students were able to conceptualize the "arithmetical" equations (Fillroy & Rojano, 1989), such as Ax + B = C simply by using their prior knowledge of arithmetic skills. However, when the students got to more complex, or "non-arithmetical," (Fillroy & Rojano, 1989) equations, such as Ax + B = Cx + D, the students were out of their comfort zone. However, the balances and visual representations of the candy pieces and rolls served as concrete models (Fillroy & Rojano, 1989) that helped them work through the complex problems as they were able to visualize what they needed to do. As the students worked through the balance problems, they were able to establish what a variable was and how the variables could be utilized as literal and non-literal symbols (Blanton, 2008).

There are many different avenues this instructional sequence could take, especially considering the ways in which it is implemented. However, this study confirms that *The Candy Shop* instructional sequence is a good start for getting students on track for conceptualizing algebraic concepts.

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APPENDIX A: IRB APPROVAL

UNC CHARLOTTE

Research and Economic Development
Office of Research Compliance
9201 University City Blvd, Charlotte, NC 28223-0001
t/ 704.687.1876 f/ 704.687.0980 http://research.uncc.edu/compliance-ethics

Institutional Review Board (IRB) for Research with Human Subjects
Approval of Exemption

Protocol #	**15-01-25**			
Title:	**Supporting Students' Conceptions of Algebraic Equations and Expressions using Realistic Mathematics Education Design Theory**			
Date:	**2/3/2015**			
Responsible Faculty	**Dr.**	**Michelle**	**Stephan**	**Middle, Secondary, K12 Educ**
Investigator	**Ms.**	**Melissa**	**Miller**	**Middle, Secondary, K12 Educ**
Co-investigator	**Dr.**	**Victor**	**Cifarelli**	**Mathematics & Statistics**
Co-investigator	**Dr.**	**Drew**	**Polly**	**Reading & Elem Educ**
Co-investigator	**Dr.**	**Diana**	**Underwood-Gregg**	**Purdue University Calumet**

The Institutional Review Board (IRB) certifies that the protocol listed above is exempt under category 1 (45 CFR 46.101).

Research conducted in established or commonly accepted educational settings, involving normal education practices, such as:

a) research on regular and special education instructional strategies, or

b) research on the effectiveness of or the comparison among instruction techniques, curricula, or classroom management methods.

This approval will expire one year from the date of this letter. In order to continue conducting research under this protocol after one year, the "Annual Protocol Renewal Form" must be submitted to the IRB. Please note that it is the investigator's responsibility to promptly inform the committee of any changes in the proposed research, as well as any unanticipated problems that may arise involving risks to subjects. Amendment and Event Reporting forms are available on our web site: http://research.uncc.edu/compliance-ethics/human-subjects/amending-your-protocol or http://research.uncc.edu/compliance-ethics/human-subjects/reporting-adverse-events

M. [signature] 2/9/15

Dr. M. Lyn Exum, IRB Chair Date

The UNIVERSITY *of* NORTH CAROLINA *at* CHARLOTTE
An Equal Opportunity/Affirmative Action Employer

APPENDIX B: PARENT CONSENT FORM

UNC CHARLOTTE
The University of North Carolina at Charlotte
9201 University City Boulevard
Charlotte, NC 28223-0001

Middle, Secondary, K-12 Education
Phone: 704-687-8875

Teachers' Parental Consent for:
Dissertation Research: Supporting Students' Conceptions of Algebraic Equations and Expressions Using Realistic Mathematics Education Design Theory

Project Title and Purpose:
Your child is invited to participate in a research study entitled "Supporting Students' Conceptions of Algebraic Equations and Expressions Using Realistic Mathematics Education Design Theory." The purpose of this study is to observe if and how the implementation of *The Candy Shop* sequence, a Realistic Mathematics Education (RME) design theory model supports students' understanding of algebraic equations and expressions.

Investigator
This study is being conducted by Melissa K. Miller as part of the requirements for my Ph.D. in Curriculum and Instruction from the University of North Carolina at Charlotte. The responsible faculty member is Dr. Michelle Stephan.

Description of Participation:
Students will be asked to participate in *The Candy Shop* instructional sequence, designed by Dr. Underwood-Gregg and Dr. Yackel prior to this study. This instructional sequence allows students to engage through a realistic scenario of a familiar fictional family, The Simpsons, and their mission to run their own Candy Store. Students will participate as one of the Simpsons' candy shop workers, helping the family solve problems using prior knowledge of arithmetical operations to advance into understanding variables and algebraic equations and expressions. In addition, five students will be selected to complete one-on-one interviews given by Dr. Michelle Stephan from UNC Charlotte. These interviews will be no longer than 30-40 minutes of the student's time and will allow them to express their individual knowledge and understanding of the mathematical concepts that are being addressed in this study. Your child's class will be the only class to participate in this study. Dr. Michelle Stephan, the chair of my dissertation committee and a mathematics professor at UNC-Charlotte will be participating in instruction approximately once a week. Data from the interviews and class sessions will be collected through video, audio, teacher field notes, and student work. All of the data collected and reported from this study will be reported with fictional names to keep your child's work anonymous and confidential. The particular steps to ensure this confidentiality include students drawing a fictional name to put on all work as well as all work samples, videos, audio tapings, and interview scripts collected by the research team. Also, all collected data will be maintained in a locked file cabinet which is only accessible by the research and immediate research staff. Data will be disposed after six years. All paper data will be shredded, and electronic data will be dismantled and, or rendered useless. Students that do not have consent will continue with regular mathematics instruction during this time.

Alternative to Research Study Participation:
Students that do not have parental consent will continue to work on 5th grade mathematics curriculum with another UA staff member. These students will be pulled into another classroom setting to prevent distraction of their learning and inclusion in the research study. The teacher-researcher will check in daily with these students to verify understanding and offer tutoring sessions as needed. Assignments completed by students without parental consent will NOT be used in the analysis.

Length of Participation:
Your child's participation in this study will begin sometime in the school year of 2014-15. The duration of this study will be a six week period. The instructional implementation will take your child's given mathematics class period for approximately 45 minutes. If you decide to grant consent for your child's participation, they will be one of approximately 20 participants in this study.

Risks and Benefits of Participation:
There are no known risks of participation for your child in this study.

The benefits of this study are the researcher will be able to see how effective RME is with supporting students' conceptualizations of mathematics and student engagement. The benefits of participation in this study include:

- Possible improvement of mathematical implementation
- Opportunity for supportive implementation
- Opportunity for students to be challenged through engaging instructional sequence

Volunteer Statement:
Your child is a volunteer. The decision for your child to participate in this study is completely up to you. If you decide for your child to be in the study, they may stop at any time. Your child will not be treated any differently if you decide for them not to participate, or if you stop their participation once you have started.

Confidentiality:
The data collected by the Investigator will not contain any identifying information or any link back to you or your participation in this study. The following steps will be taken to ensure this confidentiality:

- No participant will ever be mentioned by name in the reported results
- Participants can end their participation at any time
- Participants can choose not to respond to any question
- Participants can choose not to participate in interviews
- Only the principal investigator and research staff will have access to the raw data. All gathered data will be stored in a locked cabinet.

Fair Treatment and Respect:
UNC Charlotte wants to make sure that you are treated in a fair and respectful manner. Contact the University's Research Compliance Office (704-687-3309) if you have any questions about how you are treated as a study participant. If you have any questions about the project, please contact Melissa K. Miller (336-817-1399), or Dr. Michelle Stephan (704-687-8888).

Participant Consent:
I have read the information in this consent form. I have had the chance to ask questions about this study, and those questions have been answered to my satisfaction. I am the legal guardian of this child and I agree that they are allowed to participate in this research study. I understand that I will receive a copy of this form after it has been signed by me and the Principal Investigator.

______________________________ ____________
Name of Participant (PRINT) DATE

______________________________ ____________
Parent Signature DATE

______________________________ ____________
Investigator Signature DATE

APPENDIX C: MINOR ASSENT

Assent for Minors
(For subjects under the age of 18 unless emancipated*)

Note: Assent forms should be written in language appropriate to the subject.

My name is Mrs. Melissa Miller and I am a student at The University of North Carolina at Charlotte. I am doing a study to see how The Candy Shop unit can help students learn ideas of algebra equations, expressions and variables.

If you want to be in my study, I will ask you to participate in The Candy Shop lessons. Then you will participate in whole group and small group activities for six weeks. During the lessons you will learn ways to use what you have learned in 3rd and 4th grade to understand algebra ideas. Some of you will participate in interviews that will be done by my professor, Dr. Michelle Stephan before and after the lessons are completed in class. None of the activities that you are participating in will be counted against or for your overall mathematics grade.

You can ask questions at any time. You do not have to be in the study. If you start the study, you can stop any time you want and no one will be mad at you.

I hope that this new way of learning will help you and other students learn to understand algebra better, but I can't be sure it will. This study will not hurt you.

When I am done with the study I will write a report. I will not use your name in the report.

If you want to be in this study, please sign your name.

____________________________________ _____2/22/2015____
Signature of Participant Date

____________________________________ _____2/22/2015____
Signature of Investigator Date

Emancipated Minor (as defined by NC General Statute 7B-101.14) is a person who has not yet reached their 18th birthday and meets at least one of the following criteria: 1) has legally terminated custodial rights of his/her parents and has been declared 'emancipated' by a court; 2) is married, or 3) is serving in the armed forces of the United States.

APPENDIX D: REVISED CHLT

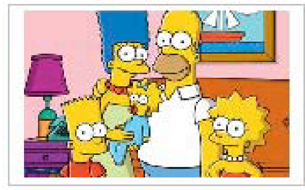
Segment	Tools	Imagery	TAS Interest	Possible Discourse	Pages
One	Candy Drawings Smarties© Candy	(un)packing and packing rolls of varying amounts of candy	Determining how many candies are packaged in a collection	• How to find the number of pieces • Writing arithmetic expressions and/or equations • Equivalence of varying expressions (3x13-4 VS. 2x13+9) and quantities • Packing loose pieces inside a roll that's missing pieces to make calculations easier	1-3
Two	Candy Drawings Smarties© candies Brainy's Balances	Unknown pieces of candy in a roll; balances with candy pieces on one side and rolls on the opposite	Describing candy quantities in terms of unknown packing rules	• Does *x* describe any unknown amount or an unknown amount of pieces per roll • Writing equations • Solving equations for unknown	4-9
Three	Candy Drawings Brainy's Balances with and without pictures	*X* is a quantity that can be organized in different ways; symbols that represent quantities of candies; negative number on roll	Determining the algebraic expression and/or equation from a picture and vice versa (Missing Pieces)	• Writing equations • How do you find the packing rule • How do you interpret a negative sign number on a candy roll • Does a variable always represent the same object and value • Distributive property	10-15
Four	Candy Drawings Balances without pictures	Expressions on each side of the balance; mixture of type of problems (simple or complex)	Determining the value of the packing rule from various balances	• What if we have both candy rolls and pieces on both sides of the balance • Compare/Contrast solutions with peers	16-21

APPENDIX E: REVISED INSTRUCTIONAL SEQUENCE ACTIVITIES

Implementation	Title	Page Number
Day 1	The Candy Shop Introduction	1
Days 1 & 2	Mrs. Simpson's Packing Rules	2
Day 3	Krazy Kustomer Chaos	3
Day 4	Mischievous Maggie and the Mystery Rolls	4
Day 4	Mystery Solved!	5
Days 5	Brainy Brian's Balances	6
Day 5	Balance Bonanza	7
Day 6	*Balance Bonanza Bonus	8
Day 6	Brainy Brian's Balances Again	9
Days 6 & 7	Rambunctious Rolls!!!	10
Day 8	*Rambunctious Roll Review	11
Day 9	*Packing Problems	12
Days 9 & 10	*Positively Preposterous Peach, Pear, and Pork Chops	13
Day 10	Missing Pieces Miasma	14
Days 10 & 11	*Missing Pieces Murk	15
Day 12	Candy Scales I	16
Day 13	Candy Scales II	17
Day 14	*Quiz	18
Day 14	*Candy Scales III	19
Day 15	Candy Scales IV	20
Day 16	Candy Worker Test	21

**Pages added or altered during the study*

The Candy Shop Introduction



Once upon a time there was an older couple named Homer and Marge Simpson. They had been married for 15 years when Marge finally said to Homer, "Homey, you HAVE to get a good paying job soon! How will we ever put Bart through college with the money you make at the nuclear power plant?!" The Simpsons always seemed to be broke, but one day Homer had a brilliant idea. Homer loved to eat JellyBellies, but he thought they were too expensive and didn't like their chewy consistency (they always get stuck in Bart's braces). He liked hard candies better, but he had not found any that could beat the flavor of JellyBellies. Together with their uncle scientist, Uncle Wiz, they developed a hard candy with almost as much flavor as a JellyBelly. They decided to make their candies disk-shaped (i.e., Lifesavers without the hole).

Uncle Wiz Retires!

As fate would have it, the Simpsons' elderly uncle retired and asked them to run his small candy and nut shop. Seeing this as a good sign, the Simpsons took over the business and soon started selling the Simpson's succulent candies in the candy shop. They named their store, ***The Simpson's Sweets Shop***.

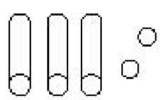
The candies were an instant hit! Knowing that people are accustomed to buying candies in rolls— like LifeSavers candies, they started to package their candies in rolls. They argued day and night about the number of candies that should be put in a roll.

Drama, Drama!!!

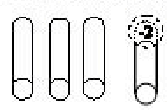
Well, the Simpsons could not come to any agreement about the number of pieces that they would put in a roll. Marge wanted to put 7 pieces in a roll of orange candy. Homer wanted to put 12 pieces in a roll of cherry candy. They did agree on one thing. If they put 10 orange candies in roll, ALL orange candy rolls would contain 10 pieces. However, they could put a different amount of candies in a Root Beer roll. But if they put 17 pieces in the Root Beer roll, ALL root beer rolls would have 17 pieces.

Representing Candy in the Candy Shop

Three rolls and 2 extra pieces of candy



Three rolls and a roll missing 2 pieces <u>or</u> Four rolls less 2 pieces

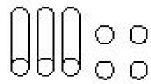
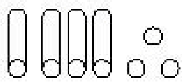
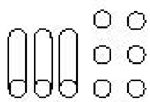
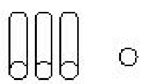
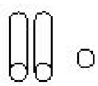
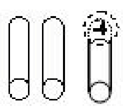
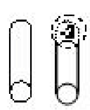
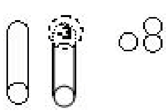


How many pieces are there altogether in each picture if...

Packing Rule Orange: 10 candies per roll
Packing Rule Rootbeer: 17 candies per roll
Packing Rule Grape: 53 candies per roll

Mrs. Simpson's Packing Rules

Mrs. Simpson had the following amount of candy in the candy shop. How many pieces does she have in all?

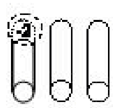
	Packing Rule	Packing Rule	Packing Rule
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.
	There are 5 pieces in each roll.	There are 13 pieces in each roll.	There are 79 pieces in each roll.

Krazy Kustomer Chaos

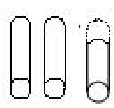
Krusty the Clown came into the store and wanted to buy 95 pieces of strawberry candy. The packing rule for strawberry candies is 5 pieces per roll. Krusty knew that Homer was not the best math student and thought he could trick him into giving him extra candy. Krusty asked Homer to give him 10 wrapped rolls and 50 loose pieces and he would pay for 95 pieces. Is this fair? Explain.

Follow up Questions:

1. What are 3 other ways to have 95 pieces of candy packaged, if the packing rule is 5 pieces per roll? Draw them below.
2. Draw 3 different ways to have 95 pieces of candy packaged if the packing rule is ***10*** pieces per roll.
3. Draw 3 different ways that the following candies can be packaged if the packing rule is 6 pieces of candy per roll.



4. Draw 3 different ways that the following candies can be packaged if there are 7 pieces of candy per roll.



Mischievous Maggie and the Mystery Rolls

The Simpson's Sweets Shop became one of the top candy stores in Springfield. However, the Simpsons' youngest daughter, Maggie, thought that she could make the Sweets Shop sell even more candy if she could come up with a clever promotion. She had a brilliant idea for a Mystery Candy Roll. The Sweets Shop would sell Mystery Rolls which contained a mystery. Each day the Sweets Shop packaged their candy, the packing rule would be a mystery.

Mystery Roll Contest

To make it interesting, Maggie decided that the packing rule for the Mystery Rolls would change each day and customers would not know how many candies were in the roll. The following advertisement appeared in the Sunday newspaper:

Sunday's Mystery Roll

X pieces inside

Each roll has exactly X pieces per roll!!!
The first person who can guess what X is
will receive the roll for free.

On other days, Maggie changed the packing rule.

Monday she changed the rule to Y pieces of candy per roll and customers had to figure out how many pieces of candy were in each roll.

On Tuesday, she said there were C candies per roll and so on.

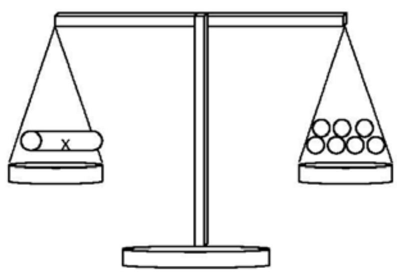
Mystery Solved!

Friday's Mystery Roll

X pieces per roll

Each roll has exactly X pieces per roll.
The first person who can guess what X is will receive the roll for free.

On Friday, Maggie placed the advertisement above in the window of the candy shop. Little did she know, her nephew, the 6th grade science brainiac, devised a way to figure out the Mystery packing rule ***without guessing***! His method uses a balance scale:



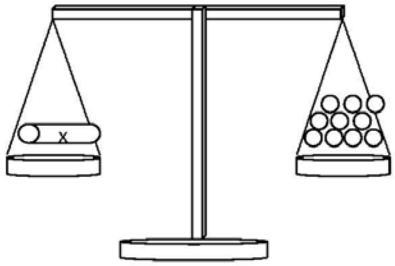
Brainy Brian's Balance Strategy

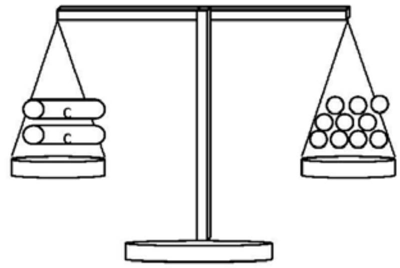
I put the Mystery Roll on one side of the balance. Then, I put single candies on the other side until both sides balanced. 1 mystery roll balanced with 7 pieces.

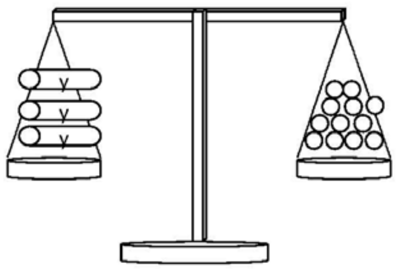
Explain how this can reveal the Mystery Roll's packing rule.

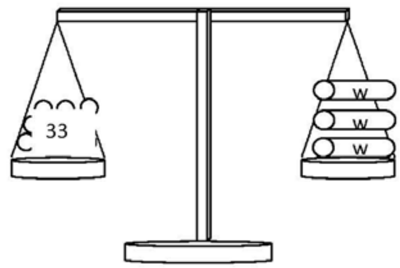
Brainy Brian's Balances

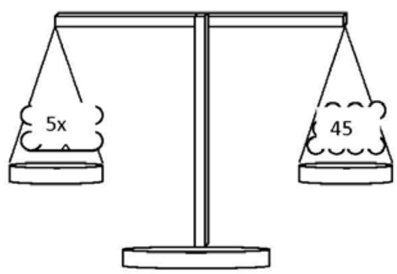
Use Brian's Balance Strategy to figure out the Mystery Roll Packing Rule for each of the following scales.





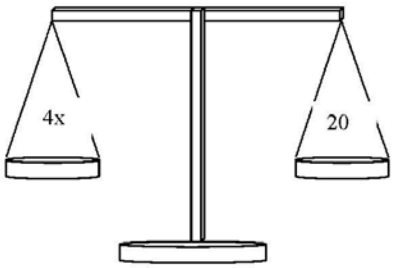


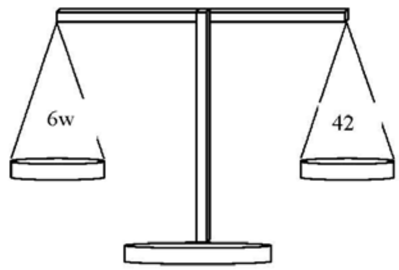


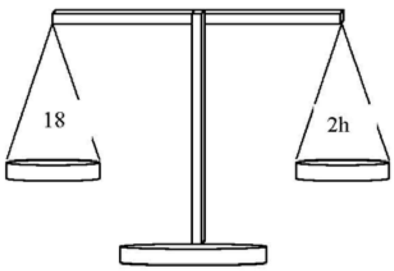


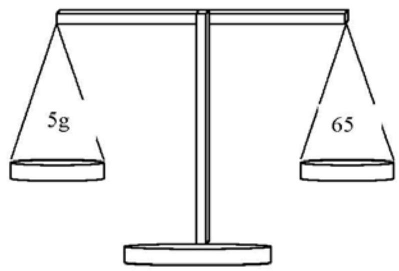
Balance Bonanza

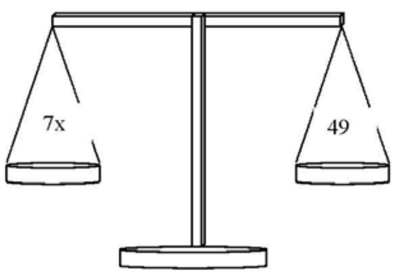
Solve for each unknown below (i.e., find the packing rule).

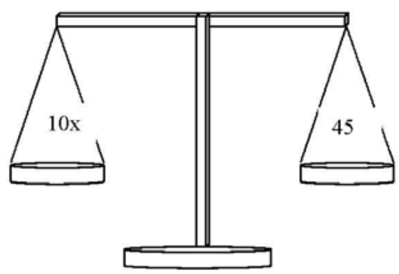








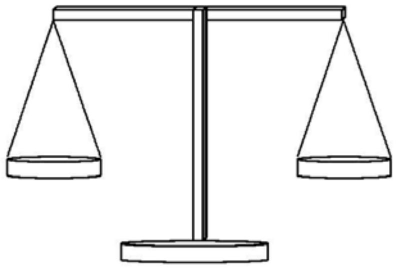




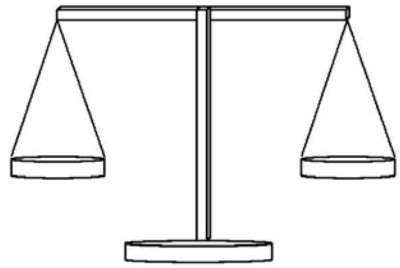
Balance Bonanza Bonus

Help Brainy Brian solve to five the packing rule for each of the following.
You may use the balance in the box if needed.

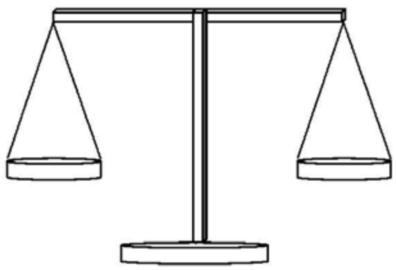
1. $6x = 36$



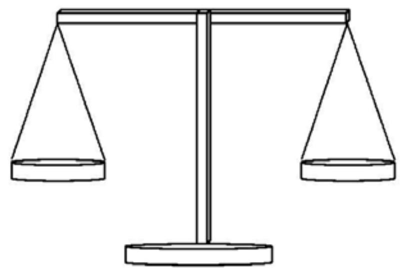
2. $10y = 90$



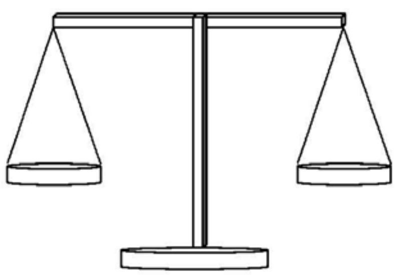
3. $15x = 60$



4. $7x = 49$



5. $8x = 72$



6. Use number 5, explain what the following represent:

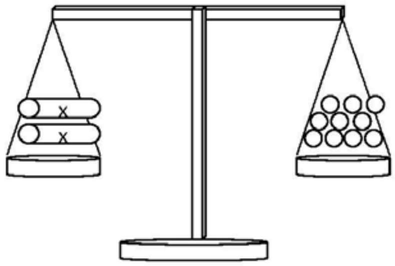
What does the 8 represent?

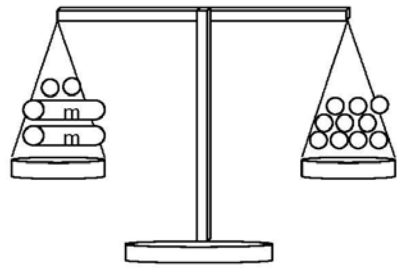
What does the x represent?

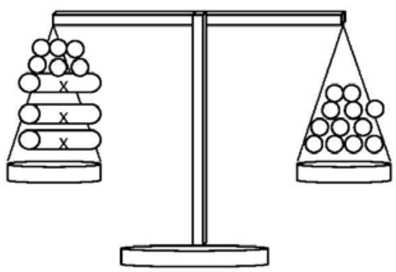
What does the 72 represent?

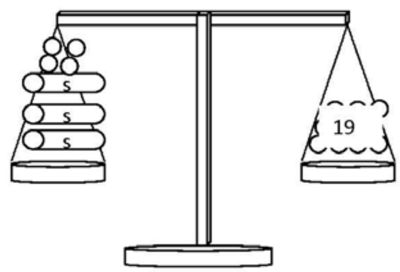
Brainy Brian's Balances Again

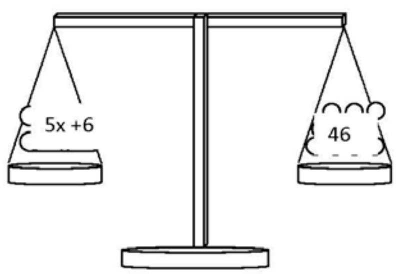
Use Brian's Balance Strategy to figure out the Mystery Roll Packing Rule for each of the following scales.





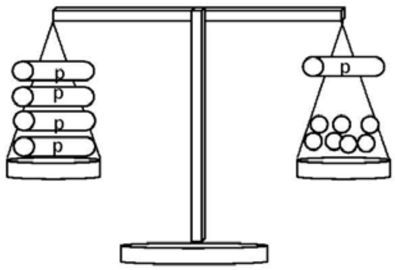


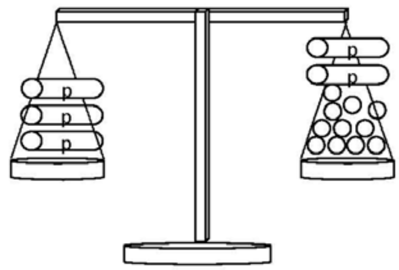


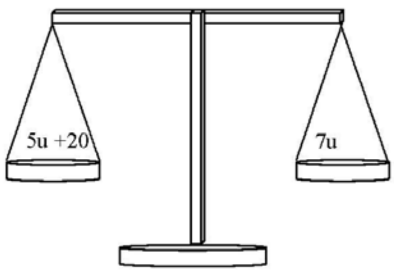


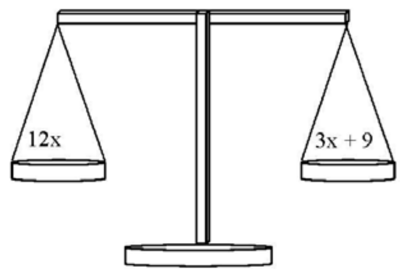
Rambunctious Rolls!!!

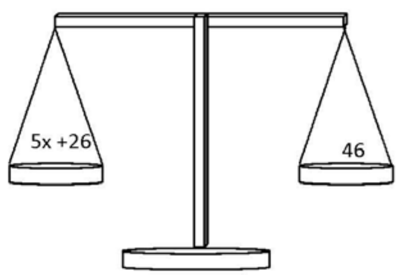
Use Brian's Balance Strategy to figure out the Mystery Roll Packing Rule for each of the following scales.







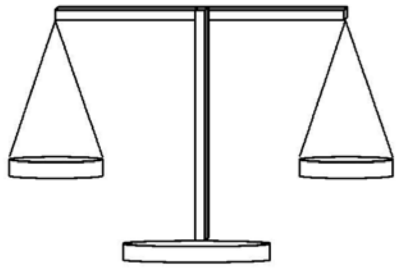




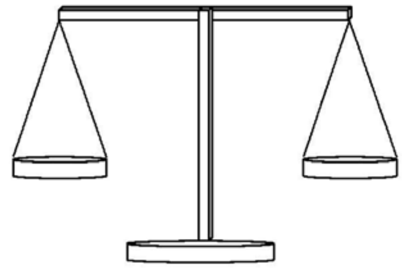
Rambunctious Roll Review

Solve for the Mystery Roll Packing Rule. The scales are there ONLY if you need to use them.
Be sure to SHOW your thinking!

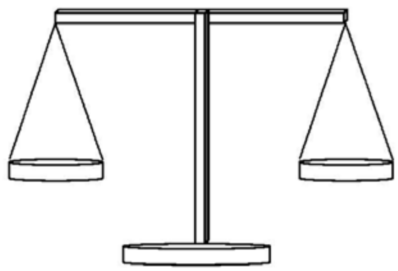
1. $8 + 2y = 18$



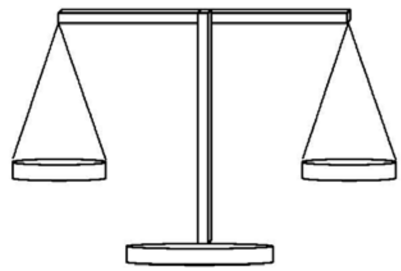
2. $12x + 7 = 18 + x$



3. $5 + 4x = 3x + 7$



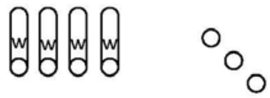
4. $5c + 8 = 3c + 4$



5. Explain how you got your answer for #4

Packing Problems

1. How much candy is in the collection below if the packing rule is w = 9?



2. How much candy is in the collection below if the packing rule is g = 15? Draw a picture, if necessary.

$$3g + 5 + g$$

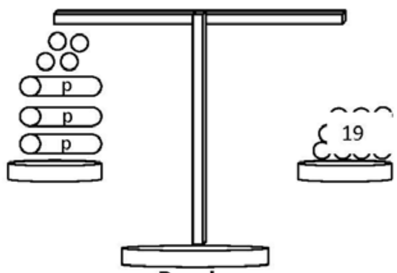
3. How could the Simpsons package 59 cherry candies if the packing rule is c = 10? Draw at least two different pictures.

4. Draw a picture to illustrate t + 7 candies. Draw a different picture to show 3t – 8 candies.

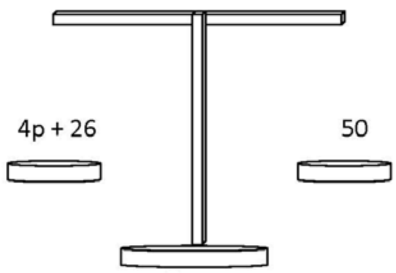
t + 7 candies	3t - 8 candies

Positively Preposterous Peach, Pear, and Pork Chops

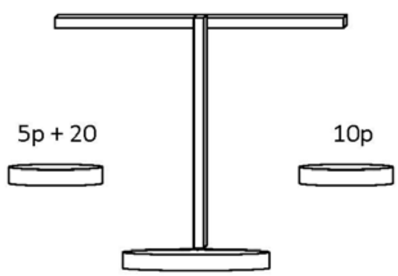
Determine the packing rule for each P-flavored candy below.



Peach



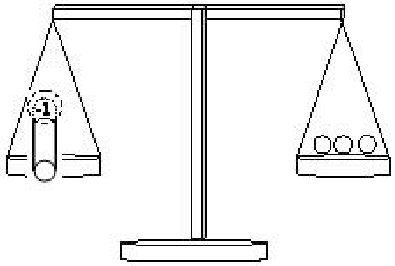
Pear

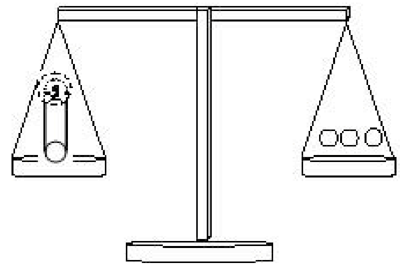


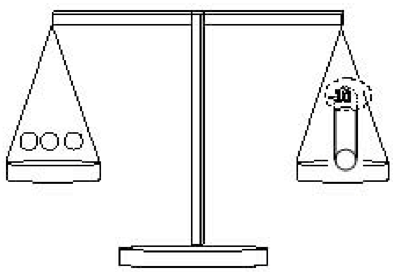
Pork Chop

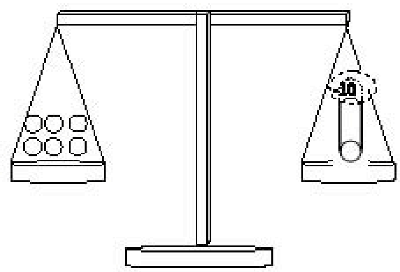
Missing Pieces Miasma

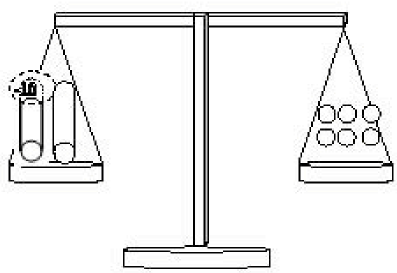
Use Brian's Balance Strategy to figure out the Mystery Roll Packing Rule for each of the following scales.





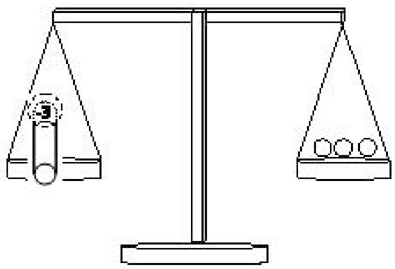


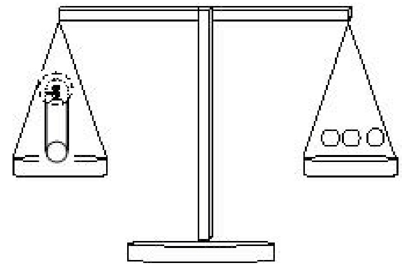


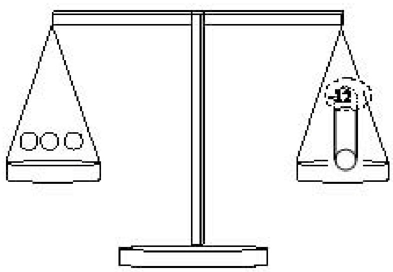


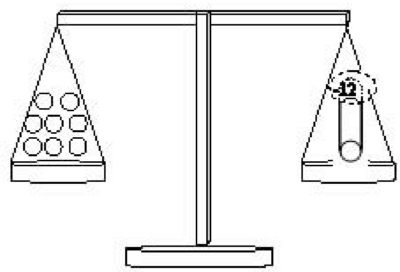
Missing Pieces Murk

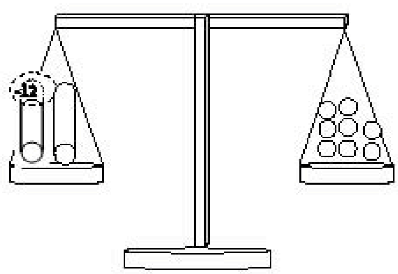
Use Brian's Balance Strategy to figure out the Mystery Roll Packing Rule for each of the following scales.





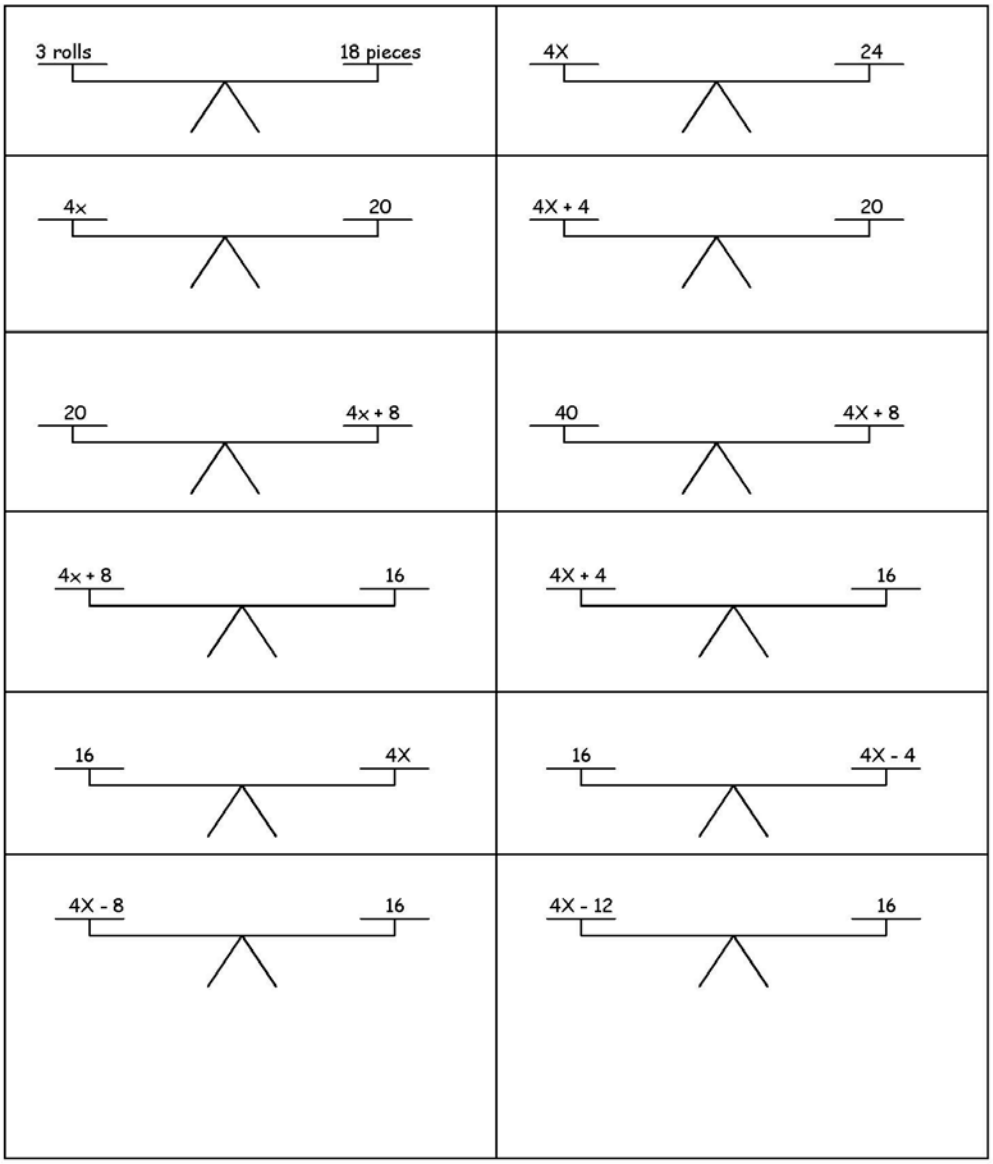






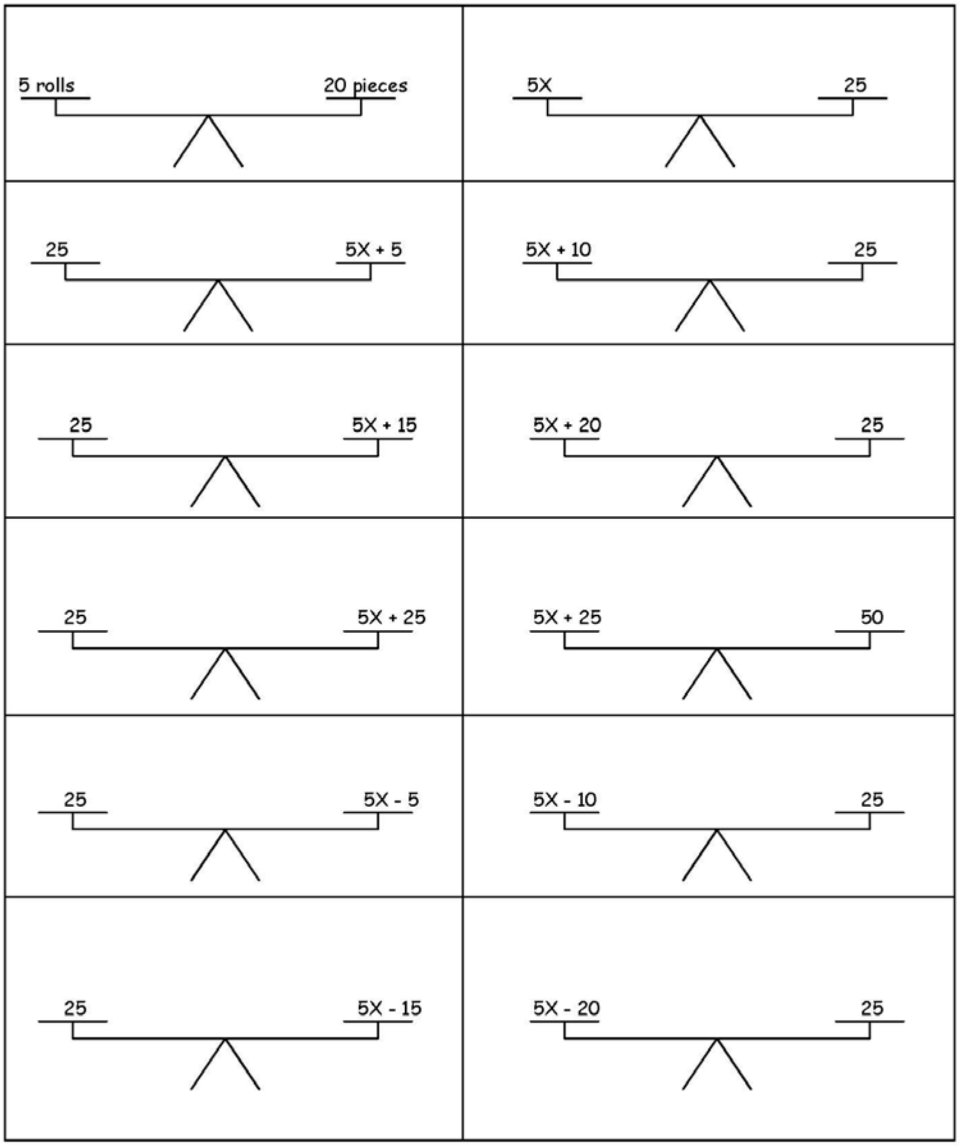
Candy Scales I

How many candies are in each roll below?



Candy Scales II

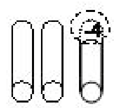
How many candies are in each roll below?



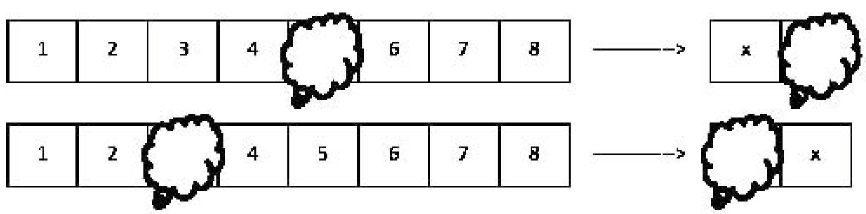
QUIZ

1. What are the packing rules for each balance below? Show your work.

2. How much candy is represented below if the packing rule is w pieces per roll? Explain.

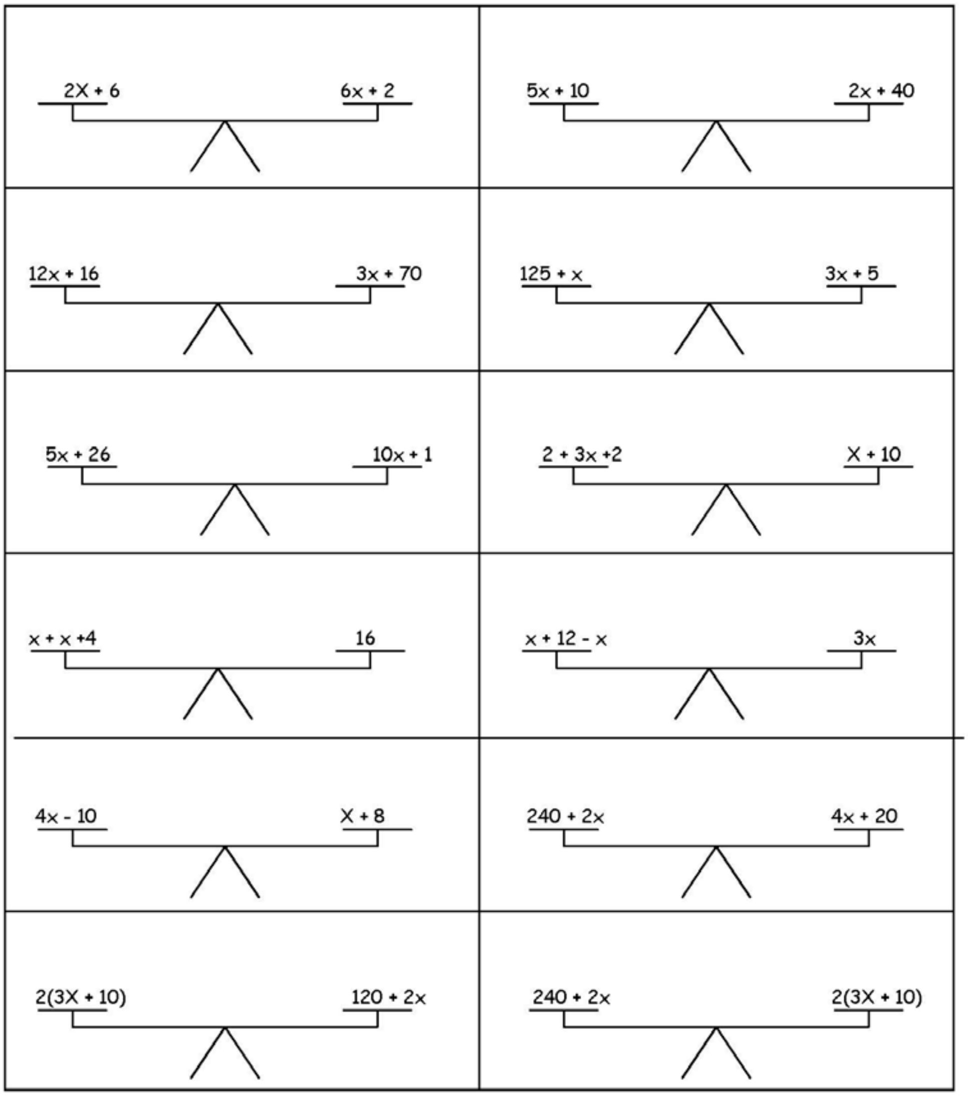


3. Krusty the Clown started counting as shown in the chart below. However, his voice was muffled in certain spots and the Simpsons could not hear him. Can you fill in the chart below wherever there is a "muffled" sound?



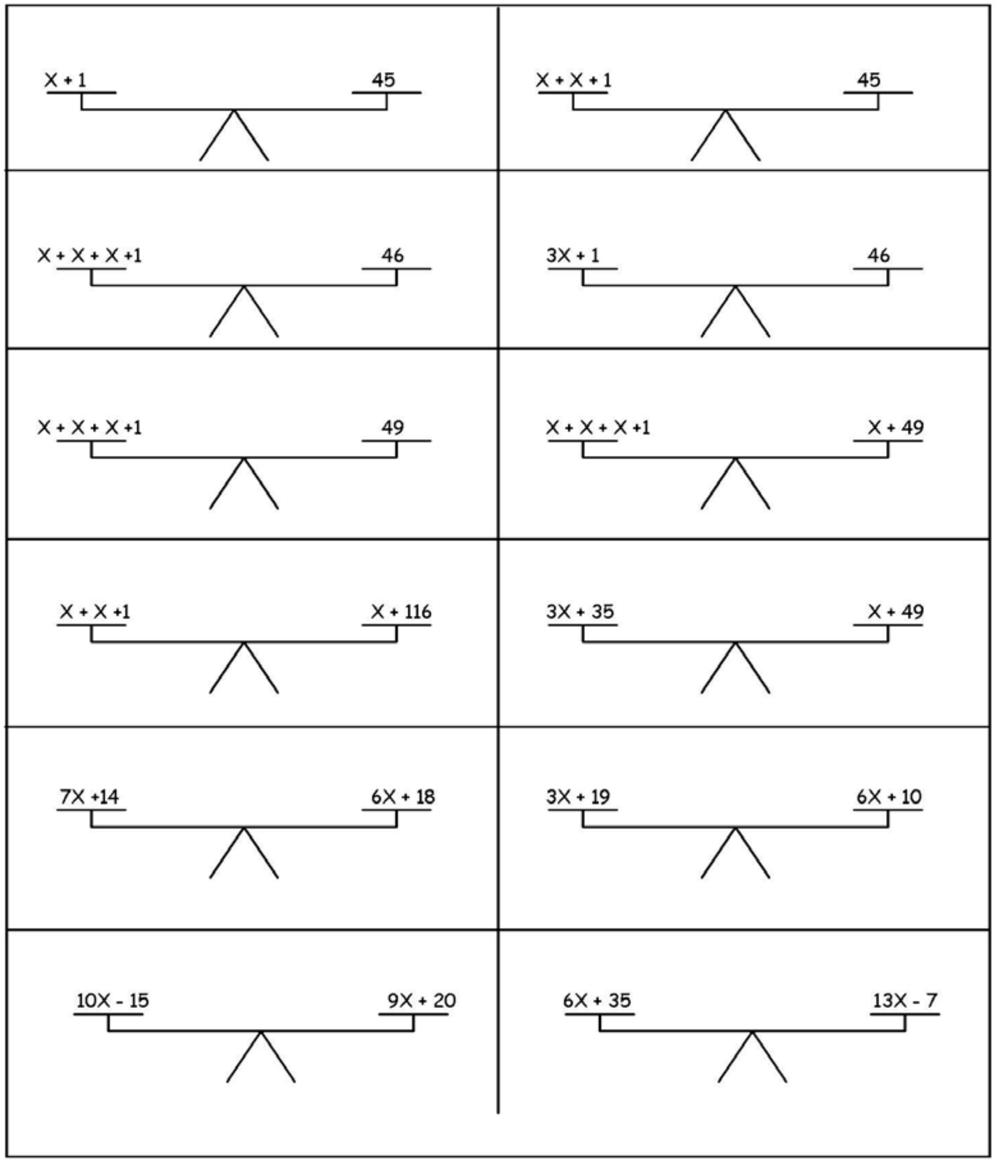
Candy Scales III

How many candies are in each roll below?



Candy Scales IV

How many candies are in each roll below?



CANDY WORKER TEST

Solve each of the equations below.
You may use a balance to help with your reasoning if you choose.

1). $a + 5 = 23$

2). $s - 7 = 15$

3). $d - 9 = 17$

4). $12 = x + 4$

5). $7 = y - 2$

6). $4 + y = 2y$

7). $8x - 4 = 9x$

8). $3x = 2x + 11$

9). $X - 3 = -1 + 4$

10). $X + 7 = 2x + 5$

11). $10x + 6 = 11x$

12). $18x + 64 = 19x$

13). $17x = 16x - 24$

14). $-8x + 9x = -1 + 7$

15). $3x - 2x + 5 = 5$

16). $2w + 10w = 48$

17). $8y + y = 45$

18). $16 = 10t - 8t$

19). $5x + 12 = 8x$

20). $24g - 22 = 2g$