CONCEPT IMAGE OF SLOPE: UNDERSTANDING MIDDLE SCHOOL MATHEMATICS TEACHERS' PERSPECTIVE THROUGH TASK BASED INTERVIEWS

by

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ABSTRACT

TIMOTHY WILLIAM HOFFMAN. Concept image of slope: Understanding middle school mathematics teachers' perspective through task-based interviews. (Under the direction of DR. ANTHONY FERNANDES and DR. MICHELLE STEPHAN)

The mathematical concept of slope has traditionally been a critical topic in the high school mathematics curriculum. However, current educational trends in the United States introduce this concept in middle school. This trend shifted the responsibility of teaching slope from high school teachers to those at the middle school level; specifically, eighth grade teachers. This qualitative study fills a void in the field of mathematics education by exploring middle school mathematics teachers' understanding of slope.

Ten middle school mathematics teachers (Grades 6-8) participated in two task-based interviews, three to five weeks apart, in which they answered questions and engaged in tasks that revolved around the mathematical concept of slope. The data was analyzed using an integrated framework of *concept image* and *concept definition* (Tall & Vinner, 1981) and eleven *conceptualizations* of slope (Stump, 1999; Moore-Russo, Conner & Rugg, 2010). The teachers' *concept image* was defined as the total number of *conceptualizations* that each evoked during their problem solving. The results indicate that the middle school teachers focused heavily on the steepness of the line to define slope and evoked real-world examples when they were tasked with explaining slope. The teachers drew heavily on geometric and algebraic ratios to calculate the slope and were challenged when slope was presented in a form that diverged from procedure-based tasks. Within their *concept image,* the number of *conceptualizations* varied among the teachers with some teachers drawing on up to eight *conceptualizations* as they engaged with the interview tasks. The results also indicate teachers with experience teaching Algebra 1 or more advanced mathematics courses had a more robust *concept image* of slope. Further, teachers with a more robust concept image had greater success in explaining their thinking, as they could draw on the flexibility of their knowledge and adapt it to the task at hand.

DEDICATION

This is dedicated to my parents. From an early age, they taught me the value of education in and outside of the classroom. To them, I am eternally grateful.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: INTRODUCTION AND RATIONALE

The mathematical concept of rate of change is a critical topic throughout the mathematics curriculum. Rate of change, as referenced in this study, is the ratio between the change in one variable over the corresponding change in another variable. If this ratio is equal for all points on a given function, then one can infer the function is linear; this constant rate of change is commonly referred to as the slope of the function (Rossi, 1970). For a nonlinear function, the rate of change can vary depending on the point or points selected. Yet, at each particular point one can calculate the slope of the function by calculating the rate of change of the line tangent to the curve. Based on this facet, I will refer to slope as the rate of change of a function and will not limit the term to only linear functions.

Steen (1990) included the concept of change as one of the five interwoven strands of mathematics with the potential to develop as early as one's informal childhood experiences and continue to develop throughout one's formal schooling. The National Council of Teachers of Mathematics, or NCTM, (2000) states that all students should enter high school with an understanding of the key components of graphs, including the concept of slope. In addition, the NCTM contends, "instructional programs from pre-kindergarten through grade 12 should enable all students to analyze change in various contexts" (NCTM, 2000, p. 40). With respect to everyday applications, rate of change helps individuals make daily choices such as choosing a phone plan or determining their mortgage. At the academic level, rate of change is needed to understand core topics in many academic courses such as physics, biology, economics, algebra, and calculus (Wilhelm & Confrey, 2003; Herbert & Pierce, 2008).

1.1 Rate of Change within the Curriculum

Within the secondary mathematics curriculum, the mathematical concept of rate of change is thoroughly investigated in two courses: Algebra I and Calculus. For many students, these two courses bookend their secondary mathematics career. In Algebra I or an equivalent course, students investigate the concept of slope primarily in a linear sense (Stump, 1999; Tuescher & Reys, 2010), while calculus provides students with the opportunity to study rate of change in both a linear and nonlinear sense.

Unfortunately, calculus is the final course in a long line of prerequisite courses (Johnson, 2010). For instance, to enroll in calculus in the state where this study took place, a student must successfully pass Math 1 (an equivalent course for Algebra 1), Math 2, Math 3, and Pre Calculus. Assuming a student takes one course a year, she would need to take Math 1 in the eighth grade in order to take a calculus course at the secondary level. Consequently, the majority of the students will never study this critical concept outside of linear functions (Kaput, 1994; Roschelle, Kaput & Stroup, 2000). According to the National Center for Educational Statistics (NCES, 2014), in 2009 only 16 percent of high school graduates completed a calculus course. Though this number grew from seven percent in 1990, 84 percent of American high school graduates will never take a calculus course at the secondary level. Even if a student wanted to take this course, it is not an option for every student. Within the United States, only 50 percent of all high schools offer a calculus course (U.S. Department of Education Office for Civil Right, 2014).

Unlike the exclusive nature of calculus**,** Algebra I, or an equivalent course, is far more accessible for students. In 2004, approximately 95 percent of all high school graduates took Algebra I, or an equivalent course, before or during high school (NCES, 2014). Within this number, 69 percent of high school graduates completed this course at the high school level. In addition, 88 percent of high school graduates completed a Geometry course, while 76 percent of high school graduates completed Algebra II (NCES, 2014). With more students completing these courses, it is essential to investigate how slope is addressed within the curriculum.

Stanton and Moore-Russo (2012) examined the standard curricula to see how the fifty states addressed the mathematical concept of slope. The two researchers completed this task separately. One read all of the documentation, looking specifically for references of slope, while the other conducted a keyword search using the following terms: slope, line, linear, gradient, straight, steepness, increasing, decreasing, parallel, perpendicular, rate of change, and displacement (Stanton & Moore-Russo, 2012). After three months, they completed their review of all documents from eighth through twelfth grade.

To analyze the documents, the researchers used the eleven conceptualizations of slope that were originally developed by Stump (1996, 2001) and extended by Moore-Russo, Conner, and Rugg (2011). Stump's (1996) original seven representations of slope are categorized as geometric ratio, algebraic ratio, physical property, functional property, parametric coefficient, trigonometric conception, and calculus conception. In a later work, Stump (1999) added an eighth category, real-world representations. Moore-Russo, Conner, and Rugg (2011) developed the final three categories: determining property, behavior indicator and linear constant. A detailed explanation of these eleven conceptualizations is located in Chapter 2 of this document.

Stanton and Moore-Russo (2012) found that all eleven conceptualizations were evident within the curriculum. Across the states, eight was the most common number of conceptualizations represented in the curriculum. Forty-four of the fifty states had at least five conceptualizations addressed in their eighth through twelfth grade mathematics curriculum. The states with under five conceptualizations were California, Colorado, Maine, Missouri, Montana, and Wyoming. Looking deeper, California addressed the concept of slope in greater detail in seventh grade, while Montana did not mention slope or linear functions (Stanton & Moore-Russo, 2012).

Even though slope is studied in the mathematics curricula, researchers have found that secondary students may not reason that slope is a constant rate of change (Johnson, 2010; Lobato, Ellis & Munoz, 2003; Moschkovich, 1996). This lack of reasoning may be the result of the approach in which students are exposed to this topic. Teuscher and Reys (2010) argued that students are introduced to the concept of slope first and study the rate of change later. Furthermore, they stated "that students tend to practice inputting numbers and calculating the slope of a line with little or no focus on interpreting the meaning of the result within a given context and with little consideration for units of measure" (p. 519).

This difficulty is consistent even when slope is present in different mediums. Given the equation $y = mx + b$, students tend to interpret the m value as a difference as opposed to a ratio (Lobato, Ellis & Munoz, 2003), and students failed to associate a change in the m value with a change in the steepness of the line (Moschkovich, 1996). When presented as a decimal, students struggled to view slope as a ratio (Barr, 1980;

Barr, 1981). In graphical form, students struggled to make a connection between the graph and the concept of rate of change (Bell & Janvier, 1981; Orton, 1984).

Even after successfully completing Algebra I, the concept of rate of change remains a difficult concept for students at the secondary level (Herbert & Pierce, 2012; Lobato, Ellis, & Munoz, 2003; Stump, 2001; Tuescher & Reys, 2010) and collegiate level (Moore-Russo & Conner & Rugg, 2010; Nagle, Moore-Russo, Viglietti & Martin, 2013; Zandieh & Knapp, 2005). In a study of pre-calculus students, Stump (2001) found students continued to have difficulty interpreting slope as a measure of rate of change. Students demonstrated misconceptions, such as opposite integers have the same rate of change (Tuescher & Reys, 2010). Even at the collegiate level, students' knowledge of slope tends to revolve around procedure with little indication of rate of change between variables or the real-world application (Nagle, Moore-Russo, Vigilietti & Martin, 2013).

1.2 Purpose of the Study

Prior research has demonstrated that rate of change continues to be a difficult concept for students to comprehend. Based on this continued issue for students, this study looks to investigate the concept image of slope of their teachers. Specifically, the focus of this study will seek to determine which conceptualizations of slope comprise middle school mathematics teachers' concept image of slope. In this study, a conceptualization will refer to a specific representation of slope, while the concept image is the total number of conceptualizations the teachers have associated with slope. Therefore, the middle school mathematics teachers' concept image of slope may consist of zero to eleven conceptualizations of slope. These conceptualizations will emerge as the teachers discuss and solve various tasks regarding slope. It must be noted that a conceptualization may be a part of the person's concept image, even if it is only evoked at a particular moment (Tall & Vinner, 1981). Furthermore, the teachers may evoke multiple conceptualizations at any moment; therefore, I carefully analyzed their words.

The decision to focus on middle school teachers was based on two facets. First, the percentage of eighth grade students taking Algebra I is on the rise (Walston & McCarroll, 2010); therefore, more students are being introduced to slope in middle school. Second, as will be discussed in Chapter 2, no prior study was found that focused on middle school mathematics teachers' understanding of slope.

In conducting the review of literature, I found most of the previously published research focused on the understanding of rate of change from the student perspective (e.g., Confrey & Smith, 1994; Herbert & Pierce, 2011; Moore-Russo & Conner, 2010; Orton, 1983; Thompson, 1994). This is not to say no research has been conducted with a focus on the teacher. Stump (1996; 1999) sought to examine the conceptualizations of slope for both pre-service and in-service high school mathematics teachers. Later, Stump (2001) continued her work by publishing an in-depth investigation into three pre-service teachers. Other researchers have looked at teachers' understanding of rate of change at the elementary level (Walter & Gerson, 2007) and secondary level (Mudaly & Moore-Russo, 2011). No study was found that focused on middle school mathematics teachers. In their study, Mudaly & Moore-Russo (2011) recommended more research was needed to understand teachers' conceptualizations of slope. Specifically, they call for another study that "includes items written to intentionally elicit conceptualizations by the teachers" (p. 6) through a qualitative approach, including the use of interviews.

1.3 Research Questions

Based on the recommendation of Mudaly and Moore-Russo (2011), this study will be qualitative and will utilize face-to-face interviews to investigate what conceptualizations make up the concept image of slope for a collection of middle school mathematics teachers. Specifically, this study seeks to answer the following research questions:

1. What is the concept definition of slope for in-service middle school mathematics teachers?
2. What conceptualizations of slope do in-service middle school mathematics teachers possess?
3. What is the concept image of slope for middle school mathematics teachers?

To answer these research questions, the qualitative research study I designed employed two task-based, semi-structured interviews that were conducted with ten in-service middle school mathematics teachers. During each of these interviews, the teachers were asked questions revolving around slope including solving pre-designed tasks and offering their definitions of terms relating to slope. For the study, middle school is defined as sixth through eighth grade. To define conceptualizations of slope, this study engages the eleven conceptualizations of slope developed by Stump (1996, 1999) and later redefined and extended by Moore-Russo, Conner and Rugg (2011). These conceptualizations aided in allowing me to infer the teachers' concept image of slope. During both interviews, each in-service teacher was asked to solve mathematical problems that revolved around each of the eleven conceptualizations. An explanation of these activities can be found in Chapter 3 of this document.

1.4 Significance of the Study

This study contributes to the field of mathematics education in two ways. First, I will examine the concept definition and the concept image of slope for a group of middle school mathematics teachers. This topic is critical to investigate. In the United States of America, the current education trend has pushed advanced mathematical topics down on younger students. This trend has demanded middle school teachers assume the responsibility for teaching advanced topics once reserved for students at the high school level. Hence, middle school teachers will be asked to integrate advanced mathematical topics such as slope in their instruction. In a review of the literature, I did not find any prior study that focused on mathematics teachers, grades 6 - 8, to determine their concept image of slope. This study, therefore, sought to determine what conceptualizations of slope are part of the middle school mathematics teachers' concept image. In addition, this study offers insight into which conceptualizations the middle school teachers understand and which conceptualizations need to be targeted.

Second, by revealing which conceptualizations of slope are part of middle school mathematics teachers' concept image, high school teachers may better understand how their students are being introduced and taught about the mathematical concept of slope in the middle grades.

1.5 Overview of the Study

This chapter serves as an introduction to the research questions as well as the rationale for this study. In Chapter 2, I will review the related literature on the mathematical concept of rate of change, offer a detailed explanation of the eleven conceptualizations, and articulate the differences between a concept definition and a concept image. In Chapter 3, I will describe the design of the study including a rationale for how the participants were selected, the process of data collection, and the analysis of the data. The results are located in Chapter 4 and Chapter 5. Chapter 6 contains my conclusions, a discussion, and how I see this study evolving in the future.

CHAPTER 2: REVIEW OF LITERATURE

In this chapter I will review the relevant literature to offer insight into why this study is warranted. First, I offer my assumptions on mathematical concepts including an explanation of concept definition and the concept image, as this will serve as the theoretical framework for this study. Second, I detail the mathematical definition of slope and the relevant mathematical terms and conceptions that are associated with slope. Following this explanation, I will review previous studies involving student and teacher understanding of slope. Finally, I will discuss the limitations with the current research.

2.1 Mathematical Concepts

Tall and Vinner (1981) made a distinction between how a mathematical concept is formally defined and the cognitive process in which the concept is conceived. They selected the term *concept definition* to refer to an individual's definition of the concept and the term *concept image* to refer to the total cognitive structure an individual associates with the concept. The concept definition is the "form of words used to specify that concept" (Tall & Vinner, 1981, p. 152) and may be learned through memorization or by relating the definition to the specific mathematical concept. The *concept definition* may vary from person to person since each individual may personally reconstruct the definition. Hence, Tall and Vinner (1981) believed one's *personal concept definition* can differ from the *formal concept definition* (the definition that is accepted by the mathematical community). Based on this assumption, it was critical that I determine the *concept definition* of slope for each of the middle school mathematics teachers participating in this study.

As with many concepts, they are not always learned in a formal education setting but through one's daily experiences (Tall & Vinner, 1981). Vygotsky also recognized this idea as he classified concepts into two categories: spontaneous concepts and scientific concepts (Panofsky, John-Steiner & Blackwell, 1990). A spontaneous concept is learned through everyday experiences; the learning is not planned nor is it arranged in a specific fashion. In contrast, a scientific concept is introduced in a systematic fashion as the learner engages in formal activities that are planned by one trusted with the task of teaching the concept.

Considering both spontaneous and scientific concepts, it is important that we recognize learning is not confined to the classroom. Even without formal instruction, individuals can develop thoughts about various mathematical concepts. Cornu (1991) stressed, "most mathematical concepts generally do not start on virgin areas" (as quoted in Dede & Soybas, 2011, p. 392). For example, prior to studying linear functions, students can develop meaningful concepts of slope, including using terms such as "steep" (Cheng & Sabinin, 2008). These prior spontaneous concepts cannot be turned off and on during formal instruction. The individual may refine the meanings and interpretations of these concepts as time progresses, and this can occur even if the individual is not exposed to the formal definition (Tall & Vinner, 1981). Based on this notion, the concept image is constructed and evolves over time as one continues to be exposed to varying stimuli. Thus, each individual forms his or her own concept image (Tall & Vinner, 1981) that includes "all mental pictures (pictorial, symbolic, and others), all mental attributes (conscious or unconscious) and associated processes" (Semadeni, 2007, p. 4).

To examine the concept image in detail, Tall and Vinner (1981) used the example of subtraction. When one first learns the process of subtraction, it typically involves only positive numbers. At this early stage, the individual may come to the realization that subtraction always reduces the number. For instance, when one takes four from six, the result is two, which is smaller. For the particular individual, this observation becomes part of her concept image and has the potential to cause an issue later when she begins to subtract negative numbers. For this reason, Tall and Vinner (1981) stress, "all mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image; they may contain the seeds of future conflict" (p. 152).

As one is exposed to various stimuli, different parts of the concept image may be activated. Tall and Vinner (1981) called the activated portion of the concept image the *evoked concept image*. This title implies anything that comes to one's mind when hearing or seeing the concept name is the concept image (Dede & Soybas, 2011). Given varying stimuli, multiple concept images may emerge. Therefore, it was critical to use tasks that present slope in a variety of different contexts and with a diverse set of representations. The selection of these tasks will be described in the next chapter.

In this study, I drew on the constructs of *concept definition* and *concept image* as a means to understand teachers' conceptions of slope. Though the constructs of concept definition and concept image were developed in the early 1980s, it "has weathered the years well and continues to be cited in the literature" (Bingolbali & Monaghan, 2007, p. 19). For instance, current mathematical researchers such as Semadeni (2008), Moore-Russo & Connor (2010), and Yanik (2014) continue to incorporate this theory into their research.

Yanik (2014) investigated middle school students' concept images of geometric translations. He contended that concept images are one of the tools for exploring understanding of a given topic. To analyze the students' concept images he used a written instrument, conducted interviews with students and teachers, and analyzed documents. Relating directly to slope, Moore-Russo and Connor (2010) classified all of the conceptualizations of slope (which will be described in detail later in this chapter) as concept images. To determine if a conceptualization was evoked by the participants, the researchers created a slope conceptualization matrix. As a participants' comments, gestures, or written work referenced a specific concept image related to the distinct conceptualizations of slope, the researchers recorded it via the matrix. Located below (TABLE 1) is the table from their study.

TABLE 1: Conceptualizations from Moore-Russo and Connor's (2010) study

	Geometry ratio	Algebraic ratio	Physical properties	Functional properties	Parametric coefficient	Trigonometric conceptualization	Calculus conceptualization	Real world	Determining property	Behavior indicator	Linear constant
Paul	X	X	X	X		–[b]	–[a]	X	X	X	X
Dee	X	X	X		–[a]	–[b]	–[a]		–[a]	X	X
Abby	X	X	X	–[a]	–[a]	–[b]	–[a]	–[a]	X	–[a]	–[b]
Keri	X	X	X			X	X	X	X	–[b]	–[b]

[a] Not evidenced until task 3
[b] Not evidenced until task 2 (revisited)

Semadeni (2008) extended the idea of a concept image by determining that when a concept image reached a certain level of development, it is defined as the level of deep intuition. This advanced level required that the basic meaning of the concept is stable even when the stimuli vary. Continuing, Semadeni (2008) broke deep intuition into three categories: 1) a deep intuition without a concept definition, 2) a deep intuition supported by a concept definition, and 3) a concept image which has not reached the level of deep intuition, whether it contains a concept definition or not (p. 9). As evident by these studies, the use of concept definition and concept image continues to be relevant in mathematical education research.

In the next section, I will discuss how the mathematics community as a whole understands the mathematical concept of slope. First, I explain the *formal* concept definition of slope as defined in a mathematics textbook. Next, I look deeper into slope by examining how mathematicians define key terms essential to understanding slope. Finally, in the next section, I analyze these critical terms by examining how a textbook tailored to middle school mathematics curriculum defines these concepts.

2.2 Slope as a Mathematical Concept

The importance of the mathematical concept of rate of change is not questioned. Within the secondary curriculum, knowledge of this particular topic is critical. Farenga and Ness (2005) contend one “cannot underestimate the significance of rate” (p. 58). The concept of constant rate of change serves as a foundation to a variety of mathematical topics including linear functions and slope (Thompson, 2008). As they move through the secondary curriculum, students are confronted with this mathematical concept in various courses such as physics, biology, economics, physical science and calculus (Farenga & Ness, 2005; Wilhelm & Confrey, 2003). Due to the variety of contexts where slope is seen, the mathematical concept of slope can take on various meanings between individuals including teachers and their students.

What is Slope?

The formal concept definition of slope, as defined in James Stewart's (2003) *Multivariable Calculus: Fifth Edition* textbook, is the measure of the steepness of the line. In more algebraic terms, the slope of a line is the ratio of the vertical change of the line compared to the horizontal change as one moves from one point to another (Kaufmann, 1992). Given the points, $(x_1, y_1)$and (x_2, y_2), the horizontal change as we move from point to point is $x_2 - x_1$,while the vertical change is calculated by $y_2 - y_1$. This vertical change is sometimes referenced as the rise, while the corresponding horizontal change is called the run (Foresman, 1987). To calculate the slope of linear function, the expression $\frac{y_2-y_1}{x_2-x_1}$can be used.

The slope of a linear function is the most basic form of rate of change that a student will encounter (Stanton & Moore-Russo, 2012). Leinhardt, Zaslavsky and Stein (1990) described slope as a "power linking concept" (p. 54) that allow students to understand a function and its graph (as cited in Stanton & Moore-Russo, 2012). Yet, in order to understand this mathematical concept, one must confront the mathematical terms quantity, ratio, and rate. Therefore, in the next section, I will address these terms.

Quantity

According to Thompson (1993), a quantity is not the same as a number, but it is a quality of an object that can be measured. When a quantity is measured, it has a defined unit and process for assigning a number that represents the proportional relationship between a particular value of the quantity and the unit (Thompson, 2011). Yet, Thompson (1993) clarifies that one does not need to measure or know the measure of the object to reason about it. For instance, in the context of slope, one can examine a graph with two lines and determine the relationship between the numerical values for slope. Hence, an important attribute in understanding quantity is the ability to compare quantities using the relationships of greater than, less than, or equal to (Davydov, 2008).

Ratio and Rate

Using two different terms, ratio and rate, suggests these are two separate ideas. Yet, Thompson and Thompson (1992) contend there is not a conventional distinction between these terms. In addition, Thompson and Thompson (1992) believe there is confusion about what makes them different from one another. They point out this confusion is not limited to the classroom but has been found in published research. Lesh, Post, and Behr (1998) stated, “there is disagreement about the essential characteristics that distinguish, for examples rates from ratios … In fact, it is common to find a given author changing terminology from one publication to another” (as cited in Thompson and Thompson, 1992, p. 2). Continuing, Thompson and Thompson (1992, p. 2) list the most frequent distinctions between ratio and rate found in literature. They are:

1) A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time (Vergnaud, 1983; Vergnaud, in press)).
2) A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
3) A ratio is a binary relation that involves ordered pairs of quantities. A rate is an intensive quantity - a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, &

Behr, 1988; Schwartz, 1988).

Thompson (1990) offered his own personal concept definition for both terms. He defined a ratio as "a multiplicative comparison between two quantities" (p. 7). In developing this definition, he emphasized the quantitative-relationship aspect of a ratio that allows one to use this concept to determine how many times bigger one quantity is than another. If the quantities in the ratio are measured in the same unit (e.g. length), then one can directly compare the quantities.

Thompson and Thompson (1992) offered the following example. A collection of three objects can be compared multiplicatively against a collection of two objects in two distinct ways. The first comparison is the two collections as wholes. This would be expressed as 3:2. In the second comparison, one quantity is measured in the units of the other. This second comparison is sometimes called a "unit rate." Therefore, instead of expressing the ratio as 3:2, it would be expressed as $\frac{3}{2}:1$. In each case, they are expressions of a multiplicative comparison of the two quantities. Thompson and Thompson (1992) contend if one focuses on the mental operation of the multiplicative comparison, the dimensions of the quantities do not matter. Instead, only the notion that the two quantities are being compared multiplicatively matter. Yet, if the quantities being compared are in the same units, then one has a direct comparison of qualities. If the quantities are measured in different units, then the measures of their qualities are being compared. In both cases, the mental operation is the multiplicative comparison of two quantities. The result of this comparison is a ratio (Thompson & Thompson, 1992).

Thompson (1990) defined rate as "a quantity that may be analyzed into a multiplicative comparison between two other quantities—where one quantity's value is conceived as varying in constant ratio with variations in the value of the other" (p. 8). In other words, "a rate is a reflectively abstracted constant ratio" which "gives prominence to the constancy of the result of the multiplicative comparison" (Thompson & Thompson, 1992, p. 7). To examine rate, they offer the example of speed. To view speed as a rate, one must recognize the distance and time accumulate simultaneously and continuously. This perspective implies the accumulated quantities maintain the same proportional relationship with their respective accumulation totals.

To make a ratio, one must conceive the two quantities in multiplicative comparison and also conceive the compared quantities in their independent and static state (Thompson, 1994). To make a rate, one must "re-conceive the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived" (Thompson, 1994, p. 19). Thompson (1994) contends it is possible that people generally first conceive a multiplicative comparison in terms of a ratio and then reconceive as a rate.

In a middle school mathematics context, this occurs when a student is asked to assume that an object continues at a constant rate given the initial situation in a manner in which there are two quantities being compared multiplicatively (Thompson, 1994). He cautions that at this moment, students are often taught by their teacher to follow procedure. Based on this example, I sought out a definition that might be shared by secondary teachers in the region where this study will take place.

Middle School Text Definition of Ratio and Rate

To find a suitable concept definition for slope that would be shared by the participants in this study, I looked towards an Algebra textbook: McDougal Littell's

Algebra 1: North Carolina Edition (Larson, Boswell, Kanold, & Stiff, 2004). This textbook was the last hard copy text adopted by one of the schools where this study will take place. Though most of the resources are digital, the school still keeps copies of this text for the teachers to use if they elect to.

The authors offer the following formal concept definition for ratio and rate. They define ratio as the relationship, $\frac{a}{b}$, between two quantities, *a* and *b*, that are measured in the same unit, while rate is defined as the relationship, $\frac{a}{b}$, between two quantities, *a* and *b*, that are measured in different units. Therefore, in these definitions, the distinguishing difference between these two terms is the corresponding units.

The textbook offers an example of both ratio and rate. For the ratio, the book references similar triangles: two triangles that have corresponding angles. Similar triangles can be confirmed by proving the ratios of the lengths of the corresponding sides are equal (Figure 1). For rate, the authors reference two real-world examples: average spending per person and miles per gallon.

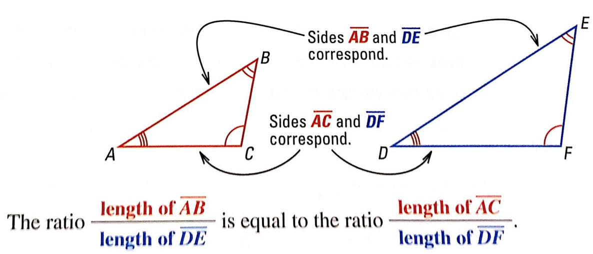


FIGURE 1: Image from textbook used by the teachers.

The distinction that a rate has different units is shared by Kaput & West (1994). They contend that rate of change is an intensive quantity, defined as the relationship between one quantity and one unit of another quantity. Confrey and Smith (1996) disagree. They contend a rate is not limited to relationships between unlike quantities. For example, they believe inflation, when expressed as the relationship between dollars to dollars, is a rate. Hence, Confrey and Smith (1996) define a rate as a ratio that can be quantitatively increased or decreased. Due to the varying definitions offered by mathematicians, this study will explore each participant's personal concept definition of the terms addressed.

2.3 Conceptualizations of Slope

Various studies (Stump, 1996; Stump, 1999; Moore-Russo, Connor & Rugg, 2010) helped to classify and define the various ways that individuals represent the mathematical concept of slope. In total, these researchers have offered 11 conceptualizations of slope. In this study, I will use these 11 conceptualizations to aid in constructing the middle school teachers' concept image (the total number of conceptualizations that each teacher associated with slope) of slope. Therefore, in the next section, I offer a description of each conceptualization.

Stump's Original Seven Conceptualizations

In her study, Stump (1996) administered a mathematical survey. Within the survey were two questions designed to investigate in-service and pre-service high school teachers' definitions of slope: "What is slope?" and "What does slope represent?" From the various responses, she formulated seven categories. The first category defined by Stump (1996) was geometric ratio. This category focused on interpreting slope as a geometric property and included responses such as "rise over run" and "vertical change over horizontal change."

The second category, algebraic ratio, was represented when the response revolved around the algebraic formula used to calculate the slope. Examples of responses in this category are “change in y over change in x” or the mathematical expression, $\frac{y_2 - y_1}{x_2 - x_1}$. Physical property, the third conceptualization, is represented by describing the line with terms such as grade, incline, pitch, steepness, or angle. The fourth conceptualization, functional property, was represented by responses that referred to slope as a rate of change between two variables. Parametric coefficient, the fifth conceptualization, was Stump’s (1999) category for those that elected to define slope as the parameter, *m,* represented in the common mathematical equations for a line including $y = mx + b$ or $y - y_1 = m(x - x_1)$.

The sixth and seventh conceptualizations are typically observed in more advanced mathematics courses. A trigonometric conceptualization of slope is related to the angle the line forms when it intersects a horizontal line. This conceptualization can also be defined as the tangent of the angle of inclination (Stump, 1999) or the direction component of a vector (Mudaly & Moore-Russo, 2011). The final conceptualization in Stump’s (1999) original categories is a calculus conception. Within this concept, an individual references a conception of the derivative such as limit, the instantaneous rate of change or a tangent to the curve.

Stump’s Additional Conceptualization

In a later work, Stump (2001) added an eighth conceptualization, real-world representations, to her list. This addition was based on teachers often implementing real-world examples to help their students better understand the concept of slope. According to Stump (2001), a real-world conceptualization of slope exists in two distinct forms: physical situations and functional situations. By physical situations, Stump (2001) refers to examples that students may encounter in their daily lives such as a road, ski slope, or wheelchair ramp. Functional situations are represented through examples that compared distance versus time or a quantity versus the cost.

Additional Three Categories

In their analysis of a dialogue that occurred throughout a graduate course with the focus of deepening the mathematical understanding of key concepts, Moore-Russo, Conner and Rugg (2011) discovered that the graduate students described slope in ways that were not represented in Stump's (2001) eight categories. According to these researchers, the following conceptualizations are presented "in order of most to least observed" (p. 8). Their first additional category and the ninth overall conceptualization addresses slope as a determining property. Under this conceptualization, the students explained that they could determine the equation of a line if given a point and the slope. In addition, the students discussed related mathematical concepts such as parallel and perpendicular lines (Moore-Russo, Conner and Rugg, 2011).

The tenth conceptualization of slope, behavior indicator, revolved around the numerical value of the slope. The students discussed characteristics such as how if the slope was positive, the line increased, if the slope was negative, the line decreased and if the slope was zero, the line was horizontal (Moore-Russo, Conner and Rugg, 2011). The students also recognized the severity of the line's inclination can be determined by calculating the absolute value of the slope. It is this property that allows an individual to know that, if the slope is not zero, the line must intersect the x-axis (Mudaly & Moore-Russo, 2011). The final conceptualization of slope relates to the students' understanding of a linear constant. For instance, an individual in this category recognizes a constant slope is unique to “straight” figures. The graduate students also realized the slope was maintained even if the line was translated (Moore-Russo, Conner and Rugg, 2011). A table of these conceptualizations, TABLE 2, is presented in the same order they are discussed in this paper. The decision to list them in this order was to maintain the order presented by Stump (1996, 1999, 2001) and Moore-Russo, Conner and Rugg (2011).

TABLE 2: List of Conceptualizations of Slope

Conceptualization	Slope as
Geometric Ratio	Rise over Run; ratio of vertical change to horizontal change
Algebraic Ratio	Change in y over change in x; represented by the expression change in y over change in x or $\frac{y_2-y_1}{x_2-x_1}$
Physical Property	Describing a line using expression like grade, pitch, slant, tilt, and “how high a line goes up”
Functional Property	Constant; Rate of change between variables
Parametric Coefficient	The parameter m found as a coefficient in either y = mx + b or in y - y_1 = m(x -x_1)
Trigonometric Conception	Property related to the angle a line makes with a horizontal line (normally the x-axis); tangent of a line’s angle of increase/decrease; direction component of a vector
Calculus Conception	Limit; Derivative; Tangent to the function; Instantaneous Rate of change
Real-World Situation	Physical situation (e.g. Ski slope); functional situation (e.g. distance versus time)
Determining Property	Property that determines parallel or perpendicular lines; property that can yield the equation of a line if given a point
Behavior Property	Property that indicates increasing/decreasing/horizontal trends of a line; the amount of increase/decrease of a line; property that determines that a line must intersect the x axis if the slope is a nonzero value
Linear Constant	Constant property independent of representation; property that is unaffected by translation of a line (Can be referenced as what makes a line “Straight”

2.4 Conceptualizations in the Curriculum

In this section, I examined how these conceptualizations are addressed in the mathematics curriculum throughout the United States of America. This aspect is critical to my study since it is important to note which conceptualizations policy makers are targeting and which conceptualizations they are not. In the current United States educational landscape, teachers are given the dual task of teaching the content and preparing their students for high stakes testing administered by their distinct state. Hence, a teacher's knowledge of slope may be dictated by the curriculum they have been entrusted to teach.

From September to November 2009, Stanton and Moore-Russo (2012) examined the standard documents for all fifty states in the United States of America to investigate how each state addressed the mathematical concept of slope throughout their secondary (eighth - twelfth grade) curriculum. To accomplish this feat, the researchers used the 11 conceptualizations of slope addressed above. After reviewing the data, they divided three of the conceptualizations (trigonometric conception, determining property, and linear constant) into two subcategories. Hence, they generated a total of 14 codes including eight categories (Geometric, Algebraic, Physical, Functional, Parametric, Calculus, Real-world, and Behavior) and six subcategories (Trigonometric 1 and 2, Determining 1 and 2, and Linear Constant 1 and 2) to represent conceptualizations of slope.

The two researchers worked independently. One read the entire document, searching for any reference to slope, while the second performed a keyword search using the terms: slope, line, linear, gradient, straight, steepness, increasing, decreasing, parallel, perpendicular, rate of change, and displacement. With the search completed, they met and discussed their findings. They found evidence throughout the fifty states that all eleven conceptualizations were represented in the standard documents. The most common number of categories and subcategories evident across the documents was eight. Only five states (California, Maine, Missouri, Colorado, Wyoming, and Montana) had less than five categories and subcategories represented in their 8-12 grade mathematics curriculum. The most common conceptualization of slope was geometric ratio, which was found within the curriculum of 45 of the 50 states. Behavior indicator, determining property, functional and algebraic ratio followed closely, being found in 43 states. The least common conceptualizations were trigonometric conception and calculus conception.

In the state where this study was conducted, Stanton and Moore-Russo (2012) stated they found evidence for seven of the categories and subcategories being represented. Specifically, they found evidence that the geometric ratio was addressed in eighth grade. The remaining six were found within the required curriculum at the high school level. These conceptualizations were the algebraic ratio, behavioral indicator, both subcategories of the determining property, functional property, and one of the two subcategories of the linear constant: constant property independent of representation. Finally, they found the calculus conception represented in the non-required coursework. Based on these findings, only one conceptualization is specifically required at the middle school level.

2.5 Students' Understanding of Slope

Although teachers and students' understanding of slope may vary, research on students' understanding of slope can serve as a starting point to investigate their teachers' understanding (Thanheiser, 2009). In recent years, researchers conducted studies on students' understanding of rate of change at all academic levels ranging from elementary (e.g., Turner, Wilhelm & Confrey, 2000) to middle school (e.g., Shternberg & Yerushalmy, 2004) to high school (e.g., Teuscher and Reys, 2010; Gomez et al, 1999;

Bezuidenhout, 1998) to the collegiate level (e.g., Carlson et al, 2002; Whitney, 2010; Kingir & Geban, 2012).

Early Introduction

For many, rate of change is isolated to the secondary curriculum. Yet, the concept of rate of change can be studied successfully at the elementary level in topics such as arithmetic (Turner, Wilhelm and Confrey, 2000). By engaging in early interactions with rate, the students positively impact their understanding of advanced topics. Nemirovsky (1993) stated that an understanding of advanced mathematical topics such as the derivative "develops from basic intuitions that children construct in their daily experiences which physical and symbolic change" (Nemirovsky as cited in Turner, Wilhelm and Confrey, 2000 p. 3).

In their investigation of elementary students, Turner, Wilhelm and Confrey (2000) found that an early introduction to rate of change was rewarding for students. They were able to understand concepts directly related to rate of change including explaining ideas related to graphs of motion and accumulation. When given the real world scenario about depositing the same amount of money each day, the students recognized these constant transactions would generate a "staircase" graph. When given a more advanced topic such as a changing rate of change, the students were challenged; however, they did recognize that if the transaction were changing, a "staircase" graph would not be produced. These results suggest that students at the elementary level have the ability to understand concepts related to rate of change.

Instrumental Understanding

Though understanding rate at an early age can be accomplished, most of the present research has focused on students' difficulty with this topic. A major finding in various studies (e.g. Hauger, 1998; Park, 2012) suggests that students have an extremely strong procedure-based approach. Skemp (1987) defined this type of understanding as instrumental: defined as knowing the "rules without reason" (p. 2). A student with instrumental understanding is one who can accurately solve various mathematical problems about a topic but lacks the knowledge of why. With respect to slope, a student with instrumental understanding can correctly calculate the numerical value for slope but lacks the understanding of what the number means or how to apply this skill to more advanced topics, such as transitioning from average to instantaneous rate of change.

Park (2012) found students had a strong procedure-based concept of average rate of change and struggled in recognizing that average rate of change is a ratio between the difference in y values over the difference in x values. Hauger (1998) found that students' thinking about rate of change is closely connected to the procedure they learned to calculate average rate of change and their knowledge was based on that procedure. Under this instrumental approach, the students in Hauger's (1998) study could not effectively transfer their knowledge of average rate of change to instantaneous rate of change. This result was directly due to the difficulty that comes in applying this learned procedure to an infinitesimal interval.

In a separate study, Thompson (1994) also found that students rely on an instrumental approach. Thompson gave a seventh grade student a problem involving the ideas of speed and acceleration. The question: How far did a car travel if its speed increased smoothly from 50 miles per hour (mph) to 60 miles over one hour? The student assumed that the acceleration was $\frac{10mi/hr}{hr}$which is equivalent to $\frac{1mi/hr}{\frac{1}{10}hr}$. Using this assumption, the student determined that the car drove one-tenth of an hour at 50 mph, then one-tenth at 51 mph, and so on. Though this approach produced a valid approximation, the student was not able to ascertain that the rate of change was constant.

Instrumental understanding is not limited to secondary students. In observing a group of graduate students, Moore-Russo and Connor (2010) noted they relied heavily on two conceptualizations of slope: geometric and algebraic ratio. Yet, this group of students favored algebraic formulas. Hence, they viewed geometric ratio as inferior to algebraic ratio. These students, to a limited extent, used functional and real world situations, but they never referenced parametric or calculus, and only one student discussed slope through a trigonometric conceptualization.

At the calculus level, students' struggles have been well documented. In recent studies, students have demonstrated multiple difficulties including being unable to write the equation of a line given a point and the slope (Habre & Abboud, 2006), viewed a positive and a negative rate of change as the same (Tuescher and Reys, 2010), and failed to explain the rate of change at a cusp (Tuescher and Reys, 2010). These results have lead researchers to conclude students have not yet developed a complete understanding of rate of change (Christensen & Thompson, 2012).

Word Choice

Park (2011) conducted a study to analyze how university level calculus students described the derivative. Of the twelve students who were interviewed, nine incorporated the word "slope" into their response to the question "What is the derivative?" Students continued to relate the derivative to slope while attempting to explain, through spoken word, their process to solve various questions. Yet, when actually solving the problems, the students did not use the terms “slope” and “derivative” synonymously. They selected other phrases such as a velocity (one student) and the tangent line (five students). Demonstrating some confusion, two students mistakenly viewed the tangent line as the derivative, while five students falsely believed the derivative increased or decreased over the same interval that the function increased or decreased. Furthermore, Park (2011) found the students lacked an understanding of the differences between the derivative function and the derivative at a point. He concluded instructors needed to articulate this distinction more clearly and stress the meaning of the various mathematical terms such as function, the derivative, the derivative function, and the derivative at a point.

Park’s (2011) conclusion that individuals need to carefully consider how they word mathematical concepts is a critical takeaway. In fact, other studies have also examined the language of the teacher. Bowers and Doerr (2001) found almost all of the participants (prospective secondary mathematics teachers) agreed that high quality teaching involves “delivering clear explanations” (p. 135). Following their study of Advanced Placement Calculus students, Tuescher and Reys (2010) noted the students’ misunderstandings about slope, rate of change, and steepness may be the direct result of the vocabulary used in textbooks and by their teachers. They recommend that both sources, textbooks and teachers, correctly use mathematical terms.

Zaslavsky, Sela and Leron (2002) addressed this issue by pointing out two specific examples. First, they contend it is commonplace for one to say the point (3,4) instead of the “point whose coordinates are (3,4)”. The second example deals directly with slope. According to the authors, we say the “the slope of the function y = 2x + 3” when it should be stated as “the slope of the line representing the function y = 2x +3” (p

137). Continuing their discussion of slope, they stress the need to distinguish between terms that are geometric versus those that are algebraic. They specifically state, “slope is a purely geometric term (a property of the position of a line relative to the horizontal direction; no coordinate system involved) and must be distinguished from the rate of change of a function” (Zaslavsky, Sela & Leron, 2002, p 137).

At the calculus level, Zandieh and Knapp (2005) examined the two common misstatements spoken by BC calculus students. Eight of the nine students participating in Zandieh’s (1997) study stated either “the derivative is the tangent line” and/or “the derivative is the change” at least once. With respect to the first misstatement, Zandieh and Knapp (2005) concluded that tangent line is explicit, visible, and easily remembered. Thus, when prompted to explain the derivative, the concept image evoked is that of a line tangent to the curve. The second misstatement occurs when one incorrectly shortens the accurate phrase “the derivative is the instantaneous rate of change.” In both examples, the students tended to believe both the accurate and inaccurate versions of the statements. Zandeih and Knapp (2005) reason students may have compartmentalized the concepts and failed to recognize the contradiction. This failure may be attributed to the fact that contradictions are found elsewhere in mathematics. For instance, many refer to “the derivative” as both the derivative function and the value of the derivative at a point (Zandeih and Knapp, 2005; Park, 2011). Park (2011) concluded when teachers were clear about the concepts, the students’ thinking became closer to the formal concept definition.

Impact of Instruction

Students’ understanding of rate may be impacted by the instruction that they receive. In a study conducted in Turkey, Bingolbali and Monaghan (2008) sought to determine if there was a difference in the understanding of the derivative between mathematics undergraduate students and mechanical engineering undergraduate students. Each group was enrolled in different courses, although they were both called calculus. The mechanical engineering calculus course devoted twenty 45-minute lessons to the derivative, while the mathematics calculus spent thirty-six 40 minute lessons on the topic. Collectively, the mechanical engineering students spent nine less hours on the topic.

To collect data, the researchers used a pre-test and a delayed post-test. The pre-test showed there was no significant difference between the mathematics and mechanical engineering students. In the post-test, both groups showed improvement. Yet, now a difference emerged. The mechanical engineering students outperformed their mathematics peers on questions on rate of change, while the mathematics students shined on questions involving tangents. The only variable that can explain this result is the instruction the students received. When asked to define the derivative, more mechanical engineering students referenced rate of change, while the mathematical students were more likely to explain using tangents. These results were consistent with a prior study (Maull & Berry, 2000).

At the secondary level, instrumental understanding may not be the teacher's desired goal. Instead, the teacher may wish to teach students such that they form a relational understanding. Skemp (1987) defined relational understanding as knowing what to do and why. Zahner (2014) documented this struggle. He observed a nationally recognized mathematics teacher for her work with English language learners as she taught a three-week unit on slope. As the teacher introduced the concept of slope, she made a point to emphasize the mathematical conception and real life applications of slope. She focused on rate of change and used terms such as “per day” rather than directly using the term “slope.” Yet, as the days passed, the teacher began to shift her focus from relational understanding to instrumental. She began assigning problems that focused on routine and lacked real world application. When the researchers questioned the teacher about this change, she referenced pressure to prepare her students for the state administered exam at the conclusion of the course.

This is a daily struggle for teachers: teach for instrumental or relational understanding. Skemp (1987) contends teaching instrumentally is easier to understand, allowing the students to experience success faster and obtain the correct answer in a more timely fashion. With state examinations gauging students’ knowledge based on the accuracy of their answers, the importance of obtaining a correct solution may be higher than ever. For example, Zahner (2014) cited that administration referred to the mathematics teachers he observed as one of their best teachers because of her students’ stellar results on standardized test. Therefore, as indicated by the teacher in this study, teachers may feel pressure to teach to the exam as opposed to teaching for a deep understanding.

2.6 Classroom Instruction

The importance of instruction on students’ understanding of a mathematical concept cannot be understated. As students’ struggles with rate of change continue to be studied (e.g. Johnson, 2015), one must also look at the instruction they are receiving. Vinner (1992) contended the teacher plays an important role in the formulation of students’ concept image. Therefore, this next section examines teachers’ understanding of slope and how instruction impacts their students’ knowledge.

Teachers' Understanding of Slope

As students' understanding of rate has been researched extensively, the same cannot be said about classroom teachers. In her review of prior studies, Stump (1999) found no research studies that focused on the teacher's knowledge of slope. Eleven years later, Mudlay & Moore-Russo (2011) stated, "very little research has investigated teachers' understanding of gradient" (p. 2) and that their "study provides a much-needed first look at teachers' conceptualizations of gradient" (p. 6). Therefore, in this section, I will examine the research that has previously been conducted and offer justification for why there is a need to focus on middle school teachers.

Stump (1999) offered a first look into classroom teachers and their knowledge of slope. In her study, she examined secondary mathematics teachers' understanding of slope by asking all of the participants to complete an eleven item survey and conducting follow up interviews with those willing. The teachers in her study included both pre-service (18) and in-service (21) teachers. The in-service teachers' teaching experience ranged from one year to 32 years. All but one of the in-service teachers had experience teaching Algebra I in the ninth grade and the vast majority had previously taught Geometry in tenth grade and Algebra II in eleventh.

Using her seven original categories, Stump (1999) found that the average number expressed in the teachers' responses to questions "What is slope?" and "What does slope represent?" was 2.6 for pre-service teachers and 2.7 for in-service teachers. First, these results indicate the number of conceptualizations for pre and in-service teachers were nearly identical. Second, on average, teachers utilized less than three of the seven conceptualizations to define slope. Looking deeper into her results, Stump (1999) found the conceptualization that dominated the thinking of both pre and in-service teachers was geometric ratio. For pre-service teachers, the second most common conceptualization was algebraic ratio followed by functional property. None of the pre-service teachers referenced a trigonometric conceptualization and only one was credited with using a calculus conception in his or her definition of slope.

In-service teachers' results varied from their pre-service peers. Following geometric ratio, the second most commonly expressed conceptualization of slope for in-service teachers was physical property. Of the 21 in-service teachers, 18 defined slope through a geometric ratio, while 17 implied a physical property. Hence, these two conceptualizations dominated in-service teachers. The third conceptualization was algebraic ratio. Unlike pre-service teachers, where two mentioned parametric coefficient, none of the in-service teachers elected to describe slope using this conceptualization.

In examining both groups collectively, the majority of teachers did not define slope as a functional conception. Hence, this implies they choose to define slope through other mediums as opposed to the rate of change between two variables. To make sense of this discovery, Stump (1999) referenced a study by Livingston and Borko (1990) where they determined pre-service teachers were limited by a lack of connections to their content. Stump's (1999) data supported this result, specifically in the manner in which in-service teachers showed a better understanding of slope and the angle of inclination. This result may be directly related to undergraduate education programs in the United States not typically teaching trigonometry. For instance, a review of the secondary mathematics curriculum at the university where I completed this dissertation requires their mathematics education students take calculus as their first required course. Thus, a pre-service teacher's last encounter with trigonometry would most likely be while they were enrolled in high school. Stump's (1999) results support this finding since the major difference between the two groups, pre-service and in-service teachers, was their ability to solve the question involving an angle of inclination.

In Stump's (1999) survey, the eleven distinct items were designed to include a variety of representations, including presenting slope as a geometric property of a line and an angle of inclination (trigonometric concept). All teachers, both in-service and pre-service, correctly answered the questions on rate of growth and equations with numerical parameters. Given a problem involving two points, all in-service teachers were correctly able to calculate the slope while only ninety percent of the pre-service teachers were successful. When asked to define the mathematical meaning of the slope in the context of the problem, the two groups of teachers performed similarly with all but two in each group correctly completing this task.

In a separate study, Coe (2006) investigated high school teachers' thinking about rate of change. He recruited three secondary mathematics teachers. The first teacher had 22 years of experience, the second taught for 11 years, while the third taught for a total of 14 years. In the study, the teachers completed four tasks that inquired about their understanding of rate of change in the context of definitions, linear, average rate of change, and changing rate of change. His research pointed to the fact that the teachers did not make connections between the meanings of slope and constant rate of change. Furthermore, the teachers failed to make connections between their meanings of the various types of rate of change including constant, instantaneous, and average. Specifically, Coe (2006) found none of the teachers had developed a quantitative understanding of rate of change. In fact, not one of the three experienced teachers could explain the reason why division is used to calculate slope. In addition, none of the teachers considered rate in terms of accumulation, and their knowledge seemed to be based in terms of calculating the slope given selected points.

In a study by Mudaly and Moore-Russo (2011), they explored the conceptualizations of slope for a group of South African secondary mathematics teachers. The teachers who participated in their study completed questions prior to attending a professional development workshop on the topic of slope. The items analyzed for this study were designed to target each of the eleven conceptualizations addressed earlier. In total, 251 teachers completed the questionnaire.

The researchers graded each of the nine questions on a scale from zero to two. Therefore, the lowest score possible was a zero and the maximum score was 18. The average score for the 251 South African secondary mathematics teachers was 9.66. As they analyzed the data, they found that teachers' understanding of slope greatly varied. Some of the teachers demonstrated very little to no understanding. For instance, 25 of the teachers did not demonstrate knowledge of any of the conceptualizations. Other teachers demonstrated a vast knowledge of this topic, including 34 teachers that demonstrated an understanding of at least six conceptualizations of slope. Breaking down the data further, the most common number of conceptualizations demonstrated by the teachers was three, followed by two. Only one teacher held nine or more conceptualizations of slope. The most common conceptualization was parametric coefficient, followed by behavior indicator. Physical property was the least common, with only nine teachers demonstrating understanding of this topic. Of this group of teachers, only 17 represented real-world situation.

At the collegiate level, Nagle, Moore-Russo, Vigilietti & Martin (2013) considered the varying conceptualizations of slope for 65 university students and 26 professors. When asked the question “what is slope?”, the students tended to respond by evoking the behavior indicator conceptualization, while the professors referenced the functional property. Overall, the students’ answers to this questions showed evidence of 10 of the 11 conceptualizations, only omitting the real-world situation. The professors mentioned seven, omitting the determining property, linear constant, real-world situation and, the students’ most popular answer, the behavior indicator. For the instructors’ most popular, less than one fifth of the students evoked the functional property. Therefore, to describe slope, university instructors were more likely to define slope via the relationship between two co-varying quantities. This trend continued when the professors evoked the functional property the most to explain how slope is used.

Asked to list all of the ways that slope can be represented, the instructors and the students produced similar results. Neither the students nor the instructors referenced the determining property, and only the instructors referenced the functional property. Both groups represented all of the other conceptualizations.

While providing three examples of slope, the professors focused heavily on real-world situations, parametric coefficient, and the calculus conception. In fact, the real-world situation dominated their responses, doubling the frequency of the next three conceptualizations. Specifically, 62 percent of the instructors referenced it compared to 31 percent for the parametric coefficient, functional property, and the calculus conception.

2.7 Gaps in Prior Studies

First, all of the prior research on teachers' conceptualizations of slope has been conducted at the high school or collegiate level. Stump's (1999) in-service teachers ranged from ninth to twelfth grade classroom teachers. The participants in Mudaly and Moore-Russo's (2011) were tenth through twelfth grade instructors. The three teachers in Coe's (2006) study were all from the high school level. Finally, the 26 instructors in Nagle, Moore-Russo, Vigilietti and Martin's (2013) study taught at 14 different colleges. Hence, none of these studies focused on mathematics teachers at the middle school level.

The present trend in education involves requiring more students to take Algebra I, or an equivalent course, prior to entering high school. In 1990, according to the National Assessment of Educational Progress, the percentage of middle school students who sat for an advanced mathematics course such as Algebra I or Algebra II in middle school was only sixteen percent (Loveless, 2013). This percentage has continued to increase. For example, in 2000, 27 percent of eighth graders took an advanced math course, and in 2011, this percentage rose all the way to 47 percent.

Based on this trend, nearly half of all students in the United States will be exposed to the mathematical concept of slope during middle school. Therefore, it is critical to investigate the teachers' conceptual knowledge at the middle school level since it will be their task to teach the students the mathematical concept of slope. Skemp (1987) believed that once a student learned a concept, it was extremely difficult to return to the underlying concept and attempt to explore it for a deeper understanding. This belief implies it is extremely important for the student to learn it relationally the first time.

With this earlier introduction of slope, it will be essential to explore how teachers believe that slope is introduced and examined in their specific mathematics curriculum. As addressed earlier, Stanton and Moore-Russo (2012) began this process, but they only reached down to the eighth grade curriculum leaving sixth and seventh grade unrepresented. Therefore, through this study, it will be vital to examine the concept image of sixth, seventh, and eighth grade teachers. Second, there has been limited research on this topic. To date, the vast majority of the research on the topic of rate of change has been conducted from the perspective of the student (e.g., Stump, 2001; Teuscher and Reys, 2010; Turner, Wilhelm & Confrey, 2000). The studies conducted on teachers' understanding of rate have been limited to a handful of researchers such as Stump (1996, 1999), Coe (2006) and Mudaly and Moore-Russo (2011). Though each study offered vital insight, there is much more that needs to be investigated on this topic. For instance, Mudaly and Moore-Russo (2011) suggested criteria for a follow-up study such as designing questions that targeted the conceptualizations that were less prevalent in their study. To accomplish this, they recommend conducting interviews.

To address these gaps, the present study will be qualitative. I will utilize two semi-structured task-based interviews with each participant: in-service middle school mathematics teachers. The decision to select middle school teachers was directly influenced by limitations found in the review of literature. The first interview explored the teachers' concept image of slope by identifying which conceptualizations they evoke to solve each task. The second interview was designed to explore their concept image more deeply and clarify points of interest to the researcher (Thanheiser, 2009). Through an analysis of the data, the researcher will seek to answer the following research questions:

1. What is the concept definition of slope for in-service middle school mathematics teachers?
2. What conceptualizations of slope do in-service middle school mathematics teachers possess?
3. What is the concept image of slope for middle school mathematics teachers?

A detailed explanation of the methodology is addressed in the following chapter.

2.8 Concluding Remarks

In this chapter, I demonstrated the importance of this study by examining the related literature. I began the chapter with an explanation of mathematical concepts. Next, I explained the vital topics in understanding slope at the secondary level including discussing ratio and rate of change and detailed the eleven conceptualizations developed by Stump (1999, 2001) and refined and extended by Mudaly, Moore-Russo and Rugg (2011). Through the teachers' use of these conceptualizations, I will glean insight into their concept image. In addition, I addressed the prior research including both student and teacher understanding of rate of change. Finally, I explained the limitations that were found in the prior research and explained how this study will address those limitations.

In the next chapter, I will explain the research design and methodology for this particular study. This will include an explanation of grounded theory and task based interviews, which will guide the collection of data. Furthermore, I will explain how the data will be analyzed.

CHAPTER 3: RESEARCH METHODOLOGIES

The purpose of this study is to explore middle school mathematics teachers' concept image of slope. In Chapter 1, I introduced the study and presented the research questions. Chapter 2 provided an insight into the related literature and offered a rationale for the significance of this particular study. In this chapter, the research design and methodology will be addressed.

As the research indicated, there is limited knowledge on teachers' concept image of slope. Previous studies (e.g. Stump, 1996; Coe, 2006; Mudlay & Moore-Russo, 2011) have begun to examine this issue. In fact, Mudaly and Moore-Russo (2011) suggested their study provided only a first look at this topic and called for more research to be conducted. Furthermore, the focus of the previous researchers was teachers at the high school level (grades 9 - 12). During my review of literature, I did not find another study that investigated middle school teachers' understanding of slope. Therefore, this study sought to answer the following research questions:

1. What is the concept definition of slope for in-service middle school mathematics teachers?
2. What conceptualizations of slope do in-service middle school mathematics teachers possess?
3. What is the concept image of slope for middle school mathematics teachers?

This study was designed to follow the recommendations of Mudaly and Moore-Russo (2011) in which they suggested a follow-up study should be qualitative in nature and utilize interviews to collect data. The methodologies of grounded theory (Glaser & Strauss, 1967) and task-based interviews (Goldin, 2000) were implemented and are discussed in the next section.

3.1 Research Methodologies

Grounded Theory

Barney Glaser and Anselm Strauss developed grounded theory as they studied individuals dying in hospitals (Charmaz, 2006; Ezzy, 2002). The theory got its name because the study is "grounded" in the data collected and, therefore, should not be influenced by any preconceived theories (Ezzy, 2002). According to Strauss and Corbin (1990), "it makes no sense to start with received theories or variables" (p. 50) because prior theories may prevent the researcher from developing new theories.

Glaser (as cited in Ezzy, 2002) argued, "the first step in gaining theoretical sensitivity is to enter the research setting with as few predetermined ideas as possible" (p. 10). Based on this recommendation, the following study was entered into without any preconceived notions about the personal concept definition, concept image, or conceptualizations of slope middle school teachers possess. To accomplish this, the utilization of a data collection protocol was enforced allowed the participants to articulate their knowledge without being influenced by the researcher.

Task-Based Interviews

A task-based interview is developed to prompt each participant to work through specifically chosen mathematical tasks (Goldin, 2000) with the purpose of observing their mathematical behavior. This approach allowed me to make inferences about the participants' possible mathematical meaning of a given topic (Goldin, 1997). In this study, the task-based interview served two purposes. First, I explored each participant's personal concept definitions slope. I also asked the participants to define this critical term in their own words. In addition, through the use of task-based interviews, I was able observe which concept definition they used in a given context since it is possible different contexts will evoke a different concept definition. Furthermore, as Tall and Vinner (1981) emphasized, a personal concept definition and concept image can vary from person to person. Hence, I used interviews of teachers as they solve carefully designed tasks about rate to evoke their concept definitions. Using task-based interviews, in which the results were analyzed with the constant comparison method, addresses the call to go beyond quantitative results.

Tall and Vinner (1981) defined the concept definition as "the form of words to specify that concept" (p 152). Specifically, it is the form of words that the individual uses for his or her explanation of the evoked concept image. Therefore, during the analysis process of this study, I carefully studied the manner in which each teacher articulated his or her problem solving and focused on the specific terminology the teacher used. For instance, a teacher may have continually referenced the rate or the steepness of the line as they solved a variety of different problems. By carefully reviewing their words, I would be able to infer their personal concept definition.

In addition to exploring their personal concept definition, I also inferred the middle school mathematics teachers' concept images of slope. A concept image is anything that comes to mind when one hears or sees the actual mathematical concept

(Dede & Soybas, 2011). As the stimuli changes, the concept image may also change. To address this, I elected to use the 11 conceptualizations of slope that will define their concept image. For instance, a teacher's concept image may be a collection of zero to 11 conceptualizations. Thus, it is extremely important that each participant be exposed to a multitude of different tasks during the interview. Hence, I addressed this concern by designing at least one mathematical task that utilized representations that might provide opportunities for each of the eleven conceptualizations to be used (Moore-Russo, Conner and Rugg, 2011; Stump 2001). This gave me the opportunity to examine their knowledge of slope in a multitude of contexts and representations. Later in this chapter, I will explain these tasks.

As discussed by Tall and Vinner (1981), the evoked concept image and concept definition may vary. This disparity may be the direct result of solving a variety of mathematical problems. To address this issue, each participant partook in two task-based interviews that occurred within a three to five week time period (Thanheiser, 2009). Following the first interview that was designed to explore the teacher's' concept definition and concept image of slope, a second interview was conducted. This interview asked follow-up questions that were individually tailored to each of the middle school mathematics teachers. Hence, the second interview served two purposes. First, I used this interview to dive deeper into each teacher's understanding allowing me to make inferences about which conceptualizations of slope are a part of their concept image (Thanheiser, 2009). Second, I was able to determine if the results from the first interview were replicated in the second interview (Goldin, 1997). Thus, the questions in the second interview asked the teachers to restate their personal concept definition of slope and engage in more personally tailored mathematical tasks.

3.2 Selection of Participants

Qualitative research often involves purposefully selecting participants since the researcher is seeking to study a unique sample based on the purpose of the research (Fraenkel, Wallen, Hyun, 2012). In this study, middle school mathematics teachers were intentionally chosen to investigate their concept definition, concept image, and conceptualizations of slope. In the United States, middle school typically consists of students in sixth through eighth grade; the students range from approximately eleven to fourteen years old. The teachers that participated in this study teach at two different middle schools located outside of a major city in the Southeastern portion of the United States.

These two schools were specifically chosen for two reasons. One, I have a professional relationship with the principal at each school. Therefore, I was comfortable approaching each principal and asking his and her permission to conduct this study at these schools. Second, these schools were selected due to their close proximity to my home residence. By choosing two local schools, I was able to ensure I would have more flexibility in setting up meetings with the various participants. For instance, one of the teachers requested that the meeting time be changed and stated she would be free in thirty minutes. Without being in close proximity to her school, I could not have honored this request to meet on such short notice.

Description of Schools

Glesne and Peshkin (1992) advise one should not conduct research in their own school unless it is some form of action research. Based on their recommendation, this study was not conducted at my own school. Instead, two local middle school principals granted permission for interviews to be conducted at their respective schools.

The schools represent two different local school districts, located in the same city. By selecting schools from different districts, I hoped to eliminate the potential of the teachers having gone through the same district level training. Furthermore, one of the schools represented a large district, while the other represented a small district. Page Middle School (pseudonym) is a part of a large district composed of 17 elementary schools, eight middle schools, eight high schools, and two alternative schools. On the other hand, G.L. Abbott Middle School (pseudonym) is from a small district composed of three elementary (kindergarten through third grade), two intermediate (fourth through sixth), one middle (seventh and eighth) and one high school (ninth – twelfth).

The first school, Page Middle School, has approximately 726 students in grades six through eight. There are nine mathematics teachers currently teaching at the school: three teach in sixth grade, three teach in the seventh grade, and the final three teach eighth grade mathematics. On the end-of-course exam for mathematics in 2013, 48.7 percent, 60.5 percent, and 53.6 percent of the students scored at or above grade level for sixth, seventh, and eighth grade, respectively. For Math I, Common Core's equivalent for Algebra I, 76 of the 84 students scored at or above grade level (NC School Report Card, 2015).

The second school, G.L. Abbott Middle School (pseudonym), houses seventh and eighth grade. Within these two grades, there are approximately 914 students. There are eight full time mathematics teachers and an additional teacher who is split between math and science. In 2013, 52.5 percent of seventh graders scored at or above grade level on the state administered mathematics exam, while eighth graders scoring at or above grade level was slightly lower at 50.3 percent. For Math I, 170 out of 188 eighth graders scored at or above grade level (NC School Report Card, 2015).

Both schools scored above the state average percentage of 38.5 percent for seventh grade mathematics and 34.2 percent for eighth grade mathematics (NC School Report Card, 2015). Based on these results, I was optimistic the teachers at each school would have a solid understanding of the content they were teaching and therefore, would provide a high quality interview. For Math I, the state average was 36.3 percent; hence, both schools again exceeded the state average percentage. The high scores for Math I were expected since it is normally the higher achieving mathematics students who enroll in Math I at the middle school level. Those students who do not take Math 1 in middle school are required by the state where this study was conducted to take this course in high school. Therefore, high school results are usually lower than the scores at the middle school level for this course.

Selection Process

To select the participants, a meeting was conducted with the principals at each school. During these meetings, I sought permission to send out an interest email to all of their mathematics teachers. This email provided basic information about the study and asked if they would be willing to participate. Those that responded affirmatively each received a consent form (Appendix A) that details the specific purpose and obligations of participating in the study. They were asked to sign and return the consent form.

Description of Participants

In total, 17 teachers were invited to participate in this study. Of the 17, ten teachers volunteered: three from sixth grade, two from seventh grade, and five from eighth grade. The teachers have varying teaching experience levels ranging from a first year teacher to a middle school mathematics teacher who has taught for 29 years. The majority of the teachers have between 10 and 18 years of classroom experience. A complete list of the teachers organized by their current grade level, years of experience, whether they earned their master's degree and/or their National Board Certification, and if they have previously taught at the high school level is located in the table below.

TABLE 3: Information about teacher participants; names are pseudonyms

Name	Grade	Years of Experience	Earned Masters	Earned National Boards	Taught high school?
Elizabeth	6	2	No	No	No
Luke	6	10	No	No	No
Sarah	6	16	No	No	No
Brianna	7	7	Yes, in Mathematics Education	No	Yes
Angela	7	29	No	No	Yes
Carrie	8	16	No	Yes	No
Deborah	8	10	No	No	No
Jackson	8	15	No	No	No
Rachel	8	18	Yes, in Mathematics Education	Yes	Yes
Liam	8	1	No	No	No

This collection of teachers allowed me to have representation from at least two teachers in each grade level. Furthermore, within this collection of teachers, there are at least two teachers who taught high school, two teachers who earned their Masters, and two who earned their National Board certification. Based on this, I was confident that this collection of teachers was a good representation of middle school mathematics teachers in the region where this study was conducted.

3.3 Data Collection

Data were collected from two task-based interviews. The first interview was designed to be approximately one-hour long. Glense and Peshkin (1992) suggest that after an hour, both parties may begin to lose focus. All of the questions and tasks in the first interview were pre-determined. Yet, I recognize there may be a need to navigate away from the script in order to ask a clarifying or follow-up question. Therefore, the interview cannot be classified as structured and must be classified as semi-structured. In this semi-structured interview, I had the freedom to add questions or offer comments to further elicit a response. I also asked the participants to explain his or her thought process while they solved problems. Through this approach, I hoped to glean insight into their concept definition and concept image of slope.

3.4 Interview Tasks

The questions and tasks selected for the first and second interview came from five sources: the interview protocol implemented by Sheryl Stump (1996), the problem set from Mudaly and Moore-Russo's (2011) study, the *Moving Straight Ahead* Textbook (Lappan et al., 2002), the Shell Center at the University of Nottingham (2012), and the researcher. To obtain Dr. Stump's documents, an email was sent to her requesting the file to which she immediately agreed to share. The questions from Mudaly and Moore-

Russo’s (2011) study were made available in Appendix A of their study. The task from the Shell Center came from the published lesson *Interpreting Distance-Time Graphs*, while the tasks from the *Moving Straight Ahead: Linear Relationships* were found online. For the remaining tasks, I relied on my thirteen years of experience teaching mathematics at both the middle and high school level.

Collectively, the tasks were selected and/or designed to provide each teacher an opportunity to evoke each of the eleven conceptualizations. It is through these conceptualizations that I was able to determine their concept image of slope. Though it was possible that each teacher reference a different conceptualization of slope while solving a specific task, the next section describes the breakdown I used to ensure at least one problem in my interview protocol targeted each of the eleven conceptualizations. The complete protocol for the first interview can be found in Appendix C of this document, while the complete list of potential questions and tasks for the second interview are located in Appendix D.

Geometric Ratio

This conceptualization is classified by the teacher who represents slope as the expression rise over run or as the vertical change to the horizontal change. To target this conceptualization, I incorporated a specific task (Figure 2) that may prompt a teacher to evoke a geometric ratio. In this task, the teachers were presented with five unique staircases and asked to order the staircases from one to five based on which one they believe to be the easiest to climb. Though the teacher may use another conceptualization, this task was selected to provide an opportunity to use a geometric ratio to justify their order. To use this conceptualization, the teacher examines the difficulty of walking up the stairs based on examining the height of each step versus the width of the step, hence, using rise over run.

Order the staircases shown below according to how easy you think they would be to climb. Please provide a reason for your answers.

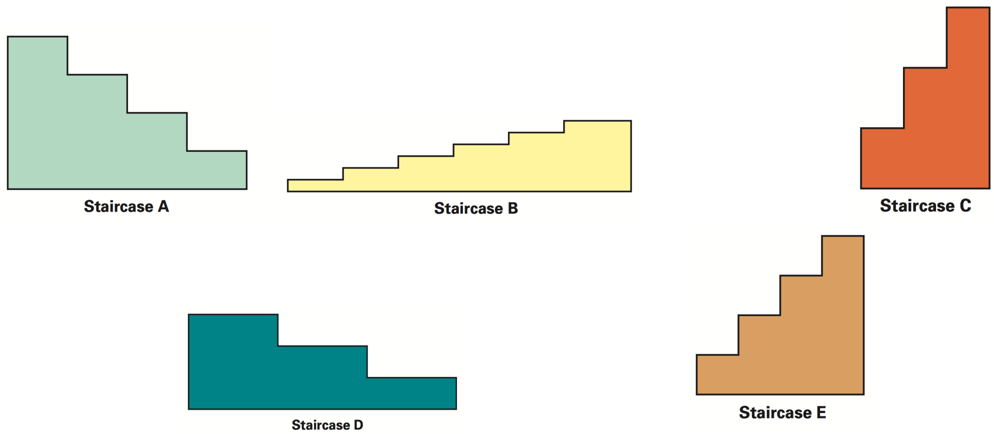


FIGURE 2: Staircases problem from first interivew protocol

Algebraic Ratio

The use of the mathematical expression "change in y over change in x" is represented in this conceptualization. Therefore, to determine if the teacher has this specific conceptualization, they must be given an opportunity to apply the expression $\frac{y_2-y_1}{x_2-x_1}$. I designed a task to target the teacher's understanding of this conceptualization. In this problem (Figure 3), the teacher was provided with three distinct tables of values and asked to determine if the function that produced the values was a linear function. Without giving the teacher a graph, he or she might be encouraged to employ this conceptualization to calculate the slope. By providing multiple points, the teacher would have to rely on this approach multiple times.

Given the following tables, please determine if any of the values were produced from a linear function?

Table A

x	F(x)
1	1
2	4
3	9
4	16

Table B

x	F(x)
1	2
2	4
3	6
4	8

Table C

x	F(x)
-2	6
-1	3
0	0
1	-3

FIGURE 3: Table task from first interview protocol

Physical Property

Evoking a physical conceptualization requires the individual to describe the line using terms such as the steepness, slant, tilt, or angle of the line. Hence, to target this conceptualization, I designed a task that would be difficult to use either the geometric or algebraic ratio. To accomplish this, I graphed two lines on different coordinate planes (Figure 4) but did not provide a scale on either the *x* or *y* axis. Without a scale, it would not be possible to extract two coordinates and use either the geometric or algebraic conceptualization. Thus, this problem encouraged the teacher to explain the difference between the two lines by describing the varying steepness, slant, tilt, or angle of the lines.

Last year, I asked my students to examine these two lines. Christine believed that the two lines have different slopes, but Ahmad disagreed. What might each student be thinking?

FIGURE 4: Two lines taks from first interview protocol

Functional Property

In this task, the teacher was asked to solve a problem that involves two variables. Thus, I chose to adapt the "Filling the Bottle" task from Carlson, Jacobs, Coe, Larsen and Hsu (2002) for the first interview. In this adaption, I picked three geometric shapes: a cylinder, a cone, and a fishbowl. The participant was instructed each object was empty but would be filled with water coming from a source at a constant rate. Their task was to match the graph that represents volume as a function of height with object (Figure 5). This problem requires the teacher to demonstrate an understanding of the rate of change between the two variables, height and time. Furthermore, this example also provides the teacher to investigate both a linear and non-linear rate of change. For the second interview, some of the teachers were asked to complete the actual task from "Filling the Bottle" (Figure 6).

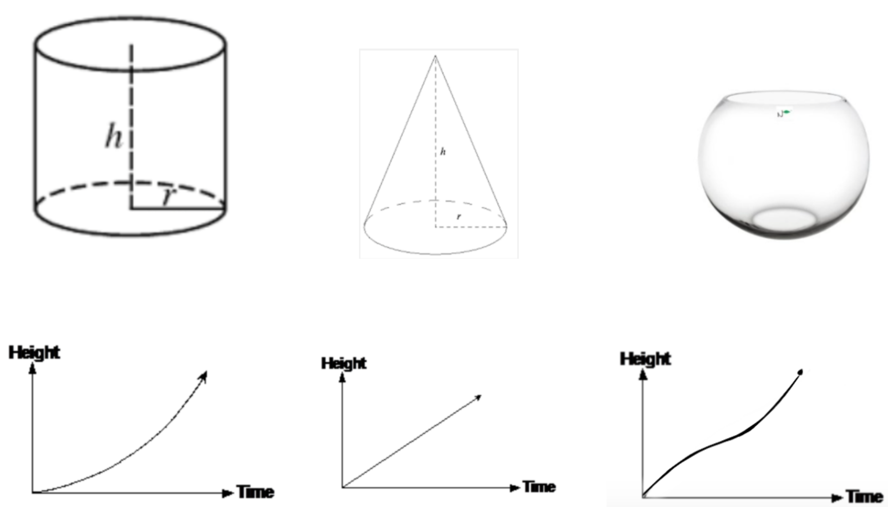


FIGURE 5: Filling mutliple container task from first interview protocol

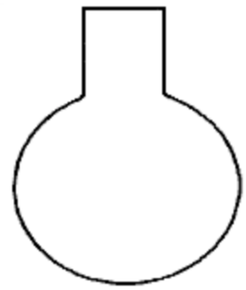


FIGURE 6: Filling a container task from second interview protocol

Parametric Conception

This conceptualization, parametric conception, is evoked when an individual references the parameter *m* in either the equation y = *m*x + b or the equation $y_2 - y_1 = m(x_2 - x_1)$. To elicit this conceptualization from the teacher, I decided to ask the participants a very direct question: Given the linear function y = ax + c, what happens to the line if the *a* value gets larger? smaller? I elected to change the letter of the parameter from *m* to *a* to avoid the potential of a teacher simply having the variable *m* memorized without understanding that the parameter is the term in front of *x.* Through this question, the teacher had the opportunity to evoke the parametric conception and was provided with the opportunity to explain how changing the parameter impacts the slope.

Trigonometric Conception

Through this conceptualization, the teacher must have the ability to understand the slope of a line when given the angle the line makes with a horizontal line. Therefore, I adapted a question from Mudaly and Moore-Russo's (2011) problem set. In this adaption, the teacher was informed that a fellow middle school teacher stated the slope of this line was 30 degrees, and I asked the teacher how he or she would respond to this teacher (Figure 7).

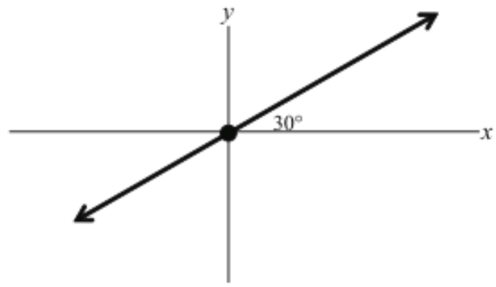


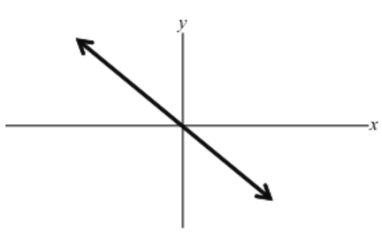
FIGURE 7: Thirty-degree angle task from first interview protocol

Calculus Conception

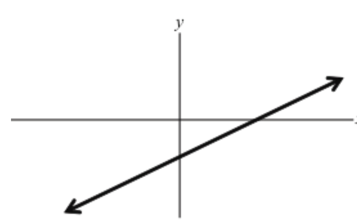
In this task, the teacher was given four functions: a line with a positive slope, a line with a negative slope, a line with a slope of zero, and a quadratic function with a positive lead coefficient (Figure 8). This task asked the teacher which function could have a slope of two (Mudlay & Moore-Russo, 2011). To correctly answer this question, the teacher must recognize the quadratic function has one x-value, where the instantaneous rate of change is positive two. Hence, their explanation to this question will provide me with insight to determine if the teacher has knowledge of instantaneous rate of change and, therefore, demonstrated a calculus conception of slope.

Explain whether any of the following functions (a - d) could have a slope of 2.

a.



b.



c. d.

FIGURE 8: Which function has a slope of two from first interview protocol

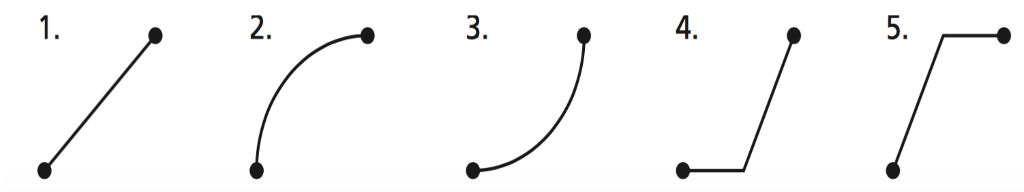
Real World Situation

This conceptualization represents either a static, physical situation or a dynamic, functional situation (Mudaly & Moore-Russo, 2011). Based on the dual nature of this conceptualization, at least two tasks were required. First, I wanted to develop a task that revolved around a linear example. Therefore, I selected the image of road sign that showed a truck going down a six percent grade (Figure 9) and asked the participants "What does the six percent mean?" As a follow-up, I asked them to explain what happens to the road as the percentage increases from six percent to 50 percent, to 100 percent, to above 100 percent.

FIGURE 9: Road sign from the first interview protocol

To evoke a dynamic, functional situation, I selected a task from the *Moving Straight Ahead* text. In this task (Figure 10), the middle school mathematics teachers were asked to match five distance versus times graph to various scenarios that describe the last five minutes of a race ran by Darren. Within the problem, the teachers had to confront both linear and nonlinear graphs, hence, targeting dynamic, functional situations.

Match the following connecting paths for the last 5 minutes of Daren's race.



a. Daren finishes running at a constant rate
b. Daren runs slowly at first and gradually increases his speed.
c. Daren runs fast and then gradually decreases his speed.
d. Daren runs very fast and reaches the finish line early.
e. After falling, Daren runs at a constant rate.

FIGURE 10: Darren problem from first interview protocol

Determining Property

To provide the teachers with an opportunity to evoke this conceptualization, I asked the teacher to define both parallel and perpendicular lines. Furthermore, I explored their understanding by asking follow up questions, such as "why does the slope of perpendicular line have to be negative reciprocals?" Moore-Russo, Connor and Rugg (2010) stated that this is the property where one understands the equation of a line can be written by knowing the slope and at least one coordinate. Hence, throughout these two interviews, the teacher was asked on multiple occasions what else would be needed to determine the equation of the line. In the second interview, I specifically ask the teacher what they need to generate an equation of a line.

Behavior Property

Under this conceptualization, an individual describes the slope by incorporating terms like decreasing, increasing, and horizontal. Hence, this conceptualization may be evoked while the teacher is solving the task described under the calculus section. The functions under A, B, and C can be described by stating each has a constant slope. Specifically, the graph under A is decreasing, B is increasing, and C is horizontal. Finally, D has at least one point on the curve where the function has all three (decreasing, increasing, and horizontal) characteristics.

Linear Constant

This conceptualization is evoked when an individual recognizes that the slope is unaffected by translation. To give the teacher an opportunity to evoke this conceptualization, I will extend the question described under parametric and will ask the teacher how changing the *c* value in the equation $y = ax+c$ impacts the slope. Through this question, the middle school mathematics teacher was given an opportunity to describe in detail why a translation does not impact the rate of change.

Multiple Conceptualizations

Finally, I incorporated tasks that were specifically designed to evoke multiple conceptualizations. For example, one task had three lines with various slopes all on the same coordinate plane and sharing a common point (Figure 11). The teacher was asked to examine the differences in the lines. By being presented with a graph, the teacher may elect to use the geometric ratio or the algebraic ratio to calculate the slope. They may also explain that the graphs have various steepness, evoking a physical conceptualization. Or, the teacher may explain the angle from the line to the x-axis varies from line to line. This reason would evoke a trigonometric conceptualization. By designing tasks that allow for multiple conceptualizations, I was able to glean which conceptualizations of slope the teachers gravitated towards.

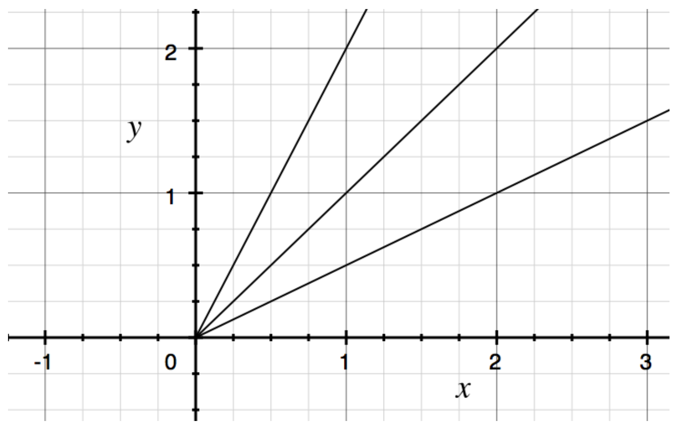


FIGURE 11: Multiple conceptualization task from second interview protocol

3.5 Data Analysis

All interviews were conducted face-to-face and recorded for both audio and video. Two cameras were used to record each interview. The focus of the first camera was on the teacher's hand. This positioning allowed for a recording of their work as they completed each task. The second camera focused on the participant. This angle captured the participants' gestures and body language that would not be visible in the first camera. To protect the teachers' anonymity, their names were changed.

The interviews took place at three locations: my office, a location at his or her school, and a local coffee shop. The individual teacher suggested all locations to ensure he or she was in a comfortable environment. Each interview was conducted at a time agreed upon by the researcher and the participant. I conducted all first round interviews, then reached out to the participants to schedule a follow up interview approximately three to five weeks later. This gap between the first and second interview was essential to allow time to analyze the data collected from the first interview and develop the interview protocol for the second interview.

Data analysis began during data collection since simultaneously collecting and analyzing data builds on the strength of qualitative methods (Ezzy, 2002). In a grounded theory approach, early data collection can be used to guide the questions asked as the research moves forward. This analysis was a critical facet of the study since the second interview will be directly influenced by the analysis of the first. Furthermore, "examining data right from the beginning of data collection for cues is what makes grounded theory grounded" (Ezzy, 2002, p. 63).

Directly following the interview, I wrote reflective notes based on the interactions. This tactic allowed reflection to occur on the current interview and aid in influencing the second interview (Johnson, 2010). Second, I watched the video and recorded my initial thoughts. Third, the interview was transcribed. This step made it possible to do a detailed reflection based on the research questions and go back and forth between the video, transcripts, and notes to develop a complete understanding of the teachers' thinking (Ezzy, 2002). Once the interview has been transcribed, my analysis continued by reading through the transcript and jotting down my initial assumptions (Maucione, 2014). These notes took the shape of a memo, or journal entry, since this encourages a systematic attempt to interpret the data being collected (Ezzy, 2002). Theories do not simply emerge; they require the researcher to continuously reflect. Furthermore, writing a memo is central to grounded theory (Ezzy, 2002).

Fourth, coding of the data was conducted. The teachers' responses and solutions to the interview questions and tasks were coded using the eleven types of conceptualizations: geometric ratio, algebraic ratio, physical property, functional property, parametric property, trigonometric conception, real-world situation, determining property, behavior property, and linear constant. Specifically, I printed up the transcript and underlined any instance when a conceptualization was evoked and wrote in the margin which conceptualization was referenced. For example, below is an excerpt from two first round interviews. First, I provided an excerpt from Jackson followed by an excerpt from Deborah.

> Jackson: Well, it depends on the type of question. Are they the same line? They're both ***positive***. You could say it that way. You could say one has a ***higher rate and the other has a lower rate of change.*** They could be different in that way. So it just depends on how you pose the question. You can see on the first one, it is a ***steeper rate of change and on the other one, it is a more lower rate of change***.
>
> Researcher: What do you mean by steeper rate of change?
>
> Jackson: ***It is going to down up higher***. ***You might have a big difference from your first one to your second one***. That is the way that I looked at it. ***The other one is not as steep as a line. If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation***.

Within this excerpt, Jackson referenced five different conceptualizations: behavioral, functional, physical, geometric, and real-world. On my copy, I underlined the distinct word or phrase and wrote the conceptualization in the margin. However, for ease of reading this document, I have used bold italics to replace my underlines. In addition, I copied the specific word or phrase and organized the results into the table below.

TABLE 4: Sample coding of Jackson

Excerpt	Conceptualization
positive	Behavioral
higher rate and the other has a lower rate of change	Functional
steeper	Physical
rate of change and on the other one, it is a more lower rate of change	Functional
It is going to down up higher	Behavioral
You might have a big difference from your first one to your second one	Geometric
The other one is not as steep as a line.	Physical
If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation	Real-World

Like Jackson, the selected excerpt from Deborah evoked multiple conceptualizations: geometric, algebraic, physical, functional, and parametric. This selection from Deborah's interview was her concluding remarks for the first interview.

> Deborah: I guess it would be what they asked me for slope in. ***If it is y=mx+b, what is slope? Well slope is the rate of change of the x***. It is how many xs do I have sometimes I would say to them. If they are looking at a graph when they ask what is the slope, ***it is the steepness of the line***. It how steep, it is how much it is changing. ***If I look at it as two points, it is the rise over the run of those two points*** or ***it is the change in y over the change in the x of those two points***. So when they ask what slope is, it depends on what we are looking at. ***If we are looking at a word problem, well then it is the rate of change***. ***If we are looking at a graph, it could be rise over run*** or ***the difference between the two points, change in y over change in x***. ***And if it is an equation, it is the coefficient of x***. So that would be my answer

TABLE 5: Sample coding of Deborah

Excerpt	Conceptualization
If it is y=mx+b, what is slope? Well slope is the rate of change of the x.	Parametric Functional
it is the steepness of the line	Physical
If I look at it as two points, it is the rise over the run of those two points	Geometric
it is the change in y over the change in the x of those two points	Algebraic
If we are looking at a word problem, well then it is the rate of change	Functional
If we are looking at a graph, it could be rise over run	Geometric
the difference between the two points, change in y over change in x	Algebraic
And if it is an equation, it is the coefficient of	Parametric

The two examples provided are a small sample of how I coded each of the interviews. Through this approach, I was able to examine which conceptualizations were incorporated and in which context.

3.6 Second Interview

Following the analysis of the first interview, a second interview was conducted three and five weeks after the first round interview. I designed this interview to further explore the teachers' understanding and to clarify points by developing tasks specifically designed for each individual teacher (Thanheiser, 2009). The tasks were selected following my detailed analysis of the first interview. Hence, the interview protocol for the second interview varied from teacher to teacher. A complete collection of the second round interview questions can be found in Appendix D of this document. To maintain consistency, the data analysis discussed for the first interview was duplicated.

For example, on the four graphs problem, Carrie was confident the increasing linear function could have a slope of two; yet, she was uncertain about the parabola. In the end, she concluded that the parabola might have a slope of two. When I asked her where it might be, she pointed to the increasing portion of the graph (Figure 12).

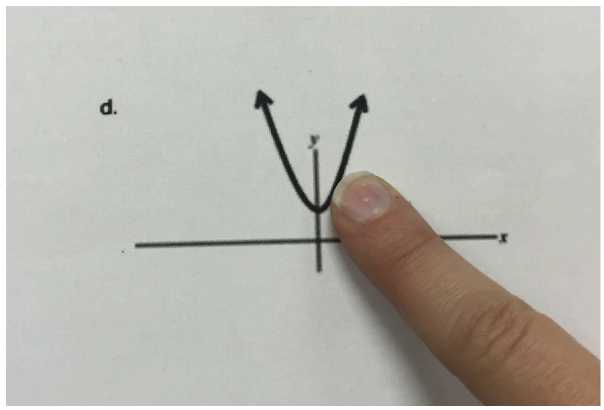


FIGURE 12: Carrie pointing to the quadratic function

During the interview, this made me believe she understood this topic. Later, I went back and watched the tape and read her words. She said:

> Carrie: "It could be right here. (Pointing to a point on the parabola to the right of zero). From that x to that x might be positive two. That would be the only part."

This made me question whether she was pointing to a single point or to range of x-values. Even in watching the video multiple times, I could not ascertain what she was implying. Therefore, in the second interview, I asked her to describe the slope of a cubic function. Though she knew the slope was not constant, when I asked her if there was anywhere on the graph she could definitely state the slope, she described the process of calculating the slope of a secant line as opposed to stating the slope of a tangent line. Revelations like this are why the second interview was so important to this study.

3.7 Grounded Theory

In the previous section, I outlined how I would analyze the data. In this section, I bring the chapter to a close by addressing how my data collection process utilized grounded theory. First, the data collection and the analysis were conducted simultaneously (Charmaz, 2011). After conducting the first interview, I began the process of data analysis. As addressed above, I initially wrote memos and coded the transcripts. As more interviews were conducted, my coding became more focused, and I was able to write advanced memos. After I completed the first round of interviews, I took the time required, three to five weeks, to thoroughly refine the theoretical concepts emerging, as these concepts directly influenced the second round interviews. As the second round interviews progressed, I continued to analyze the data by comparing the teachers' first and second round answers. For instance, in both interviews, I asked the individual to define slope. Hence, I could draw a direct comparison.

Second, I entered the study with no preconceived, logically deduced hypotheses (Charmaz, 2011). I designed the interview protocol to equally target each of the eleven conceptualizations. This decision was made to ensure that the teacher would have an opportunity to demonstrated knowledge of all eleven of the conceptualizations throughout the interview. This was critical since in this study their concept image is the total number of conceptualizations that a person has associated with slope. Third, I used the constant comparative method "which is used by the researcher to develop concepts from the data by coding and analyzing at the same time" (Kolb, 2012, p 83). Hence, as I coded, I was simultaneously analyzing the data and refining the concepts. Finally, by using grounded theory, I was able to use my analysis to identify gaps that emerged during the first interview and address those during the second interview. Without the second interview, this would not have been possible.

3.8 Validity

To achieve validity in this research study, I implemented several different strategies to establish credibility (Creswell & Miller, 2000). Specifically, triangulation, member checking, and peer debriefing were used in this study.

Creswell and Miller (2000) state, "As a validity procedure, triangulation is a step taken by researchers employing only the researcher's lens, and it is a systematic process of sorting through the data to find common themes or categories by eliminating overlapping areas" (p. 127). First, I conducted two semi-structured interviews with each participant over a three to five-week time period. During each interview, the participants were asked similar questions. This approach allowed me to directly compare their solutions to each task from interview to interview to ensure my inferences about their understanding and concept image of slope were accurate.

Second, I utilized member checking to shift the validity check from me to the participants (Creswell & Miller, 2000). After typing up and coding each interview, I sent a copy of the transcripts and coding to each participant in an email. They were asked to review so that they could "confirm the credibility of the information and narrative account" (Creswell & Miller, 2000, p. 127). Third, I discussed my results with a current mathematics teacher. Through these conversations, I began the process of justifying my inferences about the participants' understanding and concept image of slope. Finally, I shared the actual transcripts with two of my professors. Each was sent a copy of the interviews and the codes I used. This allowed me to discuss my results with them. Each of these methods discussed in this paragraph helped to remove my solo lens by encouraging others to interact with the data.

3.9 Concluding Remarks

In this chapter, I offered an explanation of the research design methodology for this study. The use of grounded theory and task-based interviews guided both the collection and analysis of data as I explored middle school teachers' personal concept definitions and their concept image of slope. In the next chapter, I will report on the results of the data that was collected via the two task-based interviews.

CHAPTER 4: RESULTS

The following chapter discloses the results of this study on middle school teachers' understanding of slope. Specifically, this chapter offers answers to the first two of the following research questions:

1. What is the concept definition of slope for in-service middle school mathematics teachers?
2. What conceptualizations of slope do in-service middle school mathematics teachers possess?
3. What is the concept image of slope for middle school mathematics teachers?

As discussed earlier in this document, the term *concept definition* was defined by Tall and Vinner (1981) as the form of words one uses to specify a concept, while the term *concept image* represents the total cognitive structure one associates with the specific concept. Hence, the concept image encapsulates all of the mental images and associated properties that revolve around the concept.

To organize and present the teachers' concept images, I will use the eleven conceptualizations of slope Stump (1996) developed and Moore-Russo, Conner and Rugg (2010) later extended and refined. Specifically, these conceptualizations are geometric ratio, algebraic ratio, physical property, functional property parametric coefficient, trigonometric conception, calculus conception, real-world situation, determining property, behavior indicator, and linear constant. Hence, their concept image is defined in this study as the total number of conceptualizations a person invokes as they work with the concept of slope as it was defined in this study. A brief explanation of each of these conceptualizations is located in the table below.

TABLE 6: List of conceptualizations of slope

Conceptualization	Slope as
Geometric Ratio	Rise over Run; ratio of vertical change to horizontal change
Algebraic Ratio	Change in y over change in x; represented by the expression change in y over change in x or $\frac{y_2-y_1}{x_2-x_1}$
Physical Property	Describing a line using expression like grade, pitch, slant, tilt, and “how high a line goes up”
Functional Property	Constant; Rate of change between variables
Parametric Coefficient	The parameter m found as a coefficient in either y = mx + b or in y - y_1 = m(x - x_1)
Trigonometric Conception	Property related to the angle a line makes with a horizontal line (normally the x-axis); tangent of a line’s angle of increase/decrease; direction component of a vector
Calculus Conception	Limit; Derivative; Tangent to the function; Instantaneous Rate of change
Real-World Situation	Physical situation (e.g. Ski slope); functional situation (e.g. distance versus time)
Determining Property	Property that determines parallel or perpendicular lines; property that can yield the equation of a line if given a point
Behavior Property	Property that indicates increasing/decreasing/horizontal trends of a line; the amount of increase/decrease of a line; property that determines that a line must intersect the x axis if the slope is a nonzero value
Linear Constant	Constant property independent of representation; property that is unaffected by translation of a line (Can be referenced as what makes a line “Straight”

4.1 Concept Definition of Slope

In this section, I explore the teachers’ concept definition of slope. This was accomplished by directly asking the teachers to define slope, asking the teachers to state what comes to mind when they hear the word slope, and analyzing the form of words they used to specify the mathematical concept of slope throughout their interviews.

Defining Slope

During the interviews, each of the middle school mathematics teachers was asked to define the word slope. Through this question, the teachers were given an opportunity to construct their own unique concept definition of slope. Their initial responses are presented in the table below.

TABLE 7: Teachers' initial concept definition of slope

Name	Grade	Concept Definition of Slope	Conceptualizations
Elizabeth	6	It is the grade of how something decreases or going down.	Physical
Luke	6	The steepness of a line. How much a line goes up or down. Positive, negative and zero slopes	Physical, Behavior
Sarah	6	The equation y = *m*x+b. The value of *m*.	Parametric
Brianna	7	Rate of change. Rise over run. Steepness of a line. Positive, negative, zero and undefined.	Functional, Geometric, Physical, Behavior
Angela	7	Rise over run. Slant of the line.	Geometric, Physical
Carrie	8	The steepness of a line	Physical
Deborah	8	Steepness. Rate of change. Snow skiing.	Physical, Functional, Real World
Jackson	8	Rate of change	Functional
Rachel	8	Steepness of a line.	Physical
Liam	8	Steepness of a line. Rise over run. Delta y over delta x	Physical, Geometric, Algebraic

Looking at initial responses, the conceptualization that dominated the teachers' responses was physical property, which was expressed by the teachers as the steepness, slant, or grade of a line. Of the ten middle school mathematics teachers who participated in this study, eight, or 80 percent, incorporated this conceptualization into their response. In addition, of these eight teachers who referenced steepness, six (Angela, Brianna, Carrie, Liam, Luke, and Rachel) focused specifically on the steepness of a line.

Angela:	I hate to say it, but rise over run. But you know it is the slant of the line.
Brianna:	Rate of change. Rise over run. Steepness of a line. Positive, negative, undefined. Zero.
Carrie:	The steepness of the line.
Liam:	It is the steepness or how far it is rise over run. Two points on that line.
Luke:	Steepness of a line. That is what I think of.
Rachel:	Like the steepness of a line.

A seventh teacher, Deborah, answered this question with the term “steepness” and no reference to a line. In her follow up interview, I asked her to define slope a second time, and she stated it was “the steepness of a line.” Hence, like the other teachers, her definition revolved around slope in a linear sense rather than extending this definition to all functions by considering the slope may vary from point to point as you move across the domain of a nonlinear function.

Following the physical property, the conceptualizations of geometric ratio and functional property were each referenced three times; the behavior property was incorporated twice, and the conceptualizations of real-world situation, parametric coefficient, and algebraic ratio were referenced once. The conceptualizations of trigonometric conception, calculus conception, determining property, and linear constant were not mentioned. The table below (Table 8) summarizes the conceptualizations brought up by the middle school mathematics teachers.

TABLE 8: Conceptualizations of teachers when asked to define slope

Geo	Alg	Phy	Funct	Para	Trig	Calc	RW	Deter	Beh	Line
3	1	7	3	1	0	0	1	0	2	0

Looking further into the initial results, it should be noted that of the three teachers who did not incorporate the physical conceptualization into their response, two were from the sixth grade, Elizabeth and Sarah. Both of these teachers' certification is kindergarten through sixth grade. Hence, at the middle school level, each of the teachers have only taught sixth grade. During their individual interviews, both teachers emphasized they have never had the opportunity to teach slope. For example, while solving one of the problems, Elizabeth stated, "I don't know how to teach slope; I've never taught it." When I asked Sarah the topics she taught, she referenced having to teach fractions, ratios, percentages, and algebra.

To confirm slope is not addressed at the sixth grade level, I located the state outlined sixth grade mathematics curriculum from the Department of Education's website, and the state does not reference slope in any capacity (North Carolina Department of Instruction, 2012). Slope is first mentioned in the seventh grade curriculum. Without having taught slope, but agreeing to participate in the study, Sarah confided in me she looked up the equation of a line prior to the interview. Thus, when asked to define slope, she referenced the formula she found on the Internet.

Hearing the Word Slope

Even though the teachers each offered own their unique definitions, it is important to remember a concept definition is the form of words one uses to explain his or her evoked concept image. Therefore, before offering my conclusion on their concept definition of slope, I examined their choice of words throughout the rest of their interviews. First, I examined their response to the specific question: “What comes to mind when you hear the word slope?” Then, I examined their responses to the various tasks they were asked to solve throughout the two interviews. From this data, I will offer my assumptions on their concept definition of slope.

During the interview, the participants were asked to articulate what images come to mind when they hear the word slope. Most of the teachers expressed more than one conceptualization for each question. A complete list of responses is organized in Table 9.

TABLE 9: Teachers’ mental images upon hearing the word slope

Name	Grade	Teacher Response	Conceptualizations
Elizabeth	6	Soil. My yard and garden.	Real-World
Luke	6	A straight line.	Linear Constant
Sarah	6	Lines on a coordinate plane. Positive and negative.	Linear Constant, Behavior
Brianna	7	A rollercoaster.	Real-World, Calculus
Angela	7	Wheelchair ramp.	Real-World
Carrie	8	The formula. Rise over run. Wheelchair ramps. Ski slopes	Geometric, Algebraic, Real-World
Deborah	8	Ski slopes.	Real-World

TABLE 9 CONTINUED

Jackson	8	Roofs on a house, planes descending, roads.	Real-World
Rachel	8	Ski slopes. Skateboard ramps.	Real-World
Liam	8	Climbing a mountain. Rise over run.	Real-World, Geometric

In their response to this question, the mean number of representations referenced by the middle schools teachers was 1.9. As shown in the figure below (Figure 13), one conceptualization dominated their thinking: real-world situation.

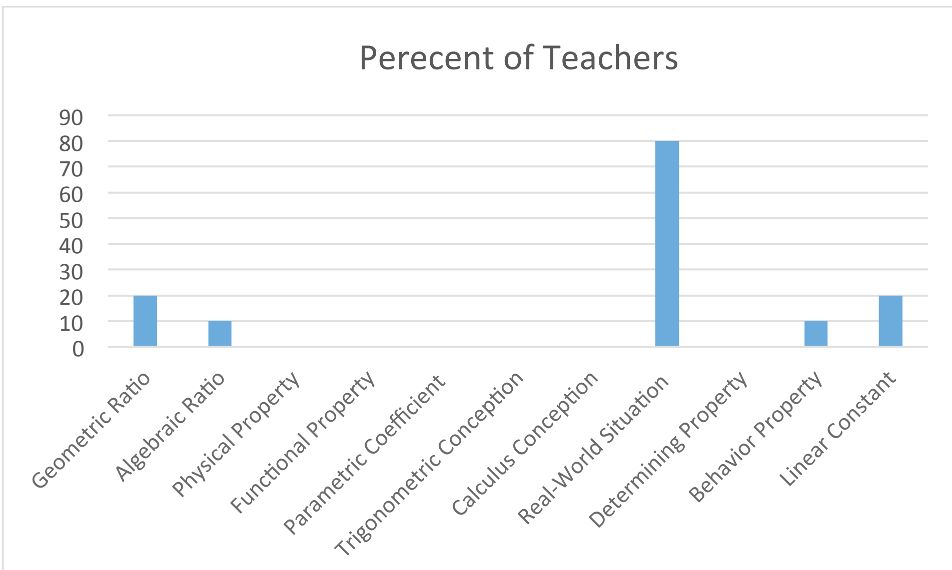


FIGURE 13: Percent of teachers that evoked a conceptualization upon hearing the word slope

Eight of the middle school mathematics teachers evoked a real-world example when asked about images that come to mind when they hear the word slope. Within these real-world examples, many of the same real-world examples were duplicated. For instance, ski slope was referenced on three occasions, while a wheelchair ramp was referenced by two of the middle school mathematics teachers. Out of the real-world examples, all were restricted to linear examples except for one teacher, Brianna, who referenced a rollercoaster. Here is how she explained the rollercoaster:

> Brianna: It (the rollercoaster) starts out at a zero slope. I always use this example with the kids. As it creeps up, cause it is going up higher, it is positive. Right before it drops and sits at the top, I tell them that at that instant pause of time, right before you go down the hill, is a zero slope. Then it goes back down, so I kinda of go through those different slopes with them. All of them expect undefined slope. Then they will tell me about the drop zone, where it takes you up and drops you straight down and I use that as the undefined slope.

Through her explanation, one can see glimmers into the calculus conceptualization. Being the only teacher to address slope in nonlinear terms, it was critical to explore her teaching background. Brianna taught high school mathematics, including teaching Advanced Placement Calculus for one year, prior to taking a teaching assignment at the seventh grade level. Yet, she never mentioned any of the keywords or phrases of a calculus conception Stump (1996) defined such as limit, derivative, or instantaneous rate of change. However, in her explanation, there is clear evidence hearing the word slope evokes her knowledge of instantaneous rate of change.

Unlike Brianna, all of the other teachers' real-world examples revolved around linear functions. For example, each of the images provided by Carrie and Jackson were rooted in the linear sense. Responding to the question: "What comes to mind when you hear the word slope?"

> Carrie: All images. I start thinking about my notebook and the formula, rise over run, the general mathematical concepts of it, then I start thinking about the pictures that I show to my students like the wheelchair ramps, and the ski slopes, and all the different visual images. So that is what comes to mind.

> Jackson: We talk about how roofs have slopes, water is positive or negative. We talk about how planes have to descend. We talk about how the

> slope of the road has to be so that water can run off. Slope is all around us.

Though Carrie says, “all images” and Jackson says, “slope is all around us,” all of their examples were based on linear terms. This idea of evoking linear functions was even more evident with two teachers, Elizabeth and Luke. Neither of these teachers referenced a real-world example. Yet, in responding to the question, both made reference to linear functions. Luke described a straight line, while Sarah went a step further and described the lines on a coordinate plane. Their responses further emphasized this collection of teachers’ understanding of slope primarily focused around linear functions, including referencing real-world examples, that are rooted in linear terms. Even Brianna, the only teacher to provide a non-linear example, referenced the steepness of a line when initially asked about slope.

Word Choice – Physical

As the teachers solved the various tasks, their word choice continued to provide insight into their collective concept definition of slope. Of the eleven conceptualizations of slope, the only conceptualization referenced by every teacher at some point during their problem solving was the physical property. Specifically, this emerged by their use of the word steep.

Throughout the tasks, the teachers were given multiple opportunities to explain how a changing variable can impact the slope. This took various forms such as increasing the value of A in the mathematical equation $y = Ax + c$, increasing the percentage of the grade of the road from six percent to 50 percent to 100 percent, and changing the angle that line intersected a horizontal line. One common theme ran through their answers. The teachers preferred to describe the change by discussing the steepness of the line. For instance, in the changing parameter problem, when the teachers were asked to explain how increasing the value of the parameter, *A*, in the equation y = *A*x+c, nine of the ten teachers responded by referencing the steepness of the line (Table 10).

TABLE 10: Teachers' responses including the physical conceptualization

Teacher	Statement
Angela	The larger the number, the more the slant it would be.
Brianna	The line gets steeper
Carrie	As A increases, your line will get steeper
Deborah	The line becomes steeper.
Jackson	If A gets bigger, it [the line] will have a steeper slope
Liam	The line gets steeper
Luke	As A gets bigger, the line gets steeper.
Sarah	It [the line] is steeper, cause A is the slope.

As these teachers were asked to explain what happens as *A* gets smaller, they continued to reference the physical property of the line. The only teacher who did not reference the steepness of the line was Elizabeth. When asked to answer this question, Elizabeth could not offer an answer. Below is the dialogue between Elizabeth and me during our first interview.

ME: So, here's a question...and this one you may not be able to answer since you have never taught slope. The equation of a line is given as...have you heard the expression y = mx+b?

Elizabeth:	Yes.
ME:	Do you know what m does?
Elizabeth:	No.
ME:	Do you know what b does?
Elizabeth:	No.
ME:	So in sixth grade, there's no slope at all?
Elizabeth:	No.

Although Elizabeth stated repeated that she never taught slope and was unable to explain how changing the parameter in the standard equation of a linear equation would impact the line, she joined the other teachers in referencing the steepness of the line in another problem. When presented with the road sign, Elizabeth discussed the steepness of the road.

Elizabeth:	The steepness is going to increase as the vehicle is going downhill and their speed is going to increase.

Elizabeth recognized as the percentage increased, the steepness of the road would also increase. The other nine teachers, who were also able to articulate the road would get steeper as the percentage increased, joined her in this revelation.

At the onset of the interview, seven of the middle school mathematics teachers elected to define slope as the steepness of a line. Yet, as they solved various problems, every teacher articulated slope via this conceptualization. In fact, many of the teachers discussed the steepness of the line on multiple occasions throughout the interview. For instance, Luke also referenced the steepness of a line while explaining how he would rank the five staircases. To attack this problem, Luke constructed a line by connecting the cusp of each stair and placed them in order based on the steepness of the line. Below is Luke's paper from his interview.

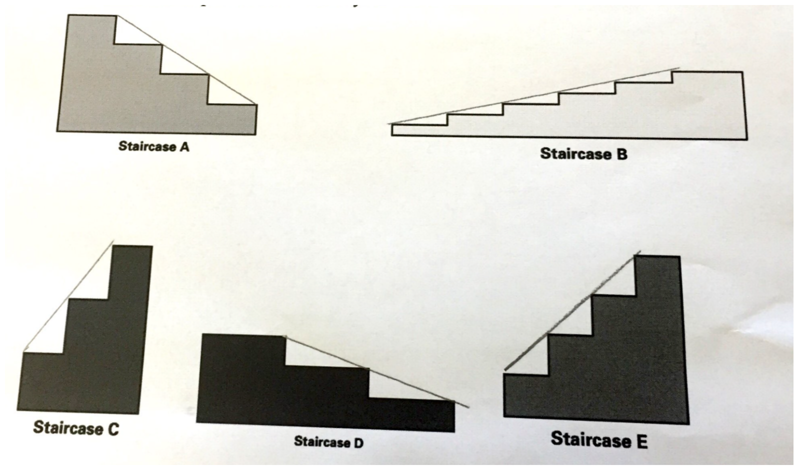


FIGURE 14: Luke's work on the staircase task

Luke described this action.

Luke:	Well, I am looking at these points (ends of stairs) here. Where I am connecting the peaks of each step. So this one is the flattest one (B) so therefore that would be the least going up the hill. The least amount overall over the longest turn, it might be longer, but I could probably handle it since I walk every night, I could probably handle that. Now, this one where you see that it is going up (C), up, it is a little further up and the increase in the last one are significantly higher. Now this one, I may switch these two around (A and E) because it is steeper. No six dozen to one half a dozen to the other. They look about the same.

Brianna also articulated the staircase problem by referencing the line one could create by connecting the peaks of the stairs. In addition, Brianna discussed the steepness of the line to explain two other problems. First, when asked to explain the difference in the two lines graphed on a coordinate plan without a scale (Question 13 in the first interview), Brianna concluded the lines were different. Her basis for this argument was that the lines had varying steepness.

ME:	How do you know that there is a different slope?
Brianna:	Because this change here (Creating a triangle on Graph 1). This triangle here is bigger than the triangle here (Graph 2). So because this line (Graph 1) is stepper or **closer to my y**

axis. This is what I tell my kids, it makes it steeper than this line which is closer to the x axis, it makes this slope less.

ME: If you were answering the question, would they be the same or different lines?

Brianna: I would tell you they are different.

ME: They are different?

Brianna: Yes, because their slopes are different.

ME: How are you making that determination?

Brianna: The steepness of the lines.

As evident in her final conclusion, Brianna placed the steepness of the line above all other reasons to conclude the slopes of the lines were different.

This trend continued when I asked Brianna how she would respond to a student or peer who stated that the slope of the line was the angle made by a line containing the origin and the x-axis.

Brianna: I would first tell them that slope is not an angle. That slope is the measurement of how steep something, a line, how steep it is. So not necessary its measurement or how wide the angle is, but basically maybe how long it is. More like the hypotenuse of this triangle (Traces the line in Q1) versus the width of the angle.

This quote from Brianna reinforces her concept definition of slope revolves around the steepness of a line. Extending this to the rest of the participants, I must conclude one facet of the concept definition of slope in this collection of teachers is the steepness of the line. As mentioned earlier, every teacher discussed the steepness during his or her problem solving at least once.

Word Choice – Constant Rate of Change

In addition to articulating slope via the physical conceptualization, the teachers' concept definition also revolved around the connection between constant rate of change and a straight line. Hence, bridging the conceptualizations of linear constant and functional property. For example, when asked to match Darren running at a constant rate to one of the five graphs, all but one of the participants immediately recognized the graph would be a linear equation. For instance, Carrie saw the word constant and immediately made the connection to a linear function. To further justify this connection, I have provided quotes from three of the teachers. Both Brianna and Luke

> Brianna: Because he is running at a constant rate. Nothing is changing. It is not saying that he is slowing down or speeds up. He is just running at a constant rate. So the constant graph is not changing.
>
> Luke: A goes with 1. It is a constant rate of change. Straight line is a constant rate.

The third teacher, Jackson, whose definition was "a rate of change," immediately said, "1 is A." After asking a follow up questions, he offered this rationale.

> ME: How did you figure that out?
> Jackson: Well, it says at a constant rate. It is just going up, they are no change.
> ME: What do you mean no change?
> Jackson: Well, he didn't fall, he didn't do anything else. It said at a constant rate, so he is going the same amount each time. So, A (the line) shows that.

The only teacher who was unable to match the scenario was Sarah. Yet, Sarah showed some understanding that a straight line was a constant rate of change. Sarah matched the graph of the straight line with D (Daren runs very fast and reaches the finish line early). She justified this decision by saying, "I read the constant rate, finishes at a constant rate. This shows the constant (Points to the positive linear portion of D)."

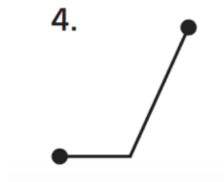


FIGURE 15: Number 4 from Darren problem

Therefore, based on the image in number 4, we can see Sarah recognized a constant rate was a straight line. She was simply unable to explain the horizontal portion of this graph.

Continuing to make the connection between constant rate of change and a linear function, all of the middle school mathematics teachers were able to explain why the rate of change of water entering a cylinder matched up with the linear equation.

TABLE 11: Teachers' quotes making connection between linear functions and constant rate of change

Teacher	Statement
Angela	This is the same size, so to me it is going to fill up like a straight line. So to me this one it is going to fill up like that.
Brianna	I am thinking that the cylinder fills up at the same rate since it is uniform. So I am thinking the first one (the line) is the cylinder. The height is the same from the bottom to the top. It is not a proportion. The base is the same on all of them in my mind but the height is the same, uniform all the way around. So, it will fill evenly. So it is not going to change or do anything like that.
Carrie	Because your circular bases, all of your bases, I believe the definition of a cylinder is a stack of cylinder bases, so that is going to be constant area that the water has to fill as it moves up the height.
Deborah	Because everything is proportion. It is round; it is going to go up the same amount. So to me the rate of change would be like that (tracing the line). You know cause it is like this (moves arms like a cylinder).

TABLE 11: CONTINUED

Elizabeth	Looking at this cylinder, I'm thinking it's going to go straight up. It's gonna slowly increase cause it's the same width across so that the water level's gonna become a continual. So it's gonna be going across the same, so it's going to be steady.
Jackson	Because it is the same (Moves hands straight up and down to demonstrate the sides of the cylinder). There is no differentiation in the sides. They are the same. The level is the same. It is going to keep filling up at a constant rate each time
Liam	Cause as you fill it, it will go up the same amount, the height and time. The shape of the object is flat on the bottom and straight on the outsides. So as you pour it, it goes up the same. It goes up the same amount; it is consistent.
Luke	Because it is the same measurement all the way down. If I put an ounce in, it is going to be equal all the way up the sides. As I continue to pour it, it is going to go at a constant rate up both sides of the object.
Sarah	I think that this one (the line) matching with the cylinder cause there is no change in the shape from the top to the bottom, so it is consistent.

Though the teachers defined slope through other means, none of the other conceptualizations were represented like physical property, linear constant, and functional property. For instance, only two teachers referenced the trigonometric conceptualization. A discussion of these conceptualizations will be discussed in the next section as I discuss the teachers' understanding of slope.

Conclusion

In conclusion, the concept definition of the middle school mathematics teachers in this study revolved around the steepness of a line and their collected understanding that a linear function is synonymous with constant rate of change. Every teacher in this study used these conceptualizations to specify the mathematical definition of slope.

4.2 Middle School Mathematics Teachers' Understanding of Slope

As indicated in Chapter 3 of this study, I selected and designed the tasks in both the first and second interview to target the eleven conceptualizations. I will be able to construct their concept image through their use of conceptualizations. As each teacher completed the collection of tasks, it became apparent the middle school mathematics teachers relied on some conceptualizations over others. In fact, some conceptualizations were used repeatedly while others were rarely, if ever, applied. This allowed me to determine which conceptualizations the teachers understood and which ones the teachers had not yet developed a mathematical understanding. In the next section, I look at each conceptualization and discuss the teachers' understanding of each.

Geometric Ratio

Though each teacher in their personal definition of slope did not cite this conceptualization, all of the teachers in this study had previously heard of the expression rise over run. Furthermore, it was applied correctly by all but two of the teachers, Elizabeth and Sarah, during the course of the interviews. For the remaining eight teachers, this conceptualization was utilized most often in three distinct scenarios. First, many of the teachers relied on this conceptualization to calculate the slope when they were provided with the graph of a line. For instance, Luke applied the geometric conceptualization to calculate the slope of each line when he was presented with the graph on a coordinate plane.

Brianna also used this conceptualization when confronted with a graph. She explained when she has "a graph, I tend to use rise over run because it is easier to see." Many of the teachers used the geometric conceptualization whether or not the graph contained a scale. For instance, Angela applied this conceptualization to argue the slopes of the two lines in Task 2 were different. While working out the problem, it was evident she projected the same scale on both graphs. Using this scale, she started at the origin and moved to another point on the linear function making comments such as "up one and over two" to apply the geometric conceptualization as a means to calculate the slope. In a similar fashion, Deborah used rise over run to calculate the slope of a line on a graph even without having a scale. When asked whether or not it was possible for a line with a positive slope to have a slope of two (Figure 16), Deborah said no. She reasoned that:

> Deborah: Even if I am just looking at this as rise over run, it [the graph] is just not proportional. To me this looks like a slope of 1/2. Up two, over 4. If I am just putting it in my own scale, it is not up two over one.

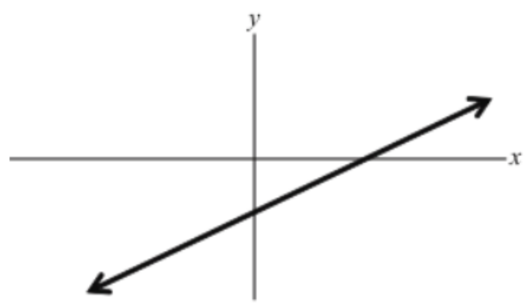


FIGURE 16: Line that Deborah was speaking about

The teachers in this study relied on their understanding of the geometric conceptualization to help them make sense of numerical values of slope that were represented in varying and unfamiliar forms. For instance, when the teachers were provided with a traffic sign that detailed the grade of the road was six percent, many of the teachers converted this percentage to a rise over run.

> Deborah: It is the rise over run of the distance of how far you are going versus the height of how far you are going.

> Carrie: A rise or decline of 6 for every 10, I mean every 100 sorry.

Rachel: I want to say it's like for every 6 feet you go down, you would go over100 feet.

Brianna: The incline of the slope of this hill is six percent. Off the ground. So, that means the height would be six, and like, if I was talking about numbers, it would be six over 100. So that would be feet, miles or whatever the units are. The height of it would be six but the horizontal bit would be 100. So that would be six percent for me, six over 100.

By transforming to a geometric conceptualization, the teachers were able to make more sense of the percentage. Other teachers tried a similar approach when they were presented with a line that intersected the x-axis at a thirty-degree angle. For instance, Liam reasoned that

Liam: The angle is thirty degrees. That means you have sixty degrees left. So you would almost go up one. It is not one half. It is not a thirty-degree slope because it is not essentially a fraction. It would be three over nine which is one third. So the whole things is ninety-degrees (Points to the right angle in the first quadrant). Thirty out of ninety is a third. So up one over three. Thirty percent, well. Thirty percent of ninety is thirty. But I don't think the slope is thirty percent. Just cause the angle is thirty percent does not mean that the slope is thirty percent. You are looking for slope. Actually the slope would be...I still say it would be one third since thirty is one third of ninety.

Like Liam, Carrie attempted a similar approach. Below is excerpt from Carrie's first interview.

Carrie: Umm, if it is a 30 degree angle then it has a 30 degree incline so basically that would be a ratio happening
ME: What ratio happening?
Carrie: A 3 to 10. A 3 to 10, if it is going up by 30 percent.
ME: Where is the 30 percent coming form?
Carrie: The 30 percent would be coming from a rise of 3 and a run of 10.

Though neither of these approaches is accurate for this specific problem, it does demonstrate these teachers were comfortable and understood the geometric conceptualization. Hence, I can conclude middle school teachers who teach seventh and eighth grade have a solid understanding of this conceptualization. As for the sixth grade teachers, neither Elizabeth nor Sarah was able to use this conceptualization to effectively calculate the slope. This is most likely the direct result of slope not being covered in their curriculum. The other sixth grade teacher, Luke, had knowledge of rise over run. Yet, it must be stated again that he has taught both seventh and eighth grade prior to moving to sixth grade.

Algebraic Ratio

Individuals who are able to accurately apply the mathematical expression change in y over change in x to calculate the slope when given two coordinates have an understanding of the algebraic ratio. Within this study, eight of the ten teachers demonstrated knowledge of this conceptualization. As with the geometric ratio discussed above, the two teachers not familiar with this conceptualization were Elizabeth and Sarah (both 6th grade teachers). All of the other teachers evoked the algebraic ratio throughout their interviews. Examining the responses of these eight teachers who used the mathematical expression change in y over change in x, a common theme emerged. This conceptualization was applied when the teachers were presented with data in a non-graphical form, such as a table or within the context of a word problem. Without having the visual stimulus of a graph, the algebraic conceptualization was applied most often. Brianna explained this sentiment when she stated, “If I do not have a graph, I tend to lean towards the formula.”

For Jackson, even having a graph evoked the use of this conceptualization. Jackson believes that change in y over change in x is a more appropriate conceptualization of slope when compared to rise over run. He argued the formula is more applicable since the mathematical formula can be used in both visual and non-visual situations. When presented with a graph, Jackson found two points and used the mathematical expression for calculating slope as opposed to counting the vertical and horizontal change from point to point. Jackson was not alone in using the formula when presented a graph; yet, he was the only teacher that never counted the rise over run to calculate the slope when presented a graph.

As with the geometric ratio, the algebraic ratio was used by the all of the seventh and eighth grade teachers. In fact, the same mathematics teachers who understood one understood the other. As I watched the teachers solve these various tasks, it became clear these two conceptualizations are intertwined. Multiple teachers referenced rise over run but used the expression change in y over change in x to calculate the rise and the run and vise versa. Carrie articulated this point during the interview.

> Carrie: When I am using rise over run, I think I am thinking about change in y over change in x. I don't really separate them very much. I feel like I am using the same method when I am using either method. To me, they are the same.

Based on this statement, it is clear why the same teachers demonstrated a dual understanding of both the algebraic and the geometric ratio. These two conceptualizations work together to allow one to calculate the slope.

Physical Property

As discussed earlier in this chapter, this conceptualization was the most common response when the middle school mathematics teachers were asked to define slope. Eight of the ten teachers elected to define slope using the phrase "the steepness of a line" as their definition of slope. Furthermore, as the teachers worked through the various tasks, all of the teachers evoked this conceptualization at least once.

The middle school mathematics teachers demonstrated a basic knowledge of the physical property, as each was able to reason that as the steepness of the line increased, the slope also increased. This reasoning allowed the teachers to make the connection that as the percentage on a road sign increased, the steepness increased. As Luke said, “The higher the grade, the steeper the road.” Yet, for some of the teachers, this logic began to collapse as the percentage approached 100. As indicated in Table 11, nine of the teachers in this study immediately described a vertical line when asked about a slope of 100 percent, while the tenth teacher, Sarah, did not offer an answer.

TABLE 12: Teachers’ descriptions of a 100 percent grade

Teacher	Response
Elizabeth	It would be straight down.
Luke	100 percent would be (Holds arm straight up and down)
Sarah	I have no idea. I am a blank on that one.
Brianna	A hundred percent incline would probably be a vertical line.
Angela	Oh my gosh. It would be, I guess just straight up and down.
Carrie	A 100 percent the truck should not be driving because it would fall off. 100 percent would technically not, almost not exist. It seems like it would be what we consider no slope, in a theoretical aspect.
Deborah	So that would be like, what would you get, would you get this (holds arm straight up and down). You’re not going to tell me. So six percent, seven percent will be steeper. Fifty percent will be steeper. A hundred percent is going to be vertical.
Jackson	Straight up and down
Rachel	A cliff. Straight down.
Liam	One hundred percent is straight down.

Of these nine teachers, three of them (Liam, Luke, and Rachel) backtracked from this initial assumption by engaging their understanding of the geometric conceptualization. Once they recognized six percent could be converted to a rise of six over a run of 100, they made the connection that 100 percent would be a rise of 100 over a run of 100 or a slope with a ratio of one to one, thus, making it non-vertical.

Sarah, who did not offer an answer, also was able to make the connection that 100 percent did not imply a vertical line. Yet, this was not done alone. Without an understanding of rise over run, I took the time during the interview to explain rise over run and how it was used to calculate the slope and then explained how to convert it to a percentage. Armed with this new information, Sarah was able to articulate the steepness of a line with percentage of 100.

ME:	What if I went all the way to 100? What would that look like?
Sarah:	Well that is not even possible, it would be here (Moves hand to indicate a vertical). Well, 100 over a 100. Now you are making me think backwards. (writes 100/) Wait, that is one to one. (Finished 100/100). But for every…(Pauses). So that would be significantly steeper, if that were even possible. I am just thinking that if it is dropping a foot every, whatever, however this is calculated, what did you say? 6 percent means 6 feet for every?
ME:	100?
Sarah:	Every 100. So if it is dropping 6 feet for every 100 feet, this one is dropping one foot for every one foot. Which is…steep.
ME:	So, 100 percent steeper?
Sarah:	Yes.
ME:	Can you go over 100 percent?
Sarah:	(Pauses) Yes.
ME:	Why do you say yes? Cause initially you said 100 was straight up and down.
Sarah:	Because you can drop two feet every feet.
ME:	So you can go over 100?
Sarah:	Yeah. Wow. I have still never seen this (indicating the road sign)

With knowledge of the geometric conceptualization, Sarah was able to determine the percentage could equal and exceed 100 percent. Yet, within her explanation, she clearly articulates herself by discussing the steepness of the line. Hence, demonstrating the physical conceptualization is part of her concept image.

Unlike Sarah and the other three teachers, the other six teachers did not make the connection to the geometric conceptualization and continued to believe a vertical line would represent a 100 percent slope. Yet, I must state I did not take the time to explain rise over run with these teachers like I did with Sarah since all of them, except Elizabeth, demonstrated knowledge of rise over run earlier in the interview. For Elizabeth, I attempted to help her determine by asking her how one would come up with six percent and she responded with

Elizabeth:	They probably measured the distance from another spot. So even if they went up the road, say, 100 yards, and measure they can see where it suddenly drops faster. And then they make the road smoother; just adjusted it.

Yet, due to time restraints, I failed to walk her through using the formula of 100 times by the rise over run.

Of these six teachers, Jackson did not believe the percentage could exceed a hundred, Elizabeth and Angela did not offer an answer, and the remaining three teachers believed as the percentage went above 100, the line would go from negative to positive. Hence, they believed zero was horizontal, between zero and 100 was negative, 100 was vertical, and over 100 went positive. This was articulate by Carrie during her interview.

ME:	How about 100 percent?
Carrie:	A 100 percent the truck should not be driving because it would fall off.
ME:	Why would it fall off?
Carrie:	(Laughs) 100 percent would technically not, almost not exist. It seems like it would be what we consider no slope, in a theoretical aspect.
ME:	So vertical?

Carrie: (NODS) Yes.
ME: Could I go over a 100 percent?
Carrie: Technically, theoretically you could. But that would just be going downhill in another positive so…
ME: What do you mean downhill?
Carrie: If I am viewing this as 50 and this (Straight up and down is a 100) then if I am going to go over 100 then I am going to be coming down.
ME: Okay, so you would be going from decreasing, to 100 percent being straight up and down to increasing as you went over a 100?
Carrie: If I am thinking about it the right way.

In a separate example, the teachers in this study continued to demonstrate an understanding of the physical property. Seven of the ten teachers referenced the steepness of a line as a reason why an individual could reach the conclusion that the two lines (Figure 17) could have different numerical slopes. These seven teachers, though they could have selected a variety of words, all elected to use the word "steep" to describe the difference in the two functions. For instance, consider Angela, Brianna, and Deborah's responses.

Angela: Because of the position the line. This one (Left) is more steep

Brianna: Because of the position the line is...this one (Left) is more steep

Deborah: I might think that it was different because this line (Left) appears to be much steeper than this line (right), so therefore they would have different slopes.

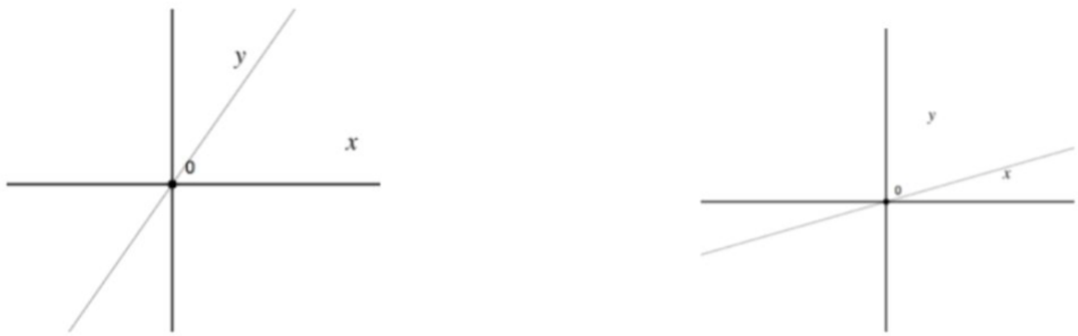


FIGURE 17: Two lines task from first interview protocol

As evident by these responses, by evoking the physical conceptualization they were able to compare the slope of two linear functions without calculating the numeric value.

This conceptualization was evoked by all of the teachers on the task involving the staircase as each discussed the “steepness” of the stairs. Yet, only half of the teachers continued to engage in this conceptualization. For those who utilized a physical conceptualization, their approach was identical. Each of the teachers drew a line connecting the tops points of each staircase and compared the steepness of each constructed line to determine the easiest and most difficult to climb. These teachers were Brianna, Deborah, Jackson, Luke, and Rachel. Luke (Figure 18) described this action: “Well, I am looking at these points here. Where I am connecting the peaks of each step.” The other four teachers provided identical reasons.

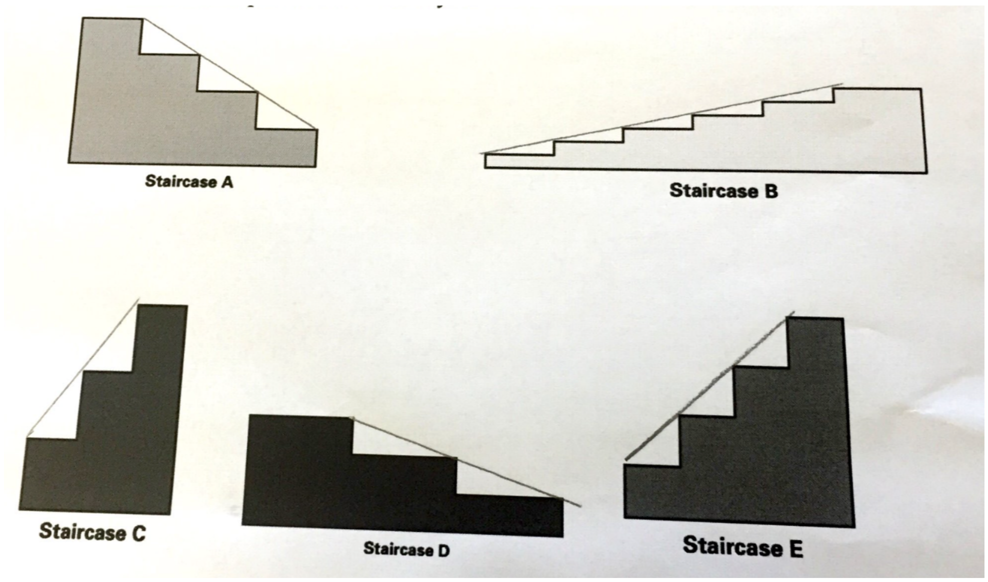


Figure 18: Luke's work on the staircase problem

The remainder of the teachers examined the staircases through a geometric conceptualization. This was accomplished by examining the rise over the run of each of the staircases. For example, Carrie stated, "E is your one to one. It looks like your rise and your run are the same." When asked how she determined the easiest to climb, Carrie continued to reference the rise over the run.

Carrie: That is simply just rise over run. So, I am comparing the height of the step with the length of the step. Cause if you are looking at this in an actual staircase, you many not have to take a big step, but you will have to take a second step, before you even get to the next step (Referring to B) which is why most people prefer E because it is just one step up at a time. Umm, so again back to your question, which one is the easiest to climb, I don't know. Are you going mathematically here? (Brief pause) I am probably going to say that B is the easiest because you are not having to exert as much force to lift your weight to the next level but I am going to feel like this (E) one is going to be the most comfortable. SO I would go that way.

From this study, it is evident that the middle school mathematics teachers have a solid understanding of the physical conceptualization of slope. All of the teachers were able to articulate the relationship between the slope and the steepness of the line. Furthermore, all of the middle school teachers used the word "steep" during their interviews.

Functional Property

Stump (1999) classified responses to slope as a rate of change between two variables as a functional property. In 2010, Moore-Russo, Conner and Rugg refined this conceptualization to only focus on constant rate of change between. Therefore, in this section, I will discuss the functional property only in terms of constant rate of change. Non-constant rate of change will be discussed in the calculus section.

During the problem solving, only two teachers continuously spoke of slope as a rate of change. These two teachers were Deborah and Jackson. For each of them, it began at the onset of the interview and carried through the duration of the interview. When asked to define slope, Jackson said slope is

> Jackson: A rate. A rate of change. That is what we talk about a lot in my classes. A lot of people think it is all about numbers, I subtract this from that. It is miles per hour. Something we see everyday. Price you pay for a vegetable. Anything at a certain rate, a cost, an amount. That is what I think it is.

Though Deborah initially defined slope as steepness, when asked to explain what she meant by the term steepness, she explained

> Deborah: How I relate it to the kids, well I was a skier, so snow skiing. How steep is the slope? Are you on a blue, a green, or a black diamond? A lot of them could ski, so they could understand that relationship. So how tough the ski slope is? But also rate of change if we are talking mathematically. And since I used to run a business, I talk to them a lot about

> how everything relates to business. So to me, it is the rate of change also. That is what we talked a lot about this year. More so the rate of change than the slope. How is it changing?

As evident in their responses, both of these teachers articulate a concept image of slope in terms of a rate of change, and this continued as they worked through the tasks. To determine the values on the table problem, Deborah justified it was linear by stating, "they go up by the same rate, so rate of change is the same. The rate is the same." In this statement, she not only spoke in terms of rate but also evoked a ratio conceptualization to confirm if the rate was the same as she moved from point to point.

Jackson continued to reference rate. For instance, on the table problem, Jackson said, "Table A, umm, it is going up at a rate of five each time." As he examined the two lines on different unmarked coordinate planes, Jackson noted, "You can see on the first one, it is a steeper rate of change and on the other one, it is a more lower rate of change." Again, Jackson discussed the slope in terms of rate. It should be noted that in his response, one could also see that Jackson evoked the physical property.

While investigating the water filling the container problem (Question #14 on the first interview), Jackson continued to articulate his concept image of slope in terms of rate. Below is an excerpt for his interview.

> Jackson: I looked at the shape of the bowl (Points to fish bowl). If I am filling this up here, it might fill up here, then go up and around (Tracing the outside of the fishbowl). This one is going to fill up the bottom then gradually come up (Pointing at the cylinder). This one is going to keep going at a different rate (The cone).
>
> ME: What do you mean by a different rate?
>
> Jackson: Well, it is going to fill up the bottom more, it will be bigger on the bottom and then smaller at the top. It is going to take

	a longer time here (pointing at the bottom of the cone). Filling up the bottom portion than the top portion.
ME:	And what about the middle one (the cylinder)?
Jackson:	It will be at a constant rate. So let me do it that way (Flips the two graphs. Correct now). Thank you.
ME:	Why do you just change two of the graphs?
Jackson:	Because it is the same (Moves hands straight up and down to demonstrate the sides of the cylinder). There is no differentiation in the sides. They are the same. The level is the same. It is going to keep filling up at a constant rate each time.

In this problem, Jackson distinguishes between "constant rate" and a "different rate." He was not alone in his ability to recognize and discuss varying versus constant rate of change. For example, in discussing the problem about students turning in their POW (Question #21 on first interview), I asked Angela if the students were turning in their assignments at a constant rate. She responded, "No. Again, I am looking at it like (Lays pencils over points). It would be a straight line. Their slope is not the same." In this example, Angela was cognizant a constant rate of change would be a straight line, hence mirroring her pencil.

To explain the difference between a line and a parabola, Carrie explained

Carrie:	Because it (the line) has a positive slope. It is the only one that has a constant positive slope. Cause there is a period in here (D) where the slope is positive two. Potentially. The slope is not constant.

From her words, she explained the slope of the parabola was not constant but recognized the line was a constant slope. Carrie continued to speak of constant rate of change when dealing with the movie ticket problem (Question #21 in the first interview).

Carrie:	Constant slope. If it were not constant, it would be steeper as the adults were going in and less steep as the kids or as seniors were coming in cause they would not be charging

as much per ticket. That would mean that the increase in the cash register would not be increasing at the same amount.

Similarly, Liam also saw the linear line modeling the amount of money in the cash register and saw constant rate of change.

Liam: Well, this one actually goes up at a constant rate. They start with x amount of money in a register. This one does not start at zero. So obviously, you have some kind of money in the register and they are increasing at a constant rate. So every single person that comes in pays the same amount, so the register increases in money. As x goes up, y goes up. The difference is the y intercept is a little higher. Obviously, if you are running a business, you want money in the register to begin with.

In fact, all of the teachers recognized in this problem the rate was constant or, as Rachel articulated, “consistent.” The excerpt from Luke’s interview encapsulates the overall viewpoint of the middle school mathematics teachers.

Luke: It looks like they charge the same rate whether you are a student or an adult.
ME: How do you know that?
Luke: Cause it is a straight line. Constant rate of change, so if it looks like it is one student say like five bucks, then two students would be ten. Three would be fifteen. Constant rate of change.

As the teachers continued to work through the tasks, a straight line prompted the teachers to think in terms of constant rate of change, while the term constant rate of change evoked a straight line. For example, when the teachers were asked to match the graph of the height of water entering a cylinder, they all recognized it would rise at a constant rate; hence, the graph would be a straight line.

Carrie: Because your circular base, all of your bases, I believe the definition of a cylinder is a stack of cylinder bases, so that is going to be constant area that the water has to fill as it moves up the height.

Deborah: Because everything is proportion. It is round, it is going to go up the same amount.

Liam: Cause as you fill it, it will go up the same amount, the height and time. The shape of the object is flat on the bottom and straight on the outsides. So as you pour it, it goes up the same. It goes up the same amount, it is consistent.

Luke: Because it is the same measurement all the way down. If I put an ounce in, it is going to be equal all the way up the sides. As I continue to pour it, it is going to go at a constant rate up both sides of the object.

Sarah: I think that this one (the line) matching with the cylinder because there is no change in the shape from the top to the bottom, so it is consistent.

Though only two of the teachers spoke about rate of change during their interviews, they were able to make the connection between constant rate of change and a linear function. The other teachers articulated their understanding through the use of other conceptualization such as physical property, geometric and algebraic ratio more often. Yet, when confronted with constant rate of change, the teachers were able to make the connection between a straight line and a constant rate of change.

Parametric Coefficient

Every middle school mathematics teacher who participated in this study was familiar with the mathematical equation of a linear function, $y = Ax + c$, which I adapted from the more standard $y = mx + b$. Even in a slightly different form, nine of the ten teachers were able to articulate the relationship between the value of parameter, A, and the slope of the line. Below are the five teachers' responses when asked what happens to the line as *A* increases and as *A* decreases.

TABLE 13: Teachers' responses to change the parameter in the equation y = mx+b

Teacher	Increase	Decrease
Carrie	As A increases, your line gets steeper.	As A decreases, your line gets flatter.
Jackson	If a gets bigger, it will have a steeper slope.	It will, less change in the line.
Liam	The line gets steeper. (Uses his arm to show steeper).	It keeps on becoming more and more horizontal instead of more and more vertical. Until you get to a zero slope which is horizontal. The less steep it gets.
Luke	As A gets bigger, the line gets steeper	The line gets flatter.
Sarah	It is steeper, cause A is the slope.	It levels out (Using her arm to indicate a horizontal line)

As evident by these teachers' responses, they recalled the relationship between a changing parameter and the slope. Also, within their responses, it should be noted all of the teachers evoked a physical conceptualization by connecting the changing parameter to the steepness of the line, directly relating back to their concept definition.

The remaining four teachers also articulated their understanding of the relationship via a physical conceptualization. Yet, unlike the previous five teachers, when asked how the value of *A* decreases, they considered that the value can enter negatives.

TABLE 14: Teachers' responses that considered negatives

Teacher	Increase	Decrease
Angela	Yeah the slope is getting larger, the line is getting steeper.	Okay, lets make it a negative. Negative x plus c and negative three x plus c. Down one over. Then if you three and it is negative three then it is (constructing graphs and talking to herself) Negative one and over one. (writes 1,1) (Talking through the problem.) mx+c. -3x+c. Okay, so as A got smaller the slope got **steeper** (comparing two lines that she drew)
Brianna	The line gets steeper	The line to me gets more flat. Closer to horizontal (Uses hand to show a line with a slope of zero). As long as you are not talking about going into negatives.
Deborah	The line becomes steeper.	The line becomes less steep. And I am assuming, we are talking about positives. If we are talking negatives, it still the steepness. Talking absolute values or whatever. (I asked her What happens if I do go negative?) It still the steepness of the line is going to change. It is still going to stay negative but as the absolute values of it becomes larger, the line will become steeper.
Rachel	It gets more steep.	It gets less steep unless you're taking about going into negatives. (I asked her What happens if I do go negative?) It goes downward. So it goes less steep and once you run into the x axis, it'll start going into a negative which means the steepness is really not…I mean if you get really steep, you'll just be going downhill, just the other way.

Like with the previous five teachers, all four of the teachers articulated the relationship of a changing parameter and the slope by discussing the steepness, hence referencing a physical conceptualization.

The only teacher who could not recall the relationship was Elizabeth. Without having taught slope, Elizabeth was not able to provide an answer to the question.

ME: So, here's a question...and this one you may...because you never taught slope. The equation of a line is given as...have you heard the expression y = mx+b?

Elizabeth: Yes.
ME: Do you know what m does?
Elizabeth: No.
ME: Do you know what b does?
Elizabeth: No.

The parametric conceptualization was never referenced by any of the participants throughout the remaining tasks. Based on this fact, I asked the following question during the second interview: How would you respond if you asked a student to define slope and she said *m*? All of the teachers responded in a very similar fashion. Below is a sample of some of their responses.

Angela: Well, it is in the equation $y = mx + b$. *m* represents the slope, but not just to say that it is the letter. I would never define slope as just *m*.

Carrie: Wow, that is like the last thing that I would think. Just because that is from the equation of a line. I would hope, and this has nothing against what anyone said, I would hope that slope, when you think of slope, you are not thinking of a formula. So for me, I hope if you asked my students what is slope, they would not say *m*.

Jackson: I would say that *m* is a letter. A symbol.

Jeff: I would probably hand it back to the student and say "Give me more detail, give me more information." Yes, you have it from a standpoint of that you know *m* is the slope in the slope-intercept form, but you did not really tell me much about what slope was.

Liam: If you ask for what slope is and they say *m*, that is just a letter. That does not really tell you the steepness of the line or anything like that.

As evident by the middle school mathematics teachers' responses, they do not believe articulating slope as the parameter in front of the variable x is a sufficient response. In each case, the teachers expressed dissatisfaction with the notion that slope can be defined by the letter. For these teachers, *m* is a symbol used to represent the slope in a specific equation, but it does not provide any insight that an individual understands what slope is. Furthermore, it shows a dependence on the formula, which is a type of understanding, instrumental, that the teachers are trying to move their students away from.

In conclusion, when the teachers were confronted with the expression y = mx+b, they were able to recall how both the variables *m* and *b* impacted the slope of the line. Yet, to explain this relationship, the middle school mathematics teachers tended to rely on an explanation rooted in the physical property.

Trigonometric Conception

According to Stump (1996), teachers who have constructed a trigonometric conceptualization of slope reference the angle a line makes with a horizontal line (normally the *x-axis).* In this study, each participant was given the opportunity to define slope in their own words and to state what hearing the word slope evokes in their thoughts, and none of the teachers made reference to the trigonometric conceptualization. As they worked through the various problems, only one teacher articulated his explanation by referencing the word. This teacher was Luke and the problem was Question 13 in the first interview which asked the teachers to explain the difference in the two lines plotted on two different coordinate planes that were not labeled.

> Luke: The coordinates would be different. They are, it would be a linear equation if you would go to solve it. But if you were looking at the, if you were doing it from a geometric standpoint where you are looking at the angles, the angles of each would be different.

I asked Luke to clarify which angle and he responded by saying:

> Luke: Here (Pointing at angle between the line and x axis) This angle here with x and y. Those measurements would be different.

Throughout the study, Luke was the only teacher to evoke this conceptualization in his problem solving. Yet, this fact does not imply that the teachers had no understanding of the trigonometric property. In fact, the middle school mathematics teachers understood changing the angle had a direct impact on the slope of the line.

> Angela: As the angle gets bigger, the slope will get larger.
>
> Liam: The less theta, the less steep the line is. The greater theta is, the larger the slope is. The larger this number is, the steeper this line will be. If the angle was one degree, it would be very level. It the angle was 89 degrees; it would be very steep.

I asked Carrie what happens as the angle goes from 30 to 40, and she responded the line "gets steeper" and "gets flatter" if the angle went from 30 to 20. To explain the relationship between the angle and the slope, she evoked a physical conceptualization.

At the beginning of the second interview, Brianna was again asked to define slope. Below is the excerpt for this portion of the interview.

> ME: How would you define slope?
>
> Brianna: I would define slope as rise over run. Like the change in y over the change in x. It's the steepness of a line. So either how far it goes up (indicates an incline with her arm), or down, or if it does it at all. It does correlate with the angle of measurement for what's happening, but it's not the angle - it's just how steep the line is.
>
> ME: And what do you mean the angle of measurement?
>
> Brianna: So if I said this was the ground (draws a horizontal line), and this was the line, or the hill that I'm talking about (draws a slanted line with a positive slope), it correlates with this angle of measurement (indicates the angle between the vertices of the two lines drawn - lower left corner). So the larger the angle is here, the bigger, or greater the slope of this line (slanted). The smaller this angle is - so if this is still the ground and this angle is more like...hmm...let's say this angle (first one drawn) is 40

degrees, and this is more like 15 degrees (second angle drawn) then I would expect the slope to be less here (second angle) and greater here (first angle), because the angle helps to justify. It correlates but it doesn't determine the slope. It just helps determine how great it is.

ME: So if I had a 30 degree angle, would that be enough information for me to determine the slope?

Brianna: No. It wouldn't. But if I had a 30 degree angle and a 50 degree angle, I should be able to tell you that the slope of the line of the 30 degree angle is smaller than the slope of the line of the 50 degree angle. I should be able to make some generalizations about it, but I cannot calculate the slope from the angle. It's just to help you see that this (the angle) is closer to the bottom, so it's (second line she drew) flatter and is going to be smaller. This one (first angle she drew) is higher and is going more vertical toward the y-axis so that slope will be greater.

It is clear from Brianna's explanation she understands how the "angle of measurement" can be used to explain the slope. Yet, there is a misconception present in her words. She argues that having a thirty-degree angle is not enough information to calculate this slope. This question was asked as a follow up to Question #15 in the first interview which asked the teachers how they would respond if another teacher stated that slope of a line with a positive slope going through the origin intersected the x-axis was 30 degrees. From her statement, Brianna does not believe one can calculate the slope with the information present. This is consistent from her first interview when she stated,

Brianna: I would first tell them that slope is not an angle. That slope is the measurement of how steep something, a line, how steep it is. So not necessarily its measurement or how wide the angle is, but basically maybe how long it is. More like the hypotenuse of this triangle (Traces the line in Q1) versus the width of the angle. So I would probably try to get them anyway from slope and angle because it [the slope] does not deal with angles. But it can but it is kinda of a stretch. I would probably try to get rid of the concept of angles because that is a whole different ballgame and try to focus them on how long it reaches (traces the x axis part of the triangle) and how high it goes (Traces the y axis part of the triangle). Using that to measure how far that line goes. That is what I would do.

Brianna was not the only teacher that struggled with the notion that the slope of a function and the angle it makes with a horizontal line were interconnected.

Jackson: I don't think an angle is going to determine the constant rate of this line. Plus, looking at the thirty here (Points to angle), the angle is going to be steeper up on this side (points to the arrow and comes down to the x-axis). So I don't think it is going to determine the constant rate of this line.

Rachel: I think I would say something like, "the slope measures the steepness so I would need a change in y over change in x." That type of value. Because degrees is what you measure a triangle in, not what you measure slope in.

Luke: Well, it is supposed to be, it is not really the angle, you can get it but it is supposed to be a, like we said, a rise over run. What is the change in the y variables or the y coordinates versus the x-coordinates. It would be more of a decimal or a fraction, not as an angle. It what, I mean. I am trying to reach back in my memory banks here. I would be more apt to say that if it is a ninety-degree angle something like this is, but for the slope of a line, I would not say to do it as an angle.

For these teachers, an angle should not be used to represent the slope of the function. Yet, as we look deeper into their reasoning, we can see this conceptualization is cast in a negative light due to their misunderstandings of angle. First, the teachers in this study struggled to take the given information of a thirty-degree and connect it to a more familiar conceptualization. For example, Liam tried to calculate the rise over the run by assuming the thirty-degree could be placed over ninety and reduced to one-third. Hence, he reasoned a line intersecting that x-axis at a thirty-degree angle had a slope equivalent to the fraction one-third. He stated, “Actually the slope would be...I still say it would be one-third since thirty is one-third of ninety.” Carrie, on the other hand, attempted to convert thirty degrees into thirty percent which she calculated as three-tenths. These teachers relied on a more familiar concept image of slope: geometric. Hence, degrees did not transform to a ratio very easily for these teachers.

The teachers expressed concern that presenting slope as a degree would prevent them from being able to generate the equation of a line. Within this equation, the slope is presented as the parameter, *m*, in the equation $y = mx+b$.

> Carrie: A lot of times, we are having to write the equation or we are having to predict the next value and 30 percent is not going to give us what we need for that.
>
> Luke: When we write it in slope intercept form, we want it in fraction or decimal form. Not as an angle.

Overall, the teachers in this study stated they had never seen the slope of a line expressed as an angle prior to agreeing to partake in this study. For example, Angela said, “I have never seen slope expressed this way,” while Deborah stated, “you can present the slope as a degree but we have not done that at all. Just being honest, I am not up to date.” Yet, even without prior experience, their concept image of slope allowed the middle school mathematics teachers to articulate a basic understanding of these conceptualizations, as all of the teachers understood the connection between the changing the measure of the angle and how this would impact the slope of the line.

Calculus Conception

According to Stump (1996), this conception makes reference to describing slope in terms of a limit, the instantaneous rate of change and/or being tangent to the curve. In fact, when asked to define a tangent line, only three of the teachers were even able to offer a definition. Carrie and Rachel, both eighth grade teachers, recalled a definition.

> Carrie: They are, oh wow, I don't know. (Laughs). Cause when you say tangent, I am thinking of a line touching a circle. I am thinking geometry not slope.

> Rachel: That is coming back from college. That is a lines that touches. I always think of a tangent line with a circle. A tangent line is a line that touches a circle at one point.

Yet each of their definitions focused on a tangent line from the geometry point of view and made no reference to slope or to the instantaneous rate of change of a function. Brianna, on the other hand, provided a definition that revolved around the notion of calculus and instantaneous rate of change.

> Brianna: A tangent line, well in my mind, it's where it touches one point on a curve. So the tangent line changes depending on where you are at on the curve. So it has a slope; it has a y-intercept; there is an equation to the line but that equation changes depending on where I am at on the curve, or circle, or something that is not linear. Umm...like when I taught it when I was in Calculus, each point had a specific tangent line. It could have similarities to another point, but because it's on a curve and it's not the same change in y/change in x because of the curve - like tangents lines have different equations.

As we examine her definition, we must remember she is the only participate in this study that taught calculus. Hence, her answers vary greatly from the two teachers who spoke about circles and the other seven teachers who could not recall a definition.

As the teachers worked through the problem, they evoked a calculus conception when confronted with functions that did not have a constant rate of change. First, the middle school teachers were able to recognize a function can have a varying rate of change as a function moves from point to point. For example, the teachers communicated this understanding on the parabola (Question #16 of the first interview)

Jackson: It is not going at a constant rate. You have some going down and going up. So, that is why I said it was not (a slope of two)

Rachel: D is a parabola, so the slope is different from one end to the other. If you start on the left and go down it's a decreasing slope which then goes to increasing when you get to the y axis.

Sarah: And then D (the parabola) is a curve so it goes negative and positive, so it cannot be consistent.

This understanding was again present as the teachers tackled the problem where water entered the cone and the fishbowl. The teachers recognized the rate of change of the water would not be constant.

Liam: It is going to go up at a decreasing rate (Pointing at the parabola). Cause after you get the bottom filled, you don't have a lot of space to fill towards the top. I am not sure that is the right one. But I will go with that anyways. I don't know if it is right.

Luke: Because, if you said this top was going to be cut off, it would be slower filling up the bottom, and then as it went up, the rate of increasing of the volume would go up. (Talking about the cylinder and the quadratic). This one, because it is smaller on the bottom, it would be at a quicker rate then when you hit the wider part where it would slow down, then go quicker as it narrow back up.

Sarah: I wanna say that this loop (the quadratic function) would match up to the cone cause in the beginning it is not really getting very far (Tracing the quadratic) because it is wider at the bottom as it smaller at the top, the height, it is going to fill up quicker. And then I would probably put this one with the fishbowl because it starts off small and that (points to the bottom of the graph) is showing where it is going pretty quick and then the graph is kinda of leveling out, that is where it (the fishbowl) is getting wider and that is going to take more time and then it gets quick again and that (points to the graph) represents the top.

Yet, many of the teachers were challenged when asked to specifically discuss the changing slope at one moment. For example, in Question #16 of the first interview, the teachers were asked whether any of the functions could have a slope of two. For the line with a positive slope, nine of the ten teachers stated they believed it could have a slope of positive two, and the only teacher who argued it could not have a slope of positive two was adamant the slope was positive one-half. For her, she was projecting a scale and the geometric conceptualization molded her answer.

In comparison, only two of the teachers believed the parabola could have a slope of positive two. These teachers were Brianna and Carrie.

Brianna: I mean, if you are going to calculus, it [the quadratic] could at a particular point like if I was trying to find the slope of the tangent line at a particular point (tracing the positive portion of the parabola) that we are talking about. But not necessarily at every point.

Carrie: Cause there is a period in here (Pointing to the graph in the first quadrant) where the slope is positive two. Potentially. The slope is not constant.

For the other teachers, they believed this function would not have a slope of positive two.

Angela: It is like an x squared (Writes x^2). We don’t deal a lot with these ones. I always tell the kids when they see the squared to know it is a parabola.

Liam:	I can't remember if technically D can have a slope. They still go up and over. It is a nonlinear line. So I would say it could not have a slope of two.
Jackson:	It is not going at a constant rate. You have some going down and going up. So, that is why I said it was not

Though not able to confirm the function had a slope of two, two of the teachers were almost able to talk themselves into believing the function could have a slope of two. First, I provide a quote from Sarah.

Sarah:	B (the line) would be the only one that could be positive two cause it is the only one that is going in a positive direction. And then D (the parabola) is a curve so it goes negative and positive, so it cannot be consistent. Well, it does end up going up. So the function could be, could be positive but you do not have it labeled so I do not know what the number are.

Next, I provide an excerpt from Rachel to demonstrate her thinking and reasoning about the question.

ME:	Does any of those graphs have a slope of a positive 2?
Rachel:	At any time?
ME:	Sure.
Rachel:	Like any little…
ME:	Sure.
Rachel:	Maybe on D but definitely B.
ME:	Definitely on B, but maybe on D. What do you mean maybe?
Rachel:	Like at a certain increment, like maybe from here to here (indicates area on graph from where x=0 to halfway up the right portion of the graph.) depending on these little…
ME:	Ok, so can the slope then change at different points?
Rachel:	Oh yeah - depending on the function. Not on a linear function, but on a function it can.
ME:	So it's possible that D could have a slope of 2?
Rachel:	At a certain little spot, yes. It doesn't look like it could, but I'm assuming it can depending on the increments as it's increasing on x and y.
ME:	So why did you say it doesn't look like it would?

Rachel:	Because it looks like it's going up rather quickly. But it could be zoomed all the way in.
ME:	So you said that it started out as a negative slope for D?
Rachel:	Well, it's going downward first, so that's a negative slope and then it increases. A parabola is going to be a U shape anyways.
ME:	So, final answer then…which of the following could have a slope of a positive 2?
Rachel:	B and D maybe.
I:	B definitely, D maybe?
Rachel:	Yes.

As evident in their responses, they both articulated the function did not have a consistent slope and might have a slope of two at a particular point.

In the follow up interview, I provided some of the teachers with a cubic function (Figure 19)

FIGURE 19: Cubic function in the second interview protocol

and asked them to describe the slope. While some of the teachers, like Angela, believed the function did not have a slope, other teachers attempted to solve the problem. For instance, Carrie discussed finding the slope.

ME:	Does that function have a slope?
Carrie:	Lots of them. (Laughs)
ME:	How so?

Carrie: Well it gets depends on your period that you are using along your x-axis. Like where along the function are you trying to determine the slope?

ME: Okay, so how would you describe the slope of that function?

Carrie: There is not a constant slope. So there is a several, an unlimited number of slopes.

ME: Is there anywhere that you can definitely say the slope is this based on that graph?

Carrie: The easiest points are the ones that have a definite integer.

ME: Why would those be the easiest?

CARRIE: The subtraction is easier. (Laughs) It is easier to do the math mentally if you are using straight integers. X is negative one to x is zero is your easiest. (Talking about calculating a secant line not a tangent line). You have another relatively easy spot from negative one to negative two.

In her approach to find the tangent line, she evoked a ratio conceptualization and calculated the secant line. She was not alone in this approach, as Liam also evoked a ratio conceptualization.

Liam: Yes, just find two points on the line itself, wherever you stop, wherever your vertices are, so you could do the top and bottom, and count rise over run. (Calculating a secant line) And she how much it increases or decreases. Still not sure if it does or not.

Brianna was the only teacher able to explain the slope of this function.

Brianna: It has slopes at points - or at sections of the graph. I would describe it as increases and decreases at phases and use that to kind of determine what the slope would kind of be or those tangent lines what kind of slopes they would have. I wouldn't necessarily be able to tell you the slope at every point because it has infinitely many points so it should have infinitely many slopes, some of them being the same. If I looked at, say, negative infinity to negative 3, that slope on that side of the graph would be positive, because it's increasing as I go from left to right. Right here across the top - I don't know what that point is called - and it may not be exactly 3. It's kind of like a little bit more like negative 2 and two thirds or something like that. And then at that point where it changes from positive to negative, it has a

> zero slope. I tell kids it's like being on a roller coaster. So you climb up the mountain, and it's that point at the top where you pause right before you go over the edge. This point where it's zero, at like -2.75 or something like that, down to this point here which I would call about positive one half - that slope would be negative because it's going down as I move from left to right. Then again where it changes from positive to negative - at this flat line that I'm drawing - it would have a zero slope. And then from that point at negative one half all the way up to positive infinity, would have a positive slope because it starts to rise again and go up and increase to the right as I move over, and so that would have a positive slope.

As the teachers worked out the various problems, it was evident this collection of middle school teachers' calculus understanding of slope revolved around the notion the slope can change and does not have to be consistent. Yet, the teachers' lack of familiarity with complex functions prevented them from being able to clearly articulate a calculus conception as described by Stump (1996). In her interview, Deborah explains how the curriculum may limit their knowledge of calculus.

> Deborah: Because we have only dealt with the linear equations, to be honest with you, and I have not dealt a lot with the, it is just something that I have not worked with. The other type of equations.

Their lack of familiarity with non-linear functions in their daily curriculum limited their knowledge to be able to accurately explain how the slope can instantaneously change from point to point as they move across a function.

Real-World Situation

In her 1999 paper, Stump added the real-world situation conceptualization to her list of seven conceptualizations individuals used to represent slope. In this, teachers use static, physical examples, such as wheelchair ramps, or dynamic, functional situations, such as distance versus time graphs, to represent slope (Moore-Russo, Conner, & Rugg, 2010). Throughout the interviews, the middle school mathematics teachers consistently related slope to real-world examples. For instance, Table 15 shows various responses when the teachers were asked “What comes to mind when you hear the word slope?”

TABLE 15: Teachers’ responses when asked “What comes to mind when you hear the word slope?”

Angela	Well I teach it going up the mountain is positive. Skiing down the mountain is negative.
Brianna	A rollercoaster
Carrie	I start thinking about the pictures that I show to my students like the wheelchair ramps, and the ski slopes, and all the different visual images. So that is what comes to mind.
Deborah	How I relate it to the kids, well I was a skier, so snow skiing. How steep is the slope? Are you on a blue, a green, or a black diamond? A lot of them could ski, so they could understand that relationship. So how tough the ski slope is?
Elizabeth	Well, I think of a map, and of landscaping.
Jackson	We talk about how roofs have slopes, water is positive or negative. We talk about how planes have to descend. We talk about how the slope of the road has to be so that water can run off. Slope is all around us.
Liam	Climbing up the mountain versus sliding down the mountain. So that would be a negative slope over a positive slope.
Rachel	I would talk about ski slopes or skateboarding or anything like that.

Of the ten teachers, only Jackson and Sarah did not make any reference to a real-world situation while answering this question. For the remaining teachers, all of the examples were physical in nature, and only Brianna provided an example outside of linear terms.

The teachers continued to evoke real-world situations to help them reason through various tasks. On problem #12 of the first interview, the teachers were asked if two lines on two unmarked coordinate planes had the same or different slopes. Four of the teachers (Angela, Brianna, Elizabeth, and Jackson) concluded the slopes of the lines were different, even though there was not enough information to calculate the actual slope. Angela and Elizabeth evoked an algebraic ratio, Brianna focused on the physical property, while Jackson focused on the rate of change and the behavior of the different lines.

> Angela: Well, they're both crossing through (0,0) at the origin. But this one here (indicates graph on left and then begins to talk through the problem by saying things like "up one over two" but does not reach a conclusion.
>
> Elizabeth: Because they're both zero for x (Referencing the origin). Um, you would think possibly, because they're both starting at the same point (the origin), but the y number is different when you do the graphing, so to me they're different numbers (Talking about a second point on the line)
>
> Brianna: Yes [they are different], because their slopes are different.
> ME: How are you making that determination?
> Brianna: The steepness of the lines.
>
> Jackson: Well, it depends on the type of question. Are they the same line? They both positive. You could say it that way. You could say one has a higher rate and the other has a lower rate of change. They could be different in that way. So it just depends on how you pose the question. You can see on the first one, it is a steeper rate of change and on the other one, it is a more lower rate of change

Yet, later in the interview, I asked them a very similar question. In this problem, the teachers were again provided with two lines on two separate graphs that were not labeled, (Question #22 on the first interview) but this task was grounded in the real-world. As the teachers read the problem, they evoked a real-world conceptualization and were able to clearly articulate why the two functions could be the same function but appear to have different slopes. In the table below, I have provided their response to each question.

TABLE 16: Teachers' responses to Question #12 and #22

Teacher	Response on Question #12	Response on Question #22
Angela	They are different because of the position the line is...this one is more steep (left), where this one is almost parallel to the x-axis I guess.	She might have been looking every five minutes (Denise) and might have been two inches and she (Takashi) might have been looking one minute. The t and v, they weren't told that they had to be the same. I just talked myself, depending on the scale for t and v, they could be the same.
Brianna	I would tell you they are different. This triangle here is bigger than the triangle here (Right). So because this line (Left) is stepper or closer to my y axis. This is what I tell my kids, it makes it steeper than this line which is closer to the x axis, it makes this slope less.	The units of time. So if I observe the graph in seconds and Denise observed in minutes, I feel like the graph would be different. There is 60 seconds in a minute, so, you have more seconds in a minute, so I would be taking more data points over that certain amount of time. So if we were both supposed to take five data points, if this one is seconds, which I think this one would be (Labeling the second graph) and this one would be in minutes (Labeling minutes on the first graph) I would see my dots closer together because I measuring them in a shorter time frame, so I would get more data points in that time frame because I have more points to measure.
Elizabeth	They are different lines. Where the endpoints are...where they end. Based on the numbers...where they align with the numbers.	The numbers they have on their graph. Say Takashi did hers by twos, and say Denise did hers by ones or even halfs. So if they used different numbers, the result would be different. They have the same information...it just depends...on the sequence of the numbers.
Jackson	It is going to down up higher. You might have a big difference from your first one to your second one. That is the way that I looked at it. The other one is not as steep as a line.	Each person validates time different.

As they reasoned through Question #12, none of these four teachers considered the scales might be different, while all of the other six teachers considered this variable as they worked out this specific question. Yet, once these four teachers evoked a real-world conceptualization by relating it to an everyday situation, they immediately considered this dynamic.

For some of the teachers, the real-world conceptualization helped them to reason and solve various problems. For instance, when I asked Angela how she knew a line with a negative slope did not have a slope of positive two, she responded by saying, “Cause it is slanting down. It has a negative slope. It is like skiing down the mountain.” In a similar fashion, Elizabeth knew the line with a negative slope could not have a slope of two because of her real-world understanding. She said

> Elizabeth: Well, I’m looking at it and I’m thinking no because I’m going from the top down to the bottom looking at the arrow, so I’m thinking this one’s negative. If I look at slope as a piece of land, this is going from my home to the bottom. So if I’m looking at it, I’m visualizing it as my landscaping.

For this collection of teachers, this conceptualization was evoked most frequently when they were attempting to explain their own reasoning or if they thought about how they would explain it to their students. I asked Deborah what she meant by steepness and she immediately began to discuss how she would explain it to her students.

> Deborah: How I relate it to the kids, well I was a skier, so snow skiing. How steep is the slope? Are you on a blue, a green, or a black diamond? A lot of them could ski, so they could understand that relationship. So how tough the ski slope is?

While varying the steepness of a line, Luke related the task to the real-world.

> Luke: Because the slope is less on. The flatter the slope, the easiest it is. Just like if you are going to walk up a hill, if you are going to walk up a mountain, that is a little bit harder. You know, like if it was more gradual, even an old guy like me can walk up a small hill

To explain a negative and a zero slope, Angela and Brianna evoked this conceptualization.

> Angela: It has a negative slope. It is like skiing down the mountain. This is a negative slope.
>
> Brianna: It is always horizontal. It is flat. It is where they walk. There is no incline. It is not the easiest part of your bike ride, if I use that analogy. There is no change in elevation or altitude when you are walking.

Jackson evoked a real-world conceptualization to explain the varying steepness of two lines.

> Jackson: The other one is not as steep as a line. If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation. The other one might be a little bit better depending how much I need. But it depends how we are looking at it.

When explaining what comes to mind when they hear the word slope, Carrie and Rachel both thought about the examples they discuss with their students.

> Carrie: I start thinking about my notebook and the formula, rise over run, the general mathematical concepts of it, then I start thinking about the pictures that I show to my students like the wheelchair ramps, and the ski slopes, and all the different visual images. So that is what comes to mind.
>
> Rachel: Like the steepness of a line. That's like a math term for me; I mean, we do it all the time. If I was talking to kids I would reference that - I would talk about ski slopes or skateboarding or anything like that, but slope to me reminds me of the steepness of a line or a path of some sort.

I asked Liam and Jackson how they would introduce slope. This question evoked a real-world conceptualization for both of them.

> Liam: Well. positive and negatives. You know Climbing up the mountain versus sliding down the mountain. So that would be a negative slope over a positive slope.

> Jackson: I really show them different videos about lines, ski slopes, roofs, miles per hour signs, price per something like a grocery ad. Those are all slopes. They are different ways to find slopes. We do talk about to find it on a graph but also how you find it in the real world.

For this collection of teachers, by evoking the real-world concept situation conceptualization, they were able to explain slope in an easier to understand manner. Collectively, they utilized a real-world conceptualization to make the mathematics more related and easier to explain. This included solving various tasks and explaining their understanding so that others can grasp.

Determining Property

During the design of this study, I generated problems that were designed to evoke the conceptualizations added to Stump's (1996, 1999) work by Moore-Russo, Connor, and Rugg (2010). The first of these conceptualizations is the determining property, which was classified as the property that determines parallel and perpendicular lines and that the equation of a line can be generated if the individual has the slope and a coordinate.

During the first interview, only two teachers made any reference to parallel lines, while none of the teachers spoke of perpendicular lines. With respect to parallel lines, Deborah and Liam used the term in different contexts. Deborah discussed a parallel line while justifying the slope of a horizontal line was zero. She explained, "But to me it looks like it is parallel to the x[axis], so if it is parallel to the x [axis] it is a horizontal line and has a slope of zero." Liam, on the other hand, evoked this conceptualization when I asked him to explain how to vary the value of *C* in the equation y = ax + *C*. He responded by informing me changing the value of c but not *A* would generate a parallel line.

With only two teachers mentioning parallel lines, I asked the teachers directly in the second interview to define parallel and perpendicular lines. A sample of the teachers' responses is located in the table below.

TABLE 17: Teachers' defining parallel and perpendicular lines

Teacher	Parallel line	Perpendicular line
Luke	Parallel lines are two lines that are on the same plane that will never intersect.	Lines that intersect at a 90 degree angle
Brianna	Parallel lines are lines that go on forever in the same direction and never touch, never intersect, never meet.	Perpendicular lines are two lines that intersect to create a 90 degree angle.
Angela	Lines where the slopes are the same.	Lines were the slopes are opposite reciprocals.
Carrie	Two lines that will never intersect	Perpendicular lines intersect at a 90-degree angle
Jackson	Lines that never intersect	The slopes are opposite flipped.

All of the teachers defined parallel lines as either lines that never intersect or lines that have the same slope. As a follow up, I asked all of the teachers who referenced the parallel lines as lines that never intersect how that related to the slope. They were all able to explain the slopes had to be equal. For perpendicular, all of the teachers, except for Elizabeth, were able to recall the relationship that exists between the slope of the two lines. It must be noted that neither Deborah nor Sarah were asked this question. Therefore, seven of the eight teachers asked could recall this relationship, but none of the middle school mathematics teachers were able to provide a rationale for why this relationship holds true. Hence, the teachers' understanding on this topic is instrumental.

Overall, this conceptualization was rarely evoked. Yet, none of the tasks asked the teachers to directly find the equation of a line given a point and a slope. I did, though, ask the teachers what they needed to calculate the equation of the line. For some of the teachers, this question evoked the algebraic ratio.

> Angela: Ordered pairs, so you would know how to graph.
>
> Brianna: I need the two ordered pairs on the line so I can determine the slope of the line, so I can determine how steep it is. And then from there I can use that and the slope to determine where it actually goes through the y-intercept and actually what the entire equation of the line is at that point.
>
> Carrie: Basically, you need the start height and the ending height. The difference between the start distance and the end distance. So a vertical and a horizontal measurement.

For others, this direct question did evoke an answer that revolved around the determining property.

> Jackson: The slope and the y – intercept.
>
> Rachel: Your slope and y-intercept. Or two ordered pairs.
>
> Liam: The slope and the y – intercept. The y-intercept is where it crosses the line. Or you could be given two points and you could calculate the slope given the two points and find the equation that way.

Overall, the middle school teachers could recall the mathematical relationship between parallel and perpendicular lines. Furthermore, the teachers had an understanding of how to calculate the equation of a line and knew the desired information required to complete this task. Yet, without a direct question about this conceptualization, the participants rarely evoked it.

Behavior Indicator

According to Moore-Russo, Conner, and Rugg (2010), this conceptualization of slope is utilized when an individual “discussed how positive, zero, and negative slopes characterized increasing, constant, and decreasing lines, respectively” (p 7). As the teachers were explaining various graphs, they evoked this conceptualization. Yet, this collection of middle school mathematics teachers was more prone to discuss the slope in terms of positive or negative as opposed to the terms increasing and decreasing. For instance, when Luke was given a decreasing line and was asked to describe the slope, he said, “it was a negative slope.” When confronted with a cubic function, Figure 20, Luke said, “Cubic. Okay. That is right it is going positive, then negative then positive.” I asked him to clarify what he meant by positive and negative, and he replied, “It is going up, when it is going down, it is going negative and when it is going back up, it is going positive.” Hence, he was describing the behavior of the cubic by utilizing the behavior indicator conceptualization.

FIGURE 20: Cubic function in the second interview protocol

In his responses, Luke did not make use of increasing or decreasing, but focused more on the positive/negative aspect of rate of change. This terminology continued for the other middle school teachers. When Angela described an upward parabola she recalled, “I know that when you have a parabola that the slope is a negative then when it goes up it is a positive.” When discussing the two lines problem, Rachel said, “Well, I’m going to say that they could be the same graphs because they both have positive slopes and they both go through the origin.” Again, like Luke and Angela, she made note of the positive nature of the slope.

On the four graphs task (question #16), the teachers were given four functions: a linear function with a positive slope, a linear function with a negative slope, a horizontal linear function and a parabola. In describing the slope of this function, the teachers referenced the behavior. In describing the decreasing line, the teachers' responses referenced a negative slope.

> Angela: Well, it has to be negative because it is going downward.
>
> Carrie: It is negative.
>
> Liam: A can't [have a positive slope] cause it has a negative slope.
>
> Luke: It is a negative slope.

For the line with the positive slope, the teachers continued to describe it by evoking a behavioral concept image and stating the slope is positive. As for the horizontal line, when describing the slope, the teachers repeatedly answered by giving the numeric value of zero. Both Carrie and Rachel stated , "it [the slope] is zero." Several teachers including Liam, Luke, and Jackson offered both the numeric value and referenced the behavior.

> Liam: C is a zero slope because it is a horizontal line.
>
> Luke: C is a zero slope. It is a flat line.
>
> Jackson: We are saying a slope of zero. It is a flat line going from left to right.

Again, it is not the word choice that is critical as Liam offered horizontal, while Luke and Jackson spoke about the line being flat. What is important is that when describing the linear function, they referenced the behavior.

Though the teachers did not use increasing, decreasing, and horizontal often, they continued to reference the behavior of the function by describing the positive, negative, and flat nature of the curves. Through these words, they were accurately able to describe the behavior of the functions. Hence, the teachers had an understanding of the behavior indicator conceptualization.

Linear Constant

Moore-Russo, Conner, and Rugg (2010) defined this conceptualization as when an individual discusses the preservation of slope when a line is persevered. In their study, this conceptualization was the least observed by the teachers. That statement holds true in this study, as the teachers made little reference to this facet of a function. In fact, the only time this conceptualization appeared to be apart of the teachers' vocabulary was when they were directly asked how changing the value of *C* in the equation y = Ax + *C* impacts the slope. For instance, when Carrie was asked how increasing the value of C alters the line, she said

> Carrie: Your y intercept increases. So you cross the y-intercept at a higher point. Or if it is a situation, you have a higher start value, or your are higher up the mountain, or you are higher up the driving board. Whatever your case may be.

As shown in her response, she was able to recall the relationship. In addition, we see as she was explaining the impact of varying *C*, she also evoked a real-world conceptualization to aid in her explanation. In fact, all but two of the teachers in this study recalled the same conclusion.

> Liam: It just changes the y-intercept. It goes up or down the y-axis. Your starting point on the y-axis.

Luke: The y variable change. (Writes down y = ax+c). As this (c) increases, my y coordinate is going to get larger. As c increases, it is just going to change my y coordinate. My slope is not going to change. The slope of the line is not going to change.

Jackson: That does not have an affect on the slope. It changes the starting point.

The two teachers who did not understand how changing the value of *C* transformed the line were Angela and Sarah. Angela reasoned as *C* gets bigger, the slope gets steeper, and as *C* gets smaller, the slope gets less steep. In her reasoning, she was evoked a ratio concept image.

Angela: Ok, so that's where we cross (drew lines on her graph). I have to draw it. So if they cross here at one, they'll cross here at two. (Draws two coordinates planes with (0,1) and (0,2) on the other). As c gets larger, the slope gets steeper.

Sarah was the other teacher who did not answer this question correctly. When asked, she did not know how either m or b impacted the line.

Overall, the teachers rarely evoked this conceptualization. It took a direct question before the teachers referenced the slope of a function was unchanged by the translation. Furthermore, they made no reference of the slope being constant if the function is moved vertically.

4.3 Concluding Remarks

In this chapter, I addressed the first two research questions. To define slope, the teachers focused on the physical property and the notion of constant rate of change. As they worked through the various tasks, the geometric and algebraic ratio came to the forefront as the preferred method to calculate the slope. When the teachers wanted to describe the slope to another person, such as a student, they tended to evoke a behavioral and/or a real-world conceptualization. The understanding of "straight" and constant were intertwined as teachers recalled a straight line would have a constant rate of change. In addition, the teachers recognized not all functions had a constant rate of change and rate of change could vary from point to point. Yet, this collection of teachers was not able to accurately explain the overall characteristics of a changing slope, and when they tried, they evoked a ratio conceptualization and attempted to calculate the slope of the secant, as opposed to the tangent line.

In the next chapter, I tackle the third research question by looking at each teacher individually to glean their concept image of slope. This will be done by looking at the total number of conceptualizations each teacher associated with slope.

CHAPTER 5: CONCEPT IMAGE

In the previous chapter, I looked at the teachers as a collective group to offer insight into the first two research questions. In this chapter, I will focus on each teacher individually in order to investigate the third research question: What is the concept image of slope for middle school mathematics teachers?

Throughout the interviews, many of the teachers demonstrated understanding of a specific conceptualization; still, they never applied it in their problem solving. For instance, take the trigonometric conception. The teachers in this study understood the relationship between changing the angle a line makes with the x-axis and the slope of that line. Yet, when solving the various tasks, a specific teacher may have never referenced this conceptualization in any instance outside of answering a direct question. Based on this, I did not consider the trigonometric conception part of the teachers' concept image.

To organize this chapter, I will begin with the teacher who had the highest number of conceptualizations in their concept image of slope and close the chapter with the teacher who represented slope through the least conceptualizations. If the teachers had the same number of conceptualizations, I organized them alphabetically. At the conclusion of this chapter, I will offer a brief summary. The next chapter offers a discussion, implications of the research, and suggestions for future research.

5.1 Brianna

Background

Brianna taught both high school and middle school mathematics. According to Brianna, she, “pretty much taught everything between seventh and twelfth grade expect for discrete. I have never taught discrete.” This included AP Calculus for one year. When she decided to make the move to G.L. Abbott Middle School, Brianna needed to take the Praxis Exam, since she was previously only certified to teach ninth through twelfth grade. Once she passed this exam, she was assigned seventh grade mathematics. At the time of her interviews, she had just completed her third year teaching at the school and her seventh overall.

Concept Image

Brianna’s knowledge of advanced mathematics, including having prior experience teaching AP calculus, helped to shape her extensive concept image of slope. In fact, Brianna represented slope via the most conceptualizations when compared to the rest of the participants. Throughout her interviews, Brianna demonstrated a deep knowledge and was able to articulate slope through both the geometric and the algebraic ratio. She used the geometric ratio when confronted with a graph and the algebraic ratio if she needed to calculate the slope from a different medium, such as a table. Brianna explained how she determines which ratio to apply.

> Brianna: So when I have a graph I tend to use rise over run because it’s easier to see. If I don’t have a graph I tend to lean towards using the formula.

This was consistent throughout her interviews. For example, on the Tom walking to the bus stop problem (second interview), which was presented as a graph, Brianna used rise over run to calculate the slope for each line segment. Yet, on the next problem, which was a word problem, Brianna used the mathematical expression for slope. Therefore, both geometric and algebraic ratio are part of her concept image.

When asked to define slope, Brianna included the steepness of a line in her definition. As she worked through the various tasks, she demonstrated the physical conceptualization was a part of her concept image. On the stairs problem, Brianna drew a line to connect the points of each stair and rank them based on the steepness. On the two lines problem, she argued they could be different "because this line (Graph 1) is stepper or closer to my y axis." As she was trying to persuade me slope could not be expressed as an angle, she again related it to the steepness of a line in saying, "the slope is the measurement of how steep something, a line, how steep it is."

To define slope, Brianna first said was it was a rate of change. As she spoke about slope, she clearly understood a straight line was a constant rate of change. With this knowledge, she was able to identify and explain why a straight line would be a constant rate of change on the Darren problem.

> Brianna: Because he is running at a constant rate. Nothing is changing. It is not saying that he is slowing down or speeds up. He is just running at a constant rate. So the constant graph is not changing.

Brianna continued to demonstrate this understanding on the movie theatre problem.

> Brianna: Because the number of customer coming to the movie theater is not going to be stacked or staying still, it will increase. Every time that I add a new customer, the money is going to increase by the same amount, the ticket price. But if we were talking about adults and children, this graph would probably not look as consistent. So you would have

> to be only taking about adults or just children. Cause if you were talking about both, that would vary your slope, but this is linear.

In her reasoning, she recognized that because the function was linear, the rate of change, or cost, would be the same for each customer, again evoking and demonstrating the functional property.

Throughout the interviews, Brianna never mentioned the parameter, m, in the equation y = mx + b unless she was asked a direct question. Therefore, the parametric coefficient was not part of her concept image of slope. In contrast, the trigonometric conception was. In her second interview, when asked to define slope, she spoke about the angle of measurement. She stated that slope "does correlate with the angle of measurement." I asked her to define the angle of measurement and she said

> Brianna: So if I said this was the ground (draws a horizontal line), and this was the line, or the hill that I'm talking about (draws an incline moving from lower left of paper to upper right), it correlates with this angle of measurement (indicates the angle between the vertices of the two lines drawn - lower left corner). So the larger the angle is here, the bigger, or greater the slope of this line (slanted). The smaller this angle is - so if this is still the ground and this angle is more like...hmm...let's say this angle (first one drawn) is 40 degrees, and this is more like 15 degrees (second angle drawn) then I would expect the slope to be less here (second angle) and greater here (first angle), because the angle helps to justify. It correlates but it doesn't determine the slope. It just helps determine how great it is.

Like Luke, Brianna evoked the trigonometric conceptualization and understood how the angle measurement was connected to the slope. Yet, her lack of trigonometric knowledge prevented her from being able to convert the angle of measurement to a ratio.

Being a former calculus teacher, Brianna understood that the rate of change could vary from point to point. This was demonstrated while she was explaining the slope of the cubic function during the second interview.

> Brianna: It has slopes at points - or at sections of the graph. I would describe it as increases and decreases at phases and use that to kind of determine what the slope would kind of be or those tangent lines what kind of slopes they would have. I wouldn't necessarily be able to tell you the slope at every point because it has infinitely many points so it should have infinitely many slopes, some of them being the same. If I looked at, say, negative infinity to negative 3, that slope on that side of the graph would be positive, because it's increasing as I go from left to right. Right here across the top - I don't know what that point is called - and it may not be exactly 3. It's kind of like a little bit more like negative 2 and two thirds or something like that. And then at that point where it changes from positive to negative, it has a zero slope. I tell kids it's like being on a roller coaster. So you climb up the mountain, and it's that point at the top where you pause right before you go over the edge. This point where it's zero, at like -2.75 or something like that, down to this point here which I would call about positive one half - that slope would be negative because it's going down as I move from left to right. Then again where it changes from positive to negative - at this flat line that I'm drawing - it would have a zero slope. And then from that point at negative one half all the way up to positive infinity, would have a positive slope because it starts to rise again and go up and increase to the right as I move over, and so that would have a positive slope.

This explanation demonstrates the calculus conception is part of her concept image. Furthermore, it shows describing slope evokes a real-life situation: the rollercoaster. Outside of this example, Brianna also spoke about riding her bike on a flat surface to explain the rate of change of a horizontal function as zero. Her ability to connect slope to real-world examples helped me to determine the real-world situation was part of her concept image.

The final conceptualization that was part of Brianna's concept image was the behavior indicator. For example, while trying to justify what happens to the line as the parameter, *A*, in the equation y = *A*x + c goes from positive to negative values, Brianna explained linear functions by discussing the trends of the line.

> Brianna: Then it kinda of changes directions. If a was two, it would be steep but it would be increasing as it goes from left to right. But if I flipped it to negative two, which means a getting smaller, it would be the same line, it would just flip directions. So instead of it increasing as it goes left to right, it would decrease as it goes left to right.

Brianna referenced the direction again to justify why the increasing line on the four-graph task could have a slope of positive two.

> Brianna: Because it is the only one going in the positive direction. The y increases the values as the x goes from left to right or smaller to larger.

Overall, Brianna represented the most conceptualizations of slope in her concept image. She was able to freely move back and forth between these conceptualizations as she worked out the different tasks. In all, Brianna's concept image contained of eight of the eleven conceptualizations: geometric ratio, algebraic ratio, physical property, functional property, trigonometric conception, calculus conception, real-world situation, and behavior indicator.

5.2 Luke

Background

For Luke, teaching is a second career. Prior to becoming a middle school mathematics teacher, Luke was a successful accountant and then started his own business. As a business owner, Luke worked nonstop and was never home with his family. He would leave before they woke up and return home well after bedtime.

Therefore, he made the decision to walk away. Luke spent the next six years as a stay at home father before a Principal friend called him and asked if he would substitute for him. Once in the classroom, Luke found another passion: teaching. He loved teaching and interacting with the students. He went back to school and earned his teaching certification in middle school mathematics. He accepted a job at Page Middle School and has been there ever since. That was ten years ago. He taught all three grade levels, but spent the last three years in sixth grade.

Concept Image

Luke's definition of slope was the steepness of a line. To demonstrate what he meant, Luke used his arm to represent a line with a positive, negative, and zero slope. Hence, he immediately focused on the behavior of the line as well as the steepness. Luke continued to reference the steepness of the line and the direction the line was traveling during his interview. For instance, when asked what happens as the parameter, m, in the equation $y = mx+b$ changes, Luke focused on the varying steepness of the line. As m gets larger, Luke stated, "the line gets steeper" and, as m approaches zero, "the line gets flatter." As Luke solved the stairs problem, he created a line by connecting the cusps of each step and ranked the staircases based on the steepness of the newly drawn lines. To Luke, the "flatter the slope, the easiest it is" to climb. While describing the four graphs problem, he described the slope of the positive line by stating he knew it was positive because that line increased and used similar phrases to described the negative and horizontal line.

Luke's concept image was not limited to just the physical property and the behavior of the line; it also included the trigonometric conceptualization. To explain the two lines problem, Luke said

> Luke: But if you were looking at the, if you were doing it from a geometric standpoint where you are looking at the angles, the angles of each would be different. Here (Pointing at angle between the line and x axis) This angle here with x and y. Those measurements would be different.

Luke was the only teacher who presented this argument to justify the two lines could have different slopes. Yet, later in the interview, Luke demonstrated his lack of trigonometric understanding as he was unable to determine how to calculate the slope of a line going through the origin that intersects the x-axis at a 30-degree angle.

> Luke: Well, it is supposed to be, it (the slope) is not really the angle, you can get it but it is supposed to be a, like we said, a rise over run. What is the change in the y variables or the y coordinates versus the x coordinates. It would be more of a decimal or a fraction, not as an angle.

In this explanation, Luke presented the argument that the slope cannot be written as an angle. Yet, based on his prior explanation, it was evident that Luke understood the angle could be used to determine two lines had varying slopes. Hence, I believe Luke's concept image contains that trigonometric conception. He merely lacks the advanced mathematics required to relate a thirty-degree angle to a ratio, which he mentions. Therefore, I contend the trigonometric conceptualization is also a part of his concept image.

To solve various problems, Luke evoked the geometric conceptualization. First, on the table problem, Luke used the rise over the run to determine if the rate of change was constant. While solving Table B, Luke spoke to himself saying, "Now, for this one (B) it is cause for every one up, I go up two. So for every one up, I go." Furthermore, in addition to using the angles on the two lines problem, Luke also used the geometric conceptualization.

Luke: If you are looking at something, the coordinates would be different. Like if I say this is over one (Drawing on the left graph) and up 2, over two and up 4. This one would be (Right graph) like over one and maybe (up) a quarter.

By calculating the rise over the run, Luke provided another reason why the two lines could have different slopes. Finally, Luke used rise over run to convert six percent into a fraction.

Luke: So for every one, maybe what you are saying is that you are dropping six inches for every 100 inches you go out.

This conceptualization allowed Luke to reason that it could have a grade of 100 percent and higher. The following excerpt demonstrates his use of this conceptualization.

Luke: 100 percent would be 1 to 1 (Total light bulb moment). So for every one, maybe what you are saying is that you are dropping six inches for every 100 inches you go out. Does that make sense?

ME: Yes.

Luke: It is taking me a while.

ME: I saw that you initially went to do a vertical line for 100 percent.

Luke: But it is not. A 1 would be over one and up one. So for every inch of fall you would have an inch of horizontal.

ME: So could you go over a hundred percent?

Luke: Well, I wouldn't advise it in a car. (Laughs)

ME: But mathematically?

Luke: Yes, I could go over one and up two. So that would be two hundred percent.

In addition to rise over run, Luke was also able to successfully evoke the algebraic ratio. He was able to solve different problems using this method. For example, in the second interview, Luke was asked to list as many reasons as he could to explain that three lines on the same coordinate plan had different slopes. His first reason was rise over run, while his second explained he could use the equation change in y over change in x. Therefore, during the interview, he demonstrated the geometric ratio is not the only ratio that is part of his current concept image of slope.

The third reason Luke gave for the lines having different slopes was that one "could create an equation. Pick two points. Get your slope. Pick a point. Generate the equation." He was the only teacher who suggested this reason, and in doing so he evoked the determining property.

Finally, Luke demonstrated the functional property. In defining rate of change, Luke said:

> Luke: Rate of change? Slope. Slope, that rate of change. You can have a constant rate of change or you can have a variable rate of change. That is one of the things that we always, you know, and in middle school we don't get into that much in depth. But in general, I always them that if it is a straight line, it is a constant. If it is a wiggly line, it is a variable rate of change
>
> .

This understanding that a straight line was a constant rate of change allowed him to explain accurately how to match the graph of the height of the water with the cylinder, identify which graph went with Darren running at a constant rate, and describe the movie theatre graph. Yet, Luke could not extend his knowledge to explain the slope of a non-linear function. When asked to explain the slope of a cubic function, he described the behavior by saying the graph "is going positive, then negative, then positive." Yet, when asked to explain the slope at a given point, Luke stated it was "beyond my reach." His response here prevents me from including the calculus conception in his concept image.

Overall, Luke showed a broad understanding of slope. His concept image included the geometric and algebraic ratio, the physical property, the functional property, the trigonometric conception, the determining property, and the behavior indicator. This suggests that Luke's conception image is composed of seven of the eleven conceptualizations.

5.3 Carrie

Background

Carrie taught eighth grade mathematics at G.L. Abbott Middle School for the past 15 years. This is the only school Carrie has ever worked at. Though Carrie has not pursued her masters, she did successfully earn her National Boards. As a student, mathematics was her favorite class. In fact, she took extremely advanced mathematics courses such as Calculus III and various engineering courses. Though she took these courses as an undergraduate, they definitely shaped her concept image.

Concept Image

From the onset of the interview, it was clear Carrie had a strong mathematics background and a solid understanding of slope. This was confirmed by two other teachers at her school who stated during their interviews they go to Carrie for advice on explaining various mathematics topics to their students. Throughout the interview, Carrie relied heavily on geometric ratio, algebraic ratio, physical property, and real-world situation. In addition, her concept image of slope included the calculus conception. Therefore, Carrie's concept image of slope was represented by five conceptualizations.

When initially asked what comes to mind when hearing the word slope, Carrie immediately thought about the mathematical formula to calculate slope, rise over run, and various real-world examples, such as a wheelchair ramp and a ski slope. As she defined slope, Carrie stated slope is "the steepness of the line, the rise over the run, how much does the y, the rise, increase or decrease, compared to the x." In this definition, Carrie again highlighted the geometric conceptualization but also pulled in the physical property.

These conceptualizations continued to emerge throughout her problem solving. On the two lines problem, she used the physical property to justify how a student could see the two lines and reason the slopes were different. As she worked on the staircase problem, Carrie referenced both a geometric ratio and the physical property.

> Carrie: Umm, the rise is greater than the run. And If I was going to put them in order of steepness, I guess technically, I am going that route if I am putting them in order of steepness

By referencing two conceptualizations, Carrie was able to validate her ranking. She duplicated this approach in discussing the road sign problem. To compare six percent to 50 percent, she first realized "50 would be one to two," and then explained that 50 percent "would be a whole lot steeper" than six percent.

During the second interview, I asked Carrie to explain all of the ways in which the three lines were different (Figure 21). First, Carrie used a combination of rise over run and the algebraic ratio to calculate the slope of each line. When asked if there was another way, Carrie also offered that the steepness of the three lines were different. Again, through this problem she demonstrated her concept image is constructed by ratio and the physical property.

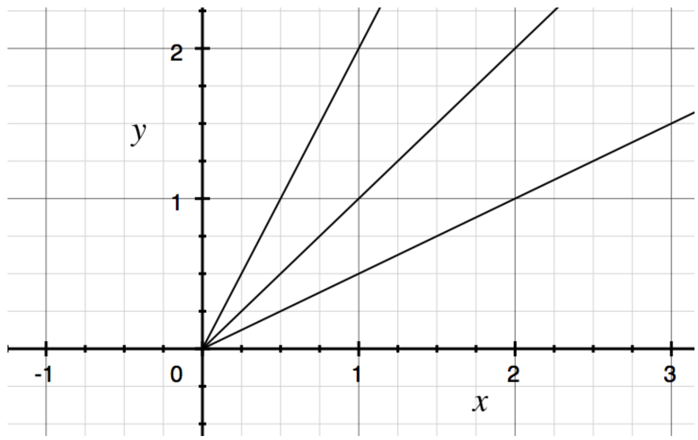


FIGURE 21: Multiple conceptualization task from second interview protocol

In tackling non-linear equations, Carrie demonstrated her concept image also included the calculus conception. Unlike most of the teachers, Carrie was able to recognize a parabola had one instance where the slope was positive two. In this example, she explained that the slope could vary from point to point. When presented with the graph of a cubic function, Carrie stated, “There is not a constant slope. So there is several, an unlimited number of slopes.”

Carrie was not able to calculate the slope of the tangent line. When asked how one would find the slope at one single value, Carrie explained how to find the slope of a secant line, evoking a ratio conceptualization. She could not recall how to transfer her knowledge of average rate of change to calculate the instantaneous rate of change at a point. Yet, it must be stressed Carrie did have a solid understanding that the slope can change from point to point, and she understood the slope can be calculated for a non-linear function.

Overall, Carrie’s concept image consisted of five conceptualizations. These were geometric ratio, algebraic ratio, physical property, calculus conception, and real-world situation. Each of these conceptualizations was used throughout the interview to solve and explain the tasks presented to her.

5.4 Deborah

Background

Teaching mathematics is a second career for Deborah. As an undergraduate, Deborah earned her degree in business and used this education to help run a family business. After she got married and started a family, Deborah decided to switch careers and become a classroom teacher. To make this transition, she went back to school and earned her master's in education. Since then, Deborah taught middle school mathematics for ten years, including spending the last seven years teaching eighth grade mathematics at Page Middle School.

Concept Image

As Deborah tackled the various tasks during her interview, it was evident her concept image was a comprised of the conceptualizations of ratio, both geometric and algebraic, the physical property, the functional property, and real-world situation.

Deborah used ratio to solve several of the problems. First, when presented a table of values, she used the mathematical formula change in y over change in x to calculate the slope. Later in the interview, she was asked if an increasing line plotted on a coordinate plane without a scale could have a slope of positive two. She responded no and offered the following justification.

> Deborah: I would say no. Even if I am just looking at this as rise over run, it (B) is just not proportional. To me this looks like a slope of 1/2. Up two, over 4. If I am just putting it in my own scale, it is not up two over one.

Hence, she tried to apply the geometric ratio even without having all of the required information to implement this specific conceptualization.

As she continued to work through the various tasks, Deborah discussed the steepness of the functions on multiple occasions. For instance, I asked Deborah how the parameter *m* in the equation y = *m*x + b changed the slope, and her answers revolved around the steepness of the line. Specifically, she said as *m* gets bigger, the "line becomes steeper," and as *m* approaches zero, "the line becomes less steep." While solving the staircase problem, Deborah ranked them based on the steepness. She explained, "It is like hiking, the steeper it is, the harder it is. The steeper it is in the shorter amount of time."

Continuing to speak in term of steepness, when asked how changing the angle the line makes with the x-axis will affect the slope, Deborah responded by saying "because if it is over here (pointing higher in quadrant 1) then it might be steeper, like a 45-degree angle. So you would have a steeper slope." When confronted with two lines in Problem #13 of the first interview, Deborah explained how they could be different.

> Deborah: I might think that it was different because this line (Left) appears to be much steeper than this line (right), so therefore they would have different slopes. So different lines.

Hence, it is clear that Deborah also thinks about slope in terms of the steepness. In fact, when I initially asked her what comes to mind when she hears the word slope, she said one word: steepness.

During the interview, Deborah explained her understanding of slope. It was in these moments she turned to real-world situations. For example, when initially asked to define slope, Deborah, like several other teachers, thought about in terms of being a classroom teacher. She explained

> Deborah: How I relate it to the kids? Well I was a skier, so snow skiing. How steep is the slope? Are you on a blue, a green, or a black diamond? A lot of them could ski, so they could understand that relationship. So how tough the ski slope is? But also rate of change if we are talking mathematically. And since I used to run a business, I talk to them a lot about how everything relates to business. So to me, it is the rate of change also. That is what we talked a lot about this year. More so the rate of change than the slope. How is it changing?

In this response, we see as soon as she thought about explaining slope to her students, she evoked personal real-world experiences: running a business and growing up as a skier. During the interview, I never asked Deborah about her students. She evoked that scenario of teaching on her own. By describing real-world situations to her students, she believed the students would better understand the complexity that is slope. Evoking a real-world example continued when she was asked to define the mathematical term “rate of change.”

> Deborah: Umm, we define rate of change, like with pizza and the pizza costs ten bucks and the toppings are $1.50. How many are you going to buy is your rate of change. If you are going to buy three times a $1.50 plus the ten dollars. The rate of change is how many toppings are you going to have. Things like that. We try to really bring it back to things, like in a food store. Things that they bought. Rate of change is the cost of something How much is it going to cost each time that I do it. So that is what rate of change is

Again, we can see she is thinking about explaining it to her students. It is in these moments that Deborah tries to relate her explanation to a real-world example.

The final conceptualization that made up Deborah’s concept image of slope was the functional property. This caused her to think about slope as a rate of change. Yet, according to Deborah, this was new for her this year due to the implementation of the Common Core curriculum. For example, after using a ratio to determine if the coordinates provided in each table were produced from a linear equation, Deborah explained her solutions by discussing the rate of change.

> Deborah: Xs going up by one and ys going up by two (referencing B). As long as they go up by the same rate, so rate of change is the same. The rate is the same. For this one (A), it (the xs) are going up by one then the ys go up by three and then five and then seven. So as soon as that was different, I eliminated that

Deborah continued to talk about the rate of change while explaining the rate of change of water entering the cylinder.

> Deborah: Because everything is proportion. It is round, it is going to go up the same amount. So to me the rate of change would be like that (tracing the line). You know cause it is like this (moves arms like a cylinder).

This idea of thinking of rate extended when she explained the money versus tickets sold at a theater.

> Deborah: To me because it is going up at a constant rate, it is like six dollars per ticket. So if I have ten customers versus two customers (Points to two different points on the graph). I would have an equal amount. The slope is the same. That is how I determined it.

At the conclusion of the interview, I asked Deborah how she would respond if a student asked, "what is slope?"

> Deborah: I guess it would be what they asked me for slope in. If it is y=mx+b, what is slope? Well slope is the rate of change of the x. It is how many xs do I have sometimes I would say to them. If they are looking at a graph when they ask what is the slope, it is the steepness of the line. It how steep, it is how much it is changing. If I look at it as two points, it is the rise over the run of those two points or it is the change in y over the change in the x of those two points. So when they ask what slope is, it depends on what we are looking

> at. If we are looking at a word problem, well then it is the rate of change. If we are looking at a graph, it could be rise over run or the difference between the two points, change in y over change in x. And if it is an equation, it is the coefficient of x. So that would be my answer.

From these responses, we can infer her concept image of slope is comprised of multiple conceptualizations. Even though she mentions parametric, she never made reference to the parameter *m* throughout her individual problem solving. Based on her previous responses, it is somewhat surprisingly she did not bring in a real world example, since this question specifically asked her about explaining it to a student. Yet, it was evident this conceptualization was part of her problem solving based on repeated reference to real-world examples throughout her interview. Thus, as mentioned at the onset of this section, Deborah's concept image is comprised of ratio, both geometric and algebraic, the physical property, the functional property, and real-world situation.

5.5 Jackson

Background

Jackson has been teaching eighth grade mathematics at G.L. Abbott Middle School for the past 16 years. As an undergraduate, Jackson elected to major in elementary education where he earned his Kindergarten through sixth grade license. After moving from another state, Jackson accepted a sixth grade teaching position at the school. After teaching sixth grade for five years, the district made the decision to restructure how the grades were organized, and G.L. Abbott Middle School became the home for seventh and eighth grade students only. Jackson did not want to leave the school; hence, he took the Praxis, and earned his sixth through ninth grade certification. For the past 11 years, Jackson has been teaching eighth grade mathematics.

Concept Image

Jackson's concept image of slope contains five conceptualizations: algebraic ratio, physical property, functional property, real-world situation, and behavioral indicator. As Jackson was defining slope, he defined it as "a rate. A rate of change." Throughout the interview, Jackson discussed the rate while solving various problems. For instance, when presented with the graph of a line, Jackson immediately described the graph as having a constant rate of change. To explain how the water filled up the different shapes, Jackson's explanation addressed the fact that the cylinder would fill up at a "constant rate," while the cone would fill up as at a "different rate." Below is an excerpt from his transcript.

ME: Why did you match them that way?

Jackson: I looked at the shape of the bowl (Points to fish bowl). If I am filling this up here, it might fill up here, then go up and around (Tracing the outside of the fishbowl). This one is going to fill up the bottom then gradually come up (Pointing at the cylinder). This one is going to keep going at a different rate (The cone).

ME: What do you mean by a different rate?

Jackson: Well, it is going to fill up the bottom more, it will be bigger on the bottom and then smaller at the top. It is going to take a longer time here (pointing at the bottom of the cone). Filling up the bottom portion than the top portion.

ME: And what about the middle one (the cylinder)?

Jackson: It will be at a constant rate. So let me do it that way (Flips the two graphs. Correct now). Thank you.

Once Jackson explained the water entering the containers by evoking the functional property, he was able to correctly match the graphs to the containers. The use of this conceptualization continued as he evaluated the tables and compared the two graphs. To determine if the values from the table were from a linear function, Jackson stated he checked each table to make sure the rate of change was a constant. As he compared the two lines on the different un-scaled coordinate planes, Jackson said

> Jackson: You could say one has a higher rate and the other has a lower rate of change. They could be different in that way. So it just depends on how you pose the question. You can see on the first one, it is a steeper rate of change and on the other one, it is a more lower rate of change.

In this response, he stressed the two lines appear to have a different rate of change. As a follow up, I asked him what he meant by a steeper rate of change. To clarify, he evoked a physical conceptualization.

> Jackson: It is going to down up higher. You might have a big difference from your first one to your second one. That is the way that I looked at it. The other one is not as steep as a line. If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation. The other one might be a little bit better depending how much I need. But it depends how we are looking at it.

As he was trying to explain his understanding to me, he drew from his real-world conceptualization, water running off a roof. The use of real-world examples was consistent for Jackson during the interview. When I asked him what comes to mind when he hears the world slope, he immediately thought about how he would teach slope and spoke about real-world examples.

> Jackson: We talk about how roofs have slopes, water is positive or negative. We talk about how planes have to descend. We talk about how the slope of the road has to be so that water can run off. Slope is all around us.

Jackson's concept image of slope also included the algebraic ratio. He used the mathematical expression change in y over change in x to calculate the slope in any context. Unlike the other teachers, Jackson never used a geometric ratio to determine the slope. Whether he was given a graph or coordinates, he continuously used the algebraic ratio. When I asked Jackson about rise over run, he expressed dissatisfaction in this approach. To Jackson, the mathematical expression is always applicable where rise over run can typically only be applied in certain contexts.

Finally, Jackson's concept image consisted of the behavior indicator. As Jackson explained the slope of a line with a positive rate of change, a negative rate of change, and a horizontal line, he described the behavior. For the positive line, Jackson said, "the line is going up from left to right." To describe the negative line, Jackson said, "because it is going down from left to right." As he described the horizontal line, he continued by discussing the behavior: "It is a flat line going from left to right."

Overall, Jackson had an understanding of other conceptualizations; yet, they were not used throughout his problem solving like the five conceptualizations (algebraic ratio, physical property, functional property, real-world situation, and behavioral indicator) referenced above.

5.6 Rachel

Background

Rachel is one of the top teachers at G.L. Abbott Middle School, if not the entire district. She won teacher of the year at her school last year, and is presently a finalist for Presidential Award for Excellence in Mathematics and Science Teaching. She has taught for a total of 18 years with the previous 14 years coming at the eighth grade level. She earned her undergraduate degree in elementary education with a middle school math extension. In addition, she earned both her masters in secondary education and her

National Boards. Her background helped to mold an in-depth concept image that contained many conceptualizations.

Concept Image

Rachel's concept image of slope is expansive and contains many conceptualizations. To begin the second interview, I asked Rachel to define slope.

Rachel:	Rise over run. Change in the y axis over change in the x axis. For me, it is the steepness of the line or a hill. So it is the steepness of something.
ME:	Anything else come to mind?
Rachel:	The formula. Change in y over change in x. The grade of a road. The slope of a mountain. How much it rises over how horizontal it goes.

In her words, she referenced four conceptualizations: geometric ratio, algebraic ratio, the physical property, and a real-world situation. As she worked through the different tasks, Rachel demonstrated each of these conceptualizations is part of her concept image of slope. When presented with the table of values, Rachel changed the mathematical formula. As she worked on the road sign problem, she converted six percent to the ratio 6 over 100. Once she made the connection to the geometric ratio, Rachel was able to determine that 100 percent was one over one as opposed to a vertical line, which she initially thought a hundred percent grade would produce.

As she worked on the stairs problem, Rachel ranked them based on the steepness of the steps. Furthermore, she evoked this conceptualization to explain how impacting the parameter, A, in the equation $y = Ax + C$ would change the slope.

ME:	What happens as A gets bigger?
Rachel:	It (the line) gets more steep.
ME:	Ok. So what happens as a gets smaller then?

Rachel: It (the line) gets less steep unless you're taking about going into negatives.

ME: What would matter if I went into negatives?

Rachel: It goes downward. So it goes less steep and once you run into the x axis, it'll start going into a negative which means the steepness is really not…I mean if you get really steep, you'll just be going downhill, just the other way.

From this excerpt, Rachel demonstrates the physical property is a part of her concept image.

When Rachel began to think about instructing her students, she evoked her real-world conceptualization. For example, she said

Rachel: If I was talking to kids I would reference that - I would talk about ski slopes or skateboarding or anything like that, but slope to me reminds me of the steepness of a line or a path of some sort.

Like many of the other teachers, when Rachel thought about explaining the topic to her students, her concept image involved the real-world situation.

In addition to these four conceptualizations, Rachel also evoked the functional property to explain slope. This conceptualization was evoked when she was confronted with the graph of a linear function. She stated on multiple occasions the linear function has a "consistent rate." For non-linear functions, Rachel expressed the slope can change; yet, she was unable to use knowledge to explain how the slope changes from point to point. Hence, the calculus conception was not part of her concept image. For example, on the parabola, she knew the slope changed but could not use this understanding to conclude whether or not the slope could be positive two at any moment, nor could she describe the slope of the cubic function.

For Rachel, her concept image of slope consisted of five conceptualizations. Given various contexts, she evoked the geometric ratio, the algebraic ratio, the physical property, the functional property, and the real-world situation.

5.7 Liam

Background

Liam earned his bachelors in business. Yet, he did not enjoy his work in this field. Therefore, he decided to change careers. He knew he always wanted to work with adolescents, so he volunteered to help coach football at G.L. Abbott Middle School and went back to school to earn his teaching certification. With a year of coursework under his belt, he had the qualifications to be hired as a mathematics teacher. Last year, he taught eighth grade mathematics. Specifically, he taught pre-algebra and Math 1. Liam has really enjoyed his first year and plans to teach and coach in the immediate future. In fact, he told me once he earns his certification, he plans to continue at the local university and pursue his master's in education.

Concept Image

Liam defined slope using three conceptualizations: geometric ratio, algebraic ratio, and the physical property. As he worked through the tasks, these conceptualizations were evoked often. For geometric, Liam used this to explain the slope of the line segments on the Tom walking to the school bus problem. Below is an excerpt for his interview.

> LIAM: He walks for every 20 meters, for the first line, for every 20 meters, it takes him 10 seconds. So he walks at 2m/s.
>
> ME: How did you figure that out?

LIAM: Rise over run. I used the first two points that I could find. You could use any two points. So rise, I rose, one block which was 20 and ran one block which was 10. So I rose 20 and ran 10. So that is 20 over 10 which 2. So he is walking at a constant rate of 2m/s

Liam also referenced rise over run as a method to explain how he knew the slopes of the three lines on the same coordinate plane had different slopes.

Liam: Well, if something crosses the origin, I would always use the origin as one point and find the next place where it crosses, intersects at a point. The first one would go up ½ over 1. The second has a slope of one. It crosses or intersects each box. This one (Steepest) has a slope of two. I would go up two and over one. Yes, Up two boxes and over one. Or you could go to the point two one.

On the thirty-degree angle problem, Liam tried to make the connection between a thirty-degree angle and a ratio. He reasoned thirty degrees could be converted to one third since thirty is one third of a right angle, further demonstrating the geometric ratio is part of his concept image.

In addition to the geometric ratio, Liam's concept image includes the algebraic ratio conceptualization. When presented the table of values, Liam immediately used the mathematical expression change in y over change in x to calculate the slope. Prior to working out the table problem, Liam explained

Liam: You would look at the x and the f(x) which is the y and you would do y_2 minus y_1 over x_2 minus x_1. For each point and make sure they are consistent. Which would then be a linear equation.

As mentioned at the beginning of this section, Liam's concept image also included the physical property. In addition to mentioning the steepness of a line in his initial definition, Liam also used this conceptualization during his problem solving. In explaining how the two lines on different un-scaled coordinate planes could have different slopes, Liam referenced the steepness of the lines.

> Liam: Well it is two different lines, but the steepness of the two. Graph 1 or the graph on the left is much steeper than the one on the right. They almost look like it would be half of the other one. It goes up about half as fast.

On the road sign problem, Liam again referenced the steepness of the road as the percent changed. This continued at the beginning of the second interview when I asked him again to define slope. Liam said, "The greater the slope, the steeper the line. The smaller the slope, the more horizontal the line." When I asked him if the variable *m* was an appropriate response to the question "what is slope," Liam stressed that he would want the individual to give a more detailed explanation, including referencing the steepness of the line. Therefore, it is evident physical property is part of his concept image.

Finally, Liam evoked the functional property. Confronted with the graph of a linear function, Liam recognized the rate of change was constant. For instance, while explaining the movie theatre problem, Liam said

> Liam: Well, this one actually goes up at a constant rate. They start with x amount of money in a register. This one does not start at zero. So obviously, you have some kind of money in the register and they are increasing at a constant rate. So every single person that comes in pays the same amount, so the register increases in money. As x goes up, y goes up. The difference is the y intercept is a little higher. Obviously, if you are running a business, you want money in the register to begin with.

He continued to show an understanding of constant rate of change. On the three lines problems (second interview), Liam recognized the three lines needed to have different slope since their rates of change are different.

Liam: How they are different? They are start at zero. The slope all three are different. They all go up at a different rate. If they were the same, there would only be one line. Since there are three lines, they all must go up at a different rate. The one closest to the x-axis being the least steep. The one closest to the y-intercept is the steepest line or the greatest slope

In his response, one can see both the functional property and the physical property.

Overall, Liam's concept image of slope consisted of four conceptualizations. Though he understood the basics of other conceptualizations, he did not utilize them in his definition or during his problem solving. They were only evoked when I asked him a direct question. The four described were evoked without being prompted.

5.8 Angela

Background

Angela has been teaching mathematics for the past 29 years. Though she started teaching ninth grade at a neighboring high school, Angela has been at Page Middle School for the past 26 years. During this time, she taught both seventh and eighth grade. Presently, she teaches seventh grade and has been in seventh for the past 10 years. She has never attempted to pursue either her masters or her National Boards.

Concept Image

Though Angela has been teaching mathematics for a long time, she struggled with many of the tasks in this study. For instance, when I gave her the three tables and asked her if the results were from a linear function, she was unable to calculate the slope for any of the tables. It became apparent that Angela did not have the algebraic ratio in her concept image. Therefore, Angela attacked the table problem by trying to plot the points. Through this graphical approach, Angela could have used one aspect of her stated definition of slope: rise over run. Yet, she never attempted to use this method on this problem. Later in the interview, Angela did use rise over run. On the two linear functions problem, Angela used rise over run to present an argument that the two lines could have varying slopes. She said

> Angela: I'm trying to think of an ordered pair on this one (graph on left) that would be the same. But when I do that...it's not going to be over here on this one (graph on right). Because if you go to the right one and up two (graph on left) and to the right one and up two; it's not gonna be on there (graph on right). So I'm trying to find an ordered pair that's going to be on both; besides (0,0) there's not going to be anything else.

To find her second ordered pair, Angela attempted to go up and over the same amount on both graphs. Yet, since the graphs appeared to have varying steepness, she was unsuccessful. The inability to use rise over to find the exact same second coordinate allowed her to argue the slopes of the lines were different. On the staircase problem, Angela again used the geometric ratio to validate her ranking of the stairs. To her, the easiest staircase to climb was the one with the smallest rise and a longer run. Based on these examples, Angela's concept image included the geometric ratio but not the algebraic ratio.

During the interviews, Angela demonstrated her concept image also included the physical property. After stating slope was defined as rise over run, she paused and stated this answer sounded like one a math teacher would give. She began to think about slope from a different point of view and evoked the physical property along with the real-world situation.

> Angela: Then I ask my husband, the accountant, - what do you think slope is? And he would say, "well, the degree that

> something is elevated." Maybe someone on the street would say, the grade, maybe the slant of a ramp, that a wheelchair goes up; the elevation of it.

As we continued through the interviews, Angela referenced these conceptualizations again. On the two lines problem, Angela contended, "this one is more steep (points to graph on left)" than the other function, and while working on the road sign problem, Angela argued six percent was not that steep when compared to 50 percent. During the second interview, I asked her directly how changing the angle a linear function makes with the x-axis would impact the line. Below is the excerpt from that interview showcasing how she evoked the physical property to help her justify her reasoning.

> Angela: As the angle gets big, the slope will get larger.
>
> ME: What does that mean?
>
> Angela: The steepness of the line will get larger. It will be more steep.

As mentioned earlier, Angela also evoked the real-world conceptualization. To explain a line with a negative slope, Angela said, "it is like skiing down the mountain." As she explained the three lines on the same coordinate problem (Question #15 on the second interview), Angela continued to use real-world examples in conjunction with the physical property.

> Angela: Well, I am looking at the steepness of them. This one (slope of two) if you were climbing that mountain, it would be almost going straight up. If I was skiing, I am not a very good skier, so I would have to start with this one (Slope of ½). It is not as steep.

In this excerpt, we observe that her concept image is constructed, in part, by the physical property and the real-world situation.

Even though Angela was able to answer some direct questions about the other conceptualizations, she did not use them to tackle any of the other tasks. Hence, her concept image of slope comprised of the geometric ratio, the physical property, and the real-world situation.

5.9 Sarah

Background

Sarah began her career in a neighboring school district teaching seventh grade. In addition to teaching mathematics, she also taught Language Arts and Social Studies. After five years, she accepted a position at Page Middle School as a sixth grade mathematics teacher. Therefore, for the past nine years, Sarah has only taught sixth grade mathematics. Sarah has not attempted to earn either her masters or her National Boards. Two weeks after conducting the first interview, Sarah informed her Principal she was resigning her position and leaving education. Based on this new development, she was unavailable to participate in the follow up interview. Thus, all of the data came from her first interview.

Concept Image

Having taught sixth grade for the past nine years, Sarah has not had the opportunity to teach the mathematical concept of slope during this time. In fact, she was unsure if she ever taught this topic even while teaching at the seventh grade level. This lack of familiarity with the topic was evident when she could not recall either the geometric or algebraic ratio to calculate the slope from any medium, including from a graph or when given two points.

Prior to the interview, Sarah prepared by using the Internet to look up the mathematical equation of a line, y = mx + b. With this equation in hand, Sarah tried to use it to solve a multitude of problems; however, Sarah did not have an understanding of how to implement this formula. The excerpt from her interview demonstrates this.

> ME: And then one final question, if you just had this right here, how would you use that table to calculate the slope? (I explain that she needs two points to use y=mx+b) Without using y=mx+b, is there a way to use this table to calculate the slope?
>
> Sarah: You would draw a grid and plot the points. That would be the starting, you would plot the points and then you would have to plug your numbers, you still have to understand how to use the formula and plug it in because this is the slope, the formula would give you the slope.
>
> ME: What part of this formula is the slope?
>
> Sarah: Well, y is y. m is the slope. I just don't remember how to use it. It is funny that you asked cause the other day a friend asked what is b? I had to say that I don't remember.

Though she looked up the formula and tried to implement it, it cannot be included in her concept image. Later in the interview, Sarah reiterated she did not know how to use this formula

> Sarah: Well let me plot points to see. (Mutters to herself) I cannot remember how to use the formula. That is what I was trying to figure it out.

At this junction, it would appear like Sarah did not have any conceptualizations of slope in her concept image. Yet, that would be untrue. As she continued to work through the tasks, I was able to identify her concept image of slope revolved around one conceptualization: the behavior indictor. When asked what mental images come to mind what hearing the word slope, Sarah said

> Sarah: Lines. Lines on a coordinate plane. Positive. Negative. If it

is going up it is positive, if it is going down it is negative. Reading it left to right.

When thinking about lines and slopes, Sarah gravitated to describing whether the line had a positive, a negative, or a zero slope, evoking the behavior indicator. For instance, while explaining the graph of a linear function, she said the line “could be a linear equation, cause I saw a line. It is positive.” By understanding how the sign of the slope impacts the direction, Sarah was able to accurately use this conceptualization to determine only one of the three linear functions could have a slope of positive two. In fact, Sarah initially tried to use the formula she memorized for the interview, but eventually relied on her conceptualization. To prompt this response, I asked her what she was trying to do.

Sarah: Just be able to plug in, my ordered pairs. To what, again, the pattern is. That would be the slope, the pattern. A is negative. C, there is no change, so there is no slope. So it is zero. B would be the only one, you said positive two?

ME: Yes

Sarah: B would be the only one that could be positive two cause it is the only one that is going in a positive direction.

Throughout the interview, Sarah hinted to other conceptualizations. For example, she was able to apply the geometric to investigate the steps, but this occurred after I explained to her how rise over run was applied. Prior to the interview, she had only heard about it. Based on the collective interview and her reasoning used throughout, her concept image was solely constructed by the behavior indicator.

5.10 Elizabeth

Background

Elizabeth is certified to teach Kindergarten through sixth grade. As an undergraduate, she had no intention of teaching mathematics. In fact, she struggled with math throughout her career as a student. This was demonstrated by having to take geometry and calculus twice to earn a passing grade. She was assigned to teach mathematics by her Principal. It was not her choice. As of next year, the Principal has made the decision to move her to Social Studies.

Concept Image

To determine Elizabeth's concept image, I had to carefully examine her words. When initially asked what comes to mind when she hears the word slope, Elizabeth stated, "Well, I think of a map, and of landscaping." After asking her to elaborate, she explained it meant "the slope of the land, how you need to plant things, or how you're going to have to move the soil around." Though she made reference to a real-world example in this one instance, she never again spoke about the slope in a real-world context connecting to the slope. Hence, I cannot conclude that her concept image included the real-world conceptualization.

Having never taught slope, Elizabeth was not able to recall either the geometric or algebraic ratio. While explaining the difference in the two linear functions on different unmarked coordinate planes, Elizabeth was the only teacher who made no reference to the steepness of the lines. Even while discussing the increasing percentage on the road task, Elizabeth did not discuss the steepness, the grade, or the pitch. She responded by saying, "It would be more dramatic" when I asked her how the road would change as the percentage increased from six percent to fifty. Hence, Elizabeth only discussed the steepness when asked a direct question. As mentioned at the onset of this chapter, I did not include a conceptualization in a teacher's concept image if it was not evoked during problem solving and was the result of a direct question. This leads me to the conclusion that the physical property is not in her concept image.

As I continued to examine her interview to find evidence of the other conceptualizations, I do not believe she demonstrated her concept image of slope contained any. She never mentioned rate unless it was spoken in the text of each task. She did not reference angle or a function rate of change could vary from point to point. Based on the tasks present, Elizabeth was challenged to evoke any conceptualization of slope.

5.11 Summary of Results

Overall, in this collection of teachers, the number of conceptualizations that comprised their concept images varied greatly. The lowest number of conceptualizations was Elizabeth, while Brianna evoked the highest. As we look at their backgrounds, Elizabeth only taught at the sixth grade level, which does not address the mathematical concept of slope. On the over hand, Brianna taught a variety of mathematics courses, including AP calculus, which covers slope in detail.

The mean number of conceptualizations was 4.3 with zero being the lowest and eight being the highest. Four of the teachers represented five conceptualizations in their concept image. A complete list of the conceptualizations that are part of their concept image is located in the table below.

TABLE 18: Overall conceptualizations in each teacher’s concept image

	Grade Level	Geo	Alg	Phy	Funct	Para	Trig	Calc	RW	Deter	Beh	Line	Total
Brianna	7	X	X	X	X		X	X	X		X		8
Luke	6	X	X	X	X		X			X	X		7
Carrie	8	X	X	X				X	X				5
Deborah	8	X	X	X	X				X				5
Jackson	8		X	X	X				X		X		5
Rachel	8	X	X	X	X				X				5
Liam	8	X	X	X	X								4
Angela	7	X		X					X				3
Sarah	6										X		1
Elizabeth	6												0
Total		7	7	8	6	0	2	2	6	1	4	0	

The conceptualization most prevalent in their concept image was the physical conceptualization, which was represented by eight of the teachers, and followed closely by the seven teachers who had geometric and algebraic ratio in their concept image. As we look at the other end, two teachers’ concept image contained calculus, one contained the determining property, and none of the middle school mathematics teachers’ concept image included the linear constant conceptualization.

5.12 Concept Image in Problem Solving

The various tasks were designed to explore the middle school teachers’ concept image of slope. Specifically, I designed at least one task that would elicit each of the eleven conceptualizations. With this design, it was possible a teacher could have a concept image consisting of all eleven conceptualizations. Yet, as the teachers each solved the various tasks, they could only evoke the specific conceptualization that comprised their own concept image. Hence, if a teacher only had one conceptualization, he or she was forced to adapt this one conceptualization to solve each problem.

This was evident in the Table task, which was designed for the teacher to use either ratio conceptualization. Eight of the ten teachers completed this task by evoking a ratio conceptualization. Without this conceptualization, the other two teachers, Elizabeth and Sarah, were unable to conceive how to apply either conceptualization to attempt this problem. This led both teachers to approach the problem in a similar fashion: they each tried to look for a pattern. Sarah explained this approach.

> Sarah: I am basing this off a function, what I know about a function. If it is a function, then it should be linear. What most of them look at is the x and just look at that (f(x)) like it is a y, input output, Do you seen a pattern? Patterns are going to make functions and functions are going to make lines. So that is the way that we would look at it with sixth graders.

Without a ratio conceptualization, the two teachers were forced to connect this problem to their background and mathematical knowledge and struggled with the task.

The thirty-degree angle problem proved to be another example where the teachers in this study had to rely on the flexibility in their unique concept image to complete this task. Only one teacher, Deborah, recognized one could calculate the slope with the given information. Yet, without having taught trigonometry recently, she was unable to recall the formula required to complete this task. The other teachers believed an angle could not be used to represent slope. This demonstrates the disconnect that exists between algebra and trigonometry for this collection of teachers.

Two teachers did try to use another conceptualizations in their concept image to attempt to tackle this problem. Carrie and Liam each evoked a ratio conceptualization to approach this problem. Carrie viewed 30 degrees as 30 percent and reasoned the slope would be 3/10, while Liam reasoned 30 degrees out of 90 would convert to one-third. Hence, having another conceptualization in their concept image allowed Carrie and Liam to make sense of this problem within their concept image.

As the teachers encountered more diverse tasks, the conceptualizations in their concept image were evoked. On the table below (Table 19), I have listed the conceptualizations that were partially or completely evoked as the teachers worked through each task. For some of the teachers, they evoked a glimmer of the conceptualization or may have used it incorrectly. Yet, I included the conceptualization on the table to demonstrate how they each attempted to tackle the various tasks in the first interview. This table should not be interpreted as the conceptualizations that make up their concept image, as this was addressed earlier in this chapter.

TABLE 19: Conceptualizations partially evoked during problem solving

TASK	Angela	Brianna	Carrie	Deborah	Elizabeth	Jackson	Liam	Luke	Rachel	Sarah
Table	NONE	ALG	GEO ALG	GEO ALG FUN	NONE	ALG	ALG	GEO	ALG	NONE
Two Lines	PHY GEO	BEH PHY	PHY ALG	PHY	NONE	FUN RW	PHY	ALG TRIG	GEO	BEH
Con-tainer	NONE	FUN	FUN	FUN	Limited FUN	FUN	FUN	FUN	FUN	NONE
Thirty Degree	NONE	PHY GEO	GEO PHY	PHY	NONE	FUN	GEO	GEO ALG	PHY ALG	NONE
Slope of Two	GEO BEH RW	BEH CALC RW	FUN CALC	GEO	Attempted RW	BEH	ALG	BEH	RW	BEH
Road Sign	PHY	GEO PHY	GEO PHY	GEO PHY	Limited PHY	FUN PHY	GEO PHY	PHY GEO	GEO	NONE
y=ax+c	PHY	PHY	PHY	PHY	NONE		PHY	PHY	PHY	NONE
Stairs	GEO	PHY	GEO PHY	GEO PHY	Limited GEO	PHY	GEO	PHY	GEO	Used GEO
Darren	GEO	FUN	FUN	FUN	Limited FUN	FUN	GEO FUN	FUN	FUN	BEH

TABLE 19 CONTINUED

Graphs	GEO BEH	RW BEH CALC	RW FUN PHY CALC	RW FUN	Limited PHY	RW BEH FUN	RW	FUN	RW	RW
Faucet	GEO RW	GEO ALG RW	ALG	RW ALG	NONE	ALG	ALG FUN	NONE	FUN GEO	NONE

The teachers with the least conceptualizations, Elizabeth and Sarah, struggled the most during the completion of the tasks. Without a robust concept image, they were unable to conceive how to represent slope in multiple ways to increase their flexibility in problem solving. For example, when confronted with the y = *A*x + c and asked how does *A* impact the line, Elizabeth was unable to offer an explanation. The other teachers in the study where able to connect changing parameter to another conceptualization, physical, and explain changing the parameter changes the steepness. An excerpt from Rachel's interview demonstrates this.

ME: Given the equation y = *A*x + c, What happens to the line as *A* gets bigger?

Rachel: It [The line] gets more steep.

ME: So what happens as a gets smaller then?

Rachel: It gets less steep.

Here Rachel showed how one could connect an explanation to another conceptualization that exists in their concept image. Without a robust concept image, Elizabeth did not have the flexibility in her reasoning to make a connection and offer a solution.

Brianna was at the other end of the spectrum. With eight conceptualizations in her concept image, she was able to freely move from conceptualization to conceptualization as she solved each problem. She used seven different conceptualizations to solve the various tasks. The only conceptualization represented in her concept image that was not used during her problem solving was trigonometric; yet, she evoked this conceptualization while discussing the definition of slope. As evident in the table, Brianna utilized multiple conceptualizations to solve six of the tasks. Therefore, when she was stuck in her problem solving, she was able to evoke another conceptualization to aid in her explanation. For example, on the two lines problems, she was asked to explain how the lines could be the same and how they could be different.

Brianna: They are similar because they both have a positive slope.

ME: What makes the slopes positive?

Brianna: Because they rise if you look at the graph from left to right. They increase as you move from left to right. As the x values move from left to right, the y values increase. Graph #1 (Label the graph on the left #1) has a greater slope than Graph 2 (One on the right). This makes them different. That is how I could see both perspectives.

ME: How do you know that there is a different slope?

Brianna: Because this change here (Creating a triangle on Graph 1). This triangle here is bigger than the triangle here (Graph 2). So because this line (Graph 1) is stepper or **closer to my y axis.** This is what I tell my kids, it makes it steeper than this line which is closer to the x axis, it makes this slope less.

In her explanation, she evoked multiple conceptualizations to aid in her explanation. Specifically, she used behavior and physical. Other teachers with multiple conceptualizations were able to duplicate Brianna's success. For instance, on the same task, Jackson reasoned:

Jackson: Well, it depends on the type of question. Are they the same line? They both positive. You could say it that way. You could say one has a higher rate and the other has a lower rate of change. They could be different in that way. So it just depends on how you pose the question. You can see on the first one, it is a steeper rate of change and on the other one, it is a more lower rate of change.

ME: What do you mean by steeper rate of change?

Jackson: It is going to down up higher. You might have a big difference from your first one to your second one. That is the way that I looked at it. The other one is not as steep as a line. If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation. The other one might be a little bit better depending how much I need. But it depends how we are looking at it.

In his reasoning, Jackson evoked almost every one of his available conceptualizations in his concept image. He spoke about the behavior of the lines, the rate of change, the steepness, and connected it to the real world. Hence, his explanation demonstrated the flexibility allotted to an individual with a robust concept image.

In contrast, Sarah had a much more difficult time explaining how the graphs could be the same and/or different.

Sarah: Well, I am go based on, hold on, let me write this out. (Writes y=mx+b). I mean based on where the line is and the first line goes, I mean, it is not labeled, you could be using different intervals, so there is a lot that could go into this. I would agree with the student that said it is not the same

ME: Why?

Sarah: If I plotted a point on the line, based on, let me think about this

ME: Sure

Sarah: I would think, based on plotting a point and seeing where x is, it x was is a point and y is a point, or is it just showing you part of an equation. I would think based on where the points are, if I were not plot a point, they would be totally different places.

With limited conceptualizations in her concept image, Sarah could not clearly articulate reasoning. She attempted to examine the points, but did not have the algebraic conceptualization required to fully explain her reasoning. When I asked her how they could be the same, she evoked the conceptualization that comprised her concept image, behavior, and offered this explanation.

> Sarah: Well, going back to, they do form a line. They do look positive. Um and the rationale could be depending on the intervals that they used. One could have used much greater intervals. This could be counting by ones (right) and this one could be showing counting by 20s or 50s (left) and it is not going to look at drastic (Uses arm to shoe steep slope). That would be my explanation.

Overall, the teachers who had the most conceptualizations in their concept image were the most successful at articulating how to approach each task. The teachers who had at least five conceptualizations were able to reason and explain their way through almost every task, while those with four or less struggled as they approach the various tasks designed to evoke conceptualizations not in their concept image.

5.13 Conclusion

In this chapter, I described the concept image of each of the ten middle school mathematics teachers who participated in this study. Overall, their concept images varied from limited to robust. Specifically, the number of conceptualizations that comprised their concept image ranged from zero to eight. At the lower end of the spectrum were Elizabeth and Sarah. Both of these teachers currently teach sixth grade and have never taught Algebra I. As we move up the list, we find Angela and Liam. Angela last taught Algebra I a decade ago, and Liam had just completed his first year of teaching. Carrie, Deborah, Jackson, and Rachel are each currently teaching Math 1 and had five conceptualizations in their concept image. Luke and Brianna were at the top of the list. Though Luke is presently teaching sixth grade, he taught eighth grade recently, while Brianna, currently in seventh grade, had experience teaching advanced mathematics at the high school level in her background. The teachers with more conceptualizations in their concept image had more flexibility in their problem solving, as they were able to more freely from one conceptualization to another to tackle each task.

In the next chapter, I close this document by offering a discussion on the results of this study. This chapter includes a comparison between my results to prior research, and I articulate my assumptions about the results. Following this discussion, I will address the limitations in this study and offer my personal reflection on completing this dissertation.

CHAPTER 6: DISCUSSION

During the first interview, all of the teachers encountered the same tasks; yet, each teacher tackled the tasks by evoking the conceptualizations that comprised their concept image. This was addressed in Chapters four and five. As a follow-up, in this chapter, I offer a discussion on the results and possible implications this study may have in the field of mathematics education. Next, I offer my vision of the emerging research that may stem from this dissertation study. Finally, I conclude this document with my personal reflections on the overall process of completing this dissertation.

6.1 Review of Results

The purpose of this qualitative study was to explore middle school mathematics teachers' concept image of slope, not to examine whether the teacher was able to solve each task correctly. Specifically, I sought to answer the following research questions by having ten middle school mathematics teachers engage in two task-based interviews.

1. What is the concept definition of slope for in-service middle school mathematics teachers?
2. What conceptualizations of slope do in-service middle school mathematics teachers possess?
3. What is the concept image of slope for middle school mathematics teachers?

In Chapter 4, I tackled the first two research questions, while Chapter 5 examined the third research question. Here, I offer a brief summary of the results for each question before launching into the discussion. The middle school mathematics teachers' concept definition of slope was rooted in a linear sense. When initially asked to define slope, the vast majority of the teachers referenced the steepness of a line. As they progressed through their interviews, the teachers continued to focus on the relationship that exists between a linear function and constant rate of change. This implies slope exists in a linear sense for this collection of teachers.

In addition, the middle school mathematics teachers demonstrated an in-depth understanding of other conceptualizations. When asked to calculate the slope, the teachers preferred to use either ratio: geometric or algebraic. This remained true for teachers who attempted to calculate the slope of a non-linear function, calculating the slope of a secant line as opposed to the tangent line. When considering describing slope to another person, the teachers relied heavily on the real-world situation and the behavioral indicator conceptualization.

The number of conceptualizations present in the teachers' concept image varied greatly. Specifically, they ranged from zero to eight conceptualizations. At the lower end were the two sixth grade teachers, Elizabeth and Sarah, who never taught Algebra I or a more advanced mathematics course. At the higher end was Brianna who, unlike the other teachers, taught multiple advanced mathematics courses including Advanced Placement Calculus. The other teachers who fell between the sixth grade teachers and Brianna had a common thread: they each had prior experience teaching at the eighth grade level. Hence, this research suggests teaching eighth grade or higher aids in developing a more robust concept image of slope.

The most prominent conceptualization in the teachers' concept image was the physical property. Eight of the teachers had this conceptualization in their concept image, and it was a key aspect of their concept definition. Following closely behind was the geometric and algebraic ratio with seven teachers' concept image being constructed in part by these ratio conceptualizations. The real-world situation and functional property was evident in six teachers' concept image, followed by the behavior indictor. Trigonometry and calculus were evident in the concept image of two teachers, while one teacher's concept image was constructed by the determining property. Finally, none of the teachers in this study had a concept image of slope comprised of either the parametric or the linear constant.

In addition to answering the research questions, this study had several findings. First, the number of conceptions that comprised a teachers' concept image corresponded to their prior teaching experience. Specifically, teaching at the eighth grade level or higher corresponded with a more robust concept image. Second, the teachers with a more robust concept image had greater success in solving the tasks. Third, the teachers recognized slope existed for non-linear functions, but lacked the ability to explain or calculate rate of change for a non-linear function. Finally, the teachers were adamant one could not use an angle to represent the slope of a line. This represented a lack of trigonometry knowledge since the teachers understood how changing the angle would impact the slope but could not use the angle and a point to calculate the slope.

6.2 Discussion

Linear Terms

The middle school mathematics teachers in this study defined slope in linear terms. Specifically, they focused on the steepness of a line and the relationship between constant rate of change and a linear function. This is a direct result of their daily curriculum. In middle school, rate of change is just being formally introduced as a scientific concept. At this early stage, it is important to begin with the slope of linear functions, since this represents the most basic form of rate of change a student will encounter (Stanton & Moore-Russo, 2012). It is not until the calculus curriculum, which is typically taught to advanced mathematics students in the twelfth grade, that mathematics teachers are responsible for teaching about the slope of non-linear functions. With the curriculum structured in this manner, middle school mathematics teachers are not required to confront this topic in their instruction, hence explaining their focus on slope in a linear sense.

The lack of familiarity with non-linear functions was evident in this collection of teachers. Only two teachers' concept image contained the calculus conceptualization. This result is comparable to those of Stump (1999) and Nagle, Moore-Russo, Viglietti and Martin (2013). In Stump's (1999) investigation of high school teachers' knowledge of slope, she found only ten percent of the in-service teachers represented the calculus conceptualization in their concept definition. Similarly, at the collegiate level, only fifteen percent of the instructors used a calculus conception in their response to the question: What is slope? (Nagle, Moore-Russo, Viglietti & Martin, 2013).

This implies that when asked to speak about slope, teachers focus in linear terms. As slope is primarily addressed throughout the curriculum in linear terms, this was an expected result. When one looks up a definition for slope, the individual will primarily come across definitions such as the rise over run, change in y over change in x, or the steepness of a line. Each of these definitions is rooted in linear terms. As one discusses slope of non-linear functions, the terminology tends to deviate away from the word slope and instead shifts to more advanced mathematical terms. For example, in Stewart's (2003) calculus book, he utilizes the terms "derivative" and "rate of change" as opposed to "slope" to discuss the change in a non-linear function. Therefore, it stands to reason if the teachers were asked to define instantaneous rate of change or the derivative as opposed to slope, they may have evoked different conceptualizations in their concept definition.

Yet, my explanation does require the teachers have knowledge of these advanced mathematical terms. In this study, only one teacher, Brianna, taught at the calculus level. This experience helped Brianna develop her calculus conception and aided her as she defined the slope of a non-linear function. She drew on this background and referenced a rollercoaster when discussing slope. For the other teachers, their lack of experience interacting with non-linear functions lead them all to speak in linear terms. This was also evident in Stump's (1999) study. Of her 21 in-service secondary teachers, only six had prior experience teaching calculus. Yet, when asked to define slope, only two mentioned the derivative. Therefore, it is reasonable to assume the teachers in her study also thought of slope in the linear sense. This leads me to wonder if more of the teachers would have mentioned the derivative if slope was replaced by a more advanced mathematical term found in the calculus curriculum.

Further demonstrating their connection between slope and linear functions, the teachers in this study elected to define slope as the steepness of the line. The word steep is important, since even at a young age individuals begin to learn about steepness by building ramps and sliding down slides (Cheng & Sabinin, 2008). Therefore, it is this definition the teachers believed allows an individual to visualize the mathematical concept of slope without relying on any mathematical skills, such as using a ratio or taking a limit. In essence, this definition is more relational than other definitions of slope, such as rise over run or change in y over change in x.

The reason why teachers may define slope in this manner lies in the adoption of a new state curriculum. Over the past four years, teachers in the state where this study took place have been asked to teach differently. They are being asked to teach relationally as opposed to instrumentally. This requires the students and their teachers to understand and explain why the mathematics is valid, not simply solve a collection of problems. Hence, the teachers have been encouraged to replace mnemonic devices, such as rise over run or change in y over change in x, with more relational definitions, such as steepness of a line or rate of change. In this study, this was evident with teachers like Jackson who do not like students using rise over run because it lacks meaning. Also, the teachers gravitated away from discussing slope as the parameter m in the equation $y = mx+b$ for similar reasons. Overall, eight of the ten teachers defined slope as the steepness of a line and three included rate of change, providing evidence that the teachers have embraced the call for curriculum change.

Procedure Knowledge of Slope

As mentioned, three teachers in this study elected to define slope as rate of change. This was slightly higher than the teachers in Stump's (1999) study, which revealed less than one fifth of each group defined slope through a functional conceptualization. In neither study, did the majority of teachers elect to define slope as a rate of change. This was similar to the result in Coe's (2006) study in which none of the high school teachers had developed a quantitative understanding of rate of change. Overall, the middle and high school teachers in these studies do not view slope as a rate of change. In contrast, instructors at the university level most commonly defined slope through a functional property. Overall, 46 percent of the instructors evoked this conceptualization. Yet, only 18 percent of the students in a first-level calculus class described slope as a rate of change. Instead, they elected to define slope most often through these three conceptualizations: behavior indicator, geometric ratio, and algebraic ratio. Hence, the students are describing slope through procedure terms.

These results imply middle and high school teachers still continue to think of slope in instrumental terms. Yet, the tide may be turning. This is evident by the vast majority of the pre-service and in-service teachers in Stump's (1999) study describing slope as a geometric ratio. In contrast, only one of the middle school teachers defined slope through a geometric ratio. Therefore, since Stump (1999) conducted her study, teachers have started to eliminate ratio in their definitions for slope. As addressed earlier, this may be a direct result of the changing curriculum to focus more on relational understanding as opposed to instrumental.

Even though the middle school teachers did not elect to define slope through ratio, they have a deep understanding and reliance on the ratio conceptualization. Six of the teachers' concept image comprised of both ratio conceptualizations. An additional two teachers had only one of the two ratio conceptualizations in their concept image. Overall, these two conceptualizations, geometric and algebraic ratio, tied for the second most represented conceptualization in middle school teachers' concept image. This indicates the teachers still rely heavily on ratio conceptualizations when asked to calculate the slope. Based on the curriculum, this is an expected result. At the middle school level, teachers are only required to confront linear functions. If the slope is presented in a common form, such as a table or a linear function plotted on a scaled graph, these conceptualizations can be utilized successfully to calculate the slope. Hence, knowledge of these conceptualizations is sufficient at the middle and, until calculus, at the high school level.

When the slope was presented in an unfamiliar form, the teachers in the study struggled. For example, none of the teachers in this study were able to calculate the slope when given an angle and a coordinate. They stated this was the first time they had ever come across such a problem. Their struggles on this problem were two-fold. First, they lacked the mathematics required to solve the problem. Without knowledge of right angle trigonometry, which is not taught until Math 2, the teachers could not solve the problem. Second, their knowledge continues to be procedure based. They were expecting the problems to be presented in a familiar manner that would allow them to use a ratio to calculate the slope. When this was not the case, they were surprised and did not know how to proceed. Overall, the teachers stated they never encountered slope represented as an angle. Even though the curriculum is calling for more advanced understanding, this implies teachers are continuing to focus on traditional problems, such as calculating the slope given two points and finding the equation of a line.

Hearing Slope

Upon hearing the word slope, the middle school teachers were quick to reference a real-world example to aid in their explanation. Evoking this conceptualization continued as they worked through various tasks and thought about explaining their reasoning to another person. This was also recognized by Stump. In her dissertation (1996), she made reference to seven conceptualizations, omitting the real-world situation conceptualization. As she continued to analyze her results, she recognized teachers had a tendency to mention real-world examples involving slope to aid in their explanation. Specifically, 75 percent of the high school teachers in her study made reference to a real-world example. Her results align with the results of this study.

Mathematics teachers are charged with helping their students learn about slope. Yet, prior to direct instruction, students may learn about this topic spontaneously. For example, they may begin to develop an understanding of slope by building ramps, biking up and down hills, or playing on a slide at their school's playground (Cheng & Sabinin, 2008). These experiences support Cornu's (1991) claim that students come into the classroom with prior knowledge of most mathematical topics. Hence, to help the students understand slope better the teachers evoked these experiences. These examples help the students to make the connection between slope and a real-world situation.

Looking deeper at real-world situation, the middle school teachers' definition of slope, the steepness of a line, makes more sense. The examples they provided, such as a ski slope or a skateboard ramp, are visual images. If the students have any prior experience with either example, they can visualize the slope. By evoking a mental picture, teachers can help students make a connection between the real-world example and its steepness. For example, students should know from prior experience that they would go down a steeper hill faster than if the hill had a gentle slope. By connecting slope to a familiar experience, teachers can help bridge the gap between spontaneous and scientific concepts for their students. Through this approach, teachers are able to explain the mathematical concept of slope to their students without the use of any mathematics. With this in mind, it is easy to understand why the majority of the teachers have the real-world conceptualization in their concept image.

Robust Concept Image

At the conclusion of Chapter 5, I addressed how teachers with a more robust concept image had greater success in explaining their thinking. This allowed them to draw on the flexibility of their knowledge and adapt it to the task at hand. Yet, I did not classify what I meant by robust. My research has led me to define one has a robust concept image if he or she has an understanding of at least five conceptualizations of slope. At this threshold, the teachers were able to freely move back and forth between their conceptualizations and offer insight into the majority of the tasks.

To calculate the slope of a linear function, it was essential the teachers had a ratio conceptualization. Of the two, an algebraic ratio provided more freedom since this expression could be used with more flexibility than the geometric ratio. Yet, the teachers in this study proved they could transform the data to use either ratio. Hence, though more beneficial to have both, a deep understanding of one allowed for the teachers to complete the tasks designed for this study. If the teachers were unable to calculate the slope of a graphed linear function, they relied on describing the steepness and the direction, which would evoke the physical property and the behavioral indicator. As they confronted a linear function, the teachers recognized the rate of change was a constant. This awareness allowed the teachers to immediately make the connection between constant and non-constants rate of change. Hence, they all had the ability to match the linear functions to situations that involved a constant rate of change, such as the changing volume of the cylinder to a linear function. Finally, having the ability to draw upon real life examples helped the teacher in their explanations. The ability to move fluidly through these conceptualizations allowed the teachers to explain the majority of the tasks in this interview.

For this collection of teachers, six had a robust concept image of slope. To draw a comparison, I applied my definition of a robust concept image to the teachers who participated in Mudaly and Moore-Russo's (2011) study and calculated that 27 percent of the South African teachers had a robust concept image of slope. In this study, all of the teachers with a robust concept image were currently or had recently taught at the eighth grade level or higher. Again, driving home the point that instruction helps develop a robust concept image. Of the four teachers who did not meet this threshold, two had never taught eighth grade, one was a first year teacher, and one had not taught eighth grade for nearly a decade. I believe if I were to have conducted this study ten years ago, Angela would have demonstrated more conceptualizations in her concept image. Similarly, if this study was conducted three years in the future, I contend Liam would have the opportunity to enrich his concept image by having more interaction with the curriculum.

6.3 Limitations

While this study provides critical insight into middle school teachers' understanding and concept image of slope, I recognize in any study there are limitations. First, this research is limited by the sample size. Overall, only ten teachers participated in this qualitative study. A larger sample size would have allowed for more statistical testing and could have introduced a quantitative aspect to this study. Second, the teachers in this study represented only two schools. Despite this, the majority of the teachers had prior experience at other schools before teaching at the Page Middle School and G.L. Abbott Middle School. Third, as mentioned in Chapter 3, both schools are high performing middle schools. Therefore, this study does not explore the concept image of teachers from a low performing school. Fourth, I only conducted two interviews with each teacher over a five-week period. Though I believe I inferred the participants' concept image from these interviews, spending more time with each participant may have allowed me to further glean insight into their concept image. For instance, I did not spend any time with the participants in the classroom. Therefore, this study did not examine their instruction or how interacting with students evoked conceptualizations in their concept image.

Another limitation was my role in the community where the study took place. Prior to taking the job as Assistant Principal, I was the Advanced Placement Calculus teacher at two local high schools. Therefore, I was known to most of the mathematics teachers in this study. Furthermore, the teachers were aware I had a professional relationship with the Principals of each school. I bring both of these facets up because it is possible the teachers were cautious I would judge their knowledge of mathematics and/or report back to their supervisor. To counteract this belief, I informed the teachers I would not speak to anyone about their interview and would change their name in my report. Yet, it is still possible my status in the community directly impacted their interview, as the teachers may have been focused on not making a mistake as opposed to answering the question.

The final limitation to this study was the overrepresentation of eighth grade teachers. At the onset of the study, I wanted to have three participants from each grade level. Yet, I was unable to obtain three seventh grade teachers to participant in this study. With slope being taught at the eighth grade level, eighth grade teachers were more willing to participant than seventh grade teachers. This was even more evident when I interviewed both seventh grade teachers and learned that each had taught a higher grade level before moving to seventh grade. Therefore, no teacher in this study had only taught seventh grade mathematics.

6.4 Future Research

This study, hopefully, marks the beginning of my career as a mathematics education researcher. As I consider this a rich topic that can be researched in greater detail, the following section outlines my proposed future research agenda. I see this study moving in two distinct directions, and I offer an explanation of those directions.

The first direction I foresee this study going is the comparison of the concept image of middle school students to their middle school mathematics teachers. In performing my review of literature, I did not find a study that existed on this topic. The most similar study was performed at the collegiate level (Nagle, Moore-Russo, Viglietti & Martin, 2013). Yet, their study did not focus on the instructors of the students who participated in the study. With this in mind, I believe a study that collected data from a large group of middle school students and their current mathematics teachers would add to the field of mathematics educations.

The second direction this study could take is to look closer at the conceptualization of middle school teachers. Specifically, I think a worthwhile study would determine the existing differences between the conceptualizations of slope for middle school teachers who are certified sixth through ninth grade versus middle school teachers who have earned their certification in sixth through twelfth grade mathematics. This distinction is worth investigating since, at the university where I completed this study, an undergraduate can pursue either his or her middle school certification or high school certification. Once armed with high school certification, a passing score on a Praxis exam will earn the teacher a 6 – 9 certification as well. This implies these mathematics education students have two different tracks and take different coursework. The middle grades mathematics major is required to take Calculus 1 and a statistics course, while the student pursing their high school certification must successfully pass Calculus 1 - 4, differential equations, linear algebra, and introduction to modern algebra. Therefore, it would be interesting to examine if taking more advanced mathematics courses develops a more diverse concept image of slope. These results could impact how universities around the country prepare secondary teachers.

6.5 Implications for Teacher Preparation

As discussed by many researchers, slope is a critical topic in secondary education (Farenga & Ness, 2005; Thompson, 2008; Wilhelm & Confrey, 2003). Though it is not limited to the mathematics curriculum, mathematics teachers are charged with the task of introducing and teaching slope to secondary students. Hence, teacher education programs must take it upon themselves to ensure our future teachers are learning about slope in a robust manner. Many pursuing a degree in mathematics education will never take an algebra course at the collegiate level. For instance, at the university where I completed this study, the first mathematics course a student will take is calculus, after which they then move on to more advanced mathematics. This was also my reality as a university student. Therefore, when I entered the classroom, my last experience interacting with slope at the introductory level was as an eighth grader. Based on this, it is highly recommended each teacher preparation program take the time required to instruct their students how to teach mathematical topics that are critical in middle and high school mathematics, including slope.

6.6 Personal Reflection

After completing this study, I feel like I can finally breathe again. This was an extremely difficult and trying process. First, I was balancing this project while trying to be the best father, husband, and son I could. Also, I am presently serving as an Assistant Principal at a large urban high school. This meant staying up late and waking up early to find time to write. Yet, I knew the entire time I had the support of those closest to me to complete this dissertation. As I look back on the process, I must say I have never been more proud of myself for completing an assignment. This project taught me I really could accomplish anything in life with the proper support and the internal motivation required.

Looking at each step of the process, I found conducting the research extremely rewarding. I read hundreds of articles about slope, teaching mathematics, student understanding of mathematics, etc. By learning and self-reflecting through these articles, I became a better teacher. I shared several of these articles with my students and my peers, and I believe this made them better students and teachers. I was able to relate to my students better because I was doing homework with them. They saw me during planning, before, and after school reading and writing papers. This helped me never lose sight of what it is like to be a student.

Once I completed my research and defended my first three chapters, I moved on to recruiting participants. This was extremely challenging. First, I was attempting to collect data over the summer. This meant I had to work around vacations, baby-sitting issues, and teachers wanting to relax and not participate in a study. Second, many teachers did not want to participate due to a lack of confidence with the concept of slope. I had to convince the teachers they could answer the tasks I designed. Finally, many teachers were reluctant to participant because I needed to meet with them twice. If it was only one interview, I believe more teachers would have agreed. The second interview proved more difficult to arrange.

As the teachers progressed through the interview, I could see my growth in the area of interviewing. At first, I would chime in with reassuring comments after each statement. For example, I had the tendency to say, "okay" after they made a point. In listening to and watching the interviews, I recognized how this detracted from their interview. It took their focus off of the task and onto me. As the interviews piled up and I continued to watch the tapes, I learned how to conduct a better interview by asking more thought provoking questions, encouraging them to think out loud, and engaging them in each question. Furthermore, as I gained experience, I learned how to introduce the tasks to ensure that each participant would better understand the questions. As I conclude this study, I am much more confident in my ability to conduct an interview.

Drafting the final three chapters was extremely stressful. I would wake up some days feeling incredibly close to being done; other days it felt like I was a million miles away from the end. I confided in my wife that I thought I would never finish; yet, I informed others I was nearly done. It seemed as though everyone was asking me when would I finish. This external pressure was tearing me up inside. I hated telling my family and friends I was not done. I wanted to achieve this goal, but I wanted the work to be quality. Luckily, I had great leadership from my two advisors who moved me forward and demanded the high quality work this degree demands.

Overall, though immensely difficult, I learned an incredible amount about myself from this process. I grew in every capacity including as a mathematics teacher, an education researcher, and as a leader.

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APPENDIX A: CONSTENT FORM

UNC CHARLOTTE

Department of
9201 University City Boulevard, Charlotte, NC 28223-0001

CONSENT TO PARTICIPATE IN A RESEARCH STUDY
"Middle School Mathematics Teachers' Understanding of Slope"

Dear ___________________,

You are being asked to participate in a research study, "Middle School Mathematics Teachers' Understanding of Slope." The purpose of this research study is to gain insight into your current understanding of the mathematical concept of slope and how you incorporate this topic into the curriculum.

Please read the information carefully. At the end, you will be asked to sign this document if you agree to participate in the study.

Mr. Timothy Hoffman, a Doctoral Student at UNCC, will be conducting the research. He will be working directly under the supervision of UNCC professors Dr. Anthony Fernandes and Dr. Michelle Stephan.

You have been contacted about this study because your Principal provided your name as someone that may be willing to participate in this study. Your participation will require you to sit down for two task-based interviews conducted by Mr. Timothy Hoffman. These two interviews will take place over the course of two to three weeks. The first interview will be approximately one hour long while the second interview will be designed to take around thirty minutes. During these interviews, you will be asked to solve various problems about slope and how you address this topic in your daily instruction. Each interview will be recorded for both audio and video. Two cameras will be used to record: one focusing on you while the second camera will focus solely on the paper to record anything that you write while solving each mathematical problem. Using the recording, Mr. Timothy Hoffman will transcribe the interview, of which you will be given a copy of the transcript.

Mr. Timothy Hoffman will make every effort to protect your privacy. All your responses to the interview questions will be kept confidential with the digital audio recording files kept on a password protected computer in a password protected folder. The recordings will not be stored on a public network folder and they will be coded by a number rather than your name. After the audio recording is transcribed, it will be destroyed. The

transcriptions will contain no identifying information. During the study, all transcription materials will be kept in a locked filing cabinet in a locked office. When the results of this study are published, participants will be referred to by a pseudonym.
The decision to participate in this study is completely voluntary. You will not be treated any differently by the researcher if you decide not to be in this study. If you decide to be in the study, you have the right to withdraw from the study at any time.

UNC Charlotte wants to make sure that all research participants are treated in a fair and respectful manner. Contact the university's Office of Research Compliance at (704)-687-1871 if you have questions about your rights as a study participant. If you have any questions about the purpose, procedures, and outcome of this project, please contact either Mr. Hoffman, Dr. Fernandes or Dr. Stephan.

This form was approved for use on **June, 30, 2015** for a period of one (1) year.

I have read the information in this consent form. I have had the chance to ask questions about this study, and those questions have been answered to my satisfaction. I am at least 18 years of age, and I agree to participate in this research project. I understand that I will receive a copy of this form after it has been signed by both me and the principal investigator of this research study.

____________________________ __________________________ __________
Printed name of participant Signature of participant Date

____
Person obtaining consent

APPENDIX B: FIRST INTERVIEW PROTOCOL

Warm Up Questions

1. How long have you been a mathematics teacher?
2. Why did you decide to teach mathematics?
3. Why did you elect to teach middle school?
4. What mathematics courses have you taught?
5. What courses or grades are you licensed to teach?
6. Have you earned your National Boards?
7. Have you earned your Masters? In what?
8. Have you ever taken calculus? When?
9. What is the highest level math course that you have taken?
10. What comes to your mind when I say the word "slope?"

First Interview Protocol

11. Given the following tables, determine if any of the values were produced from a linear function.

Table A

x	F(x)
1	1
2	4
3	9
4	16

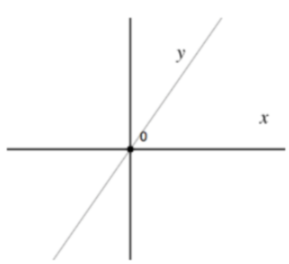
Table B

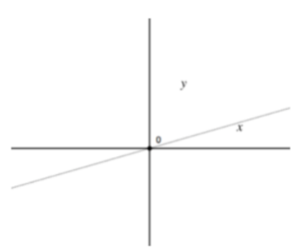
x	F(x)
1	2
2	4
3	6
4	8

Table C

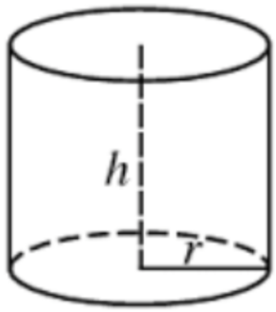
x	F(x)
-2	6
-1	3
0	0
1	-3

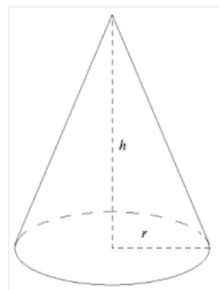
12. Last year, I asked my students to examine these two lines. Christine believed that the two lines have different slopes, but Ahmad disagreed. What might each student be thinking?

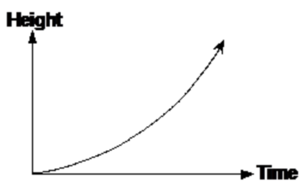


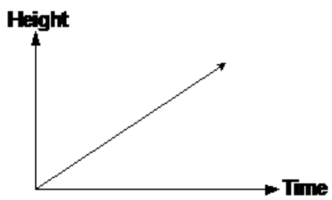


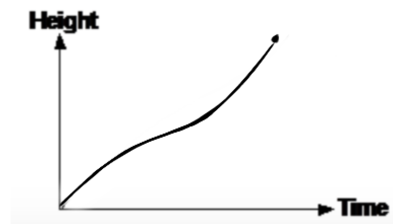
13. Let's imagine that we have three empty containers (a cylinder, a cone and a fish bowl). They each will be filled with water from the same source. Please match the container to the graph showing how height of the water changes with time. Explain your reasoning.







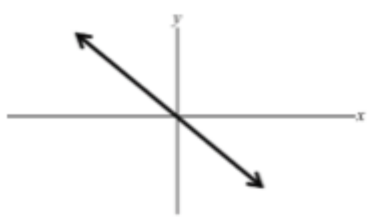




14. During a previous interview, a teacher stated that the slope of this line is thirty degrees. How would you respond?

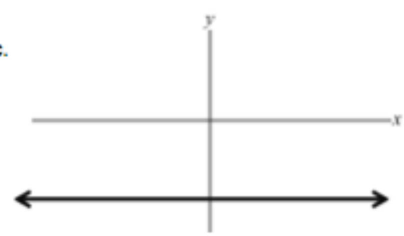
15. Explain whether any of the following functions (a - d) could have a slope of 2?

a.

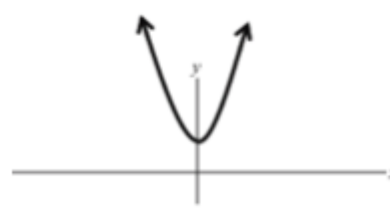


b.

c.



d.

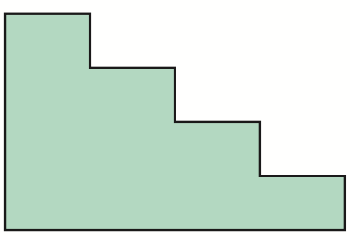


16. As I was searching the internet, I found this image

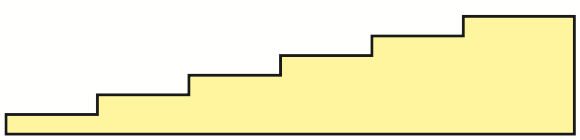
   a. What do you think they mean by six percent?
   b. What would sign look like if the percentage was
      i. 50 ii. 100 iii. 200

17. Given the linear function y = ax + c,
   a. what happens to the line if the *a* value gets larger? smaller?
   b. what happens to the line if the *c* value gets larger? smaller?

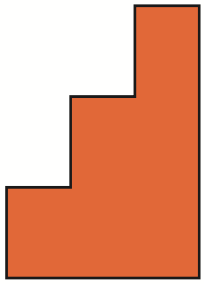
18. Order the staircases shown below according to how easy you think they would be to climb. Please provide a reason for your answers.



Staircase A

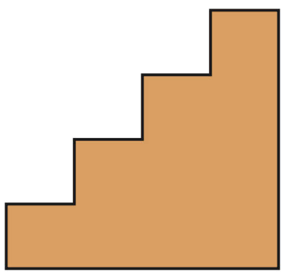


Staircase B



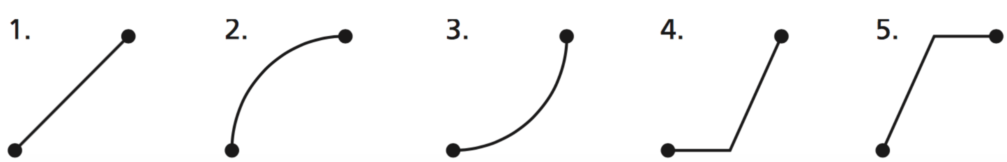
Staircase C

Staircase D



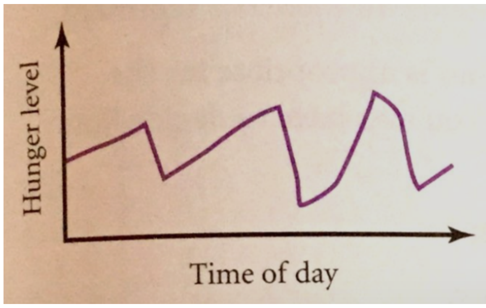
Staircase E

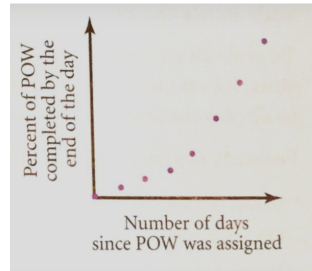
19. Match the following connecting paths for the last 5 minutes of Daren's race.

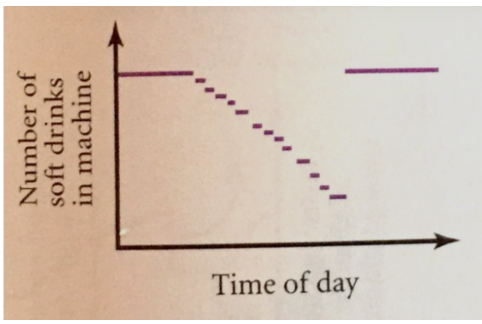


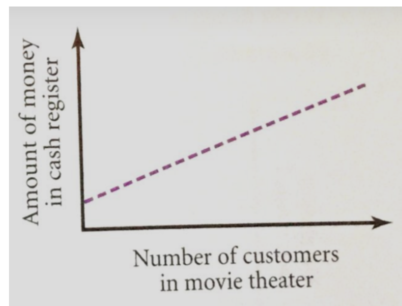
a. Daren finishes running at a constant rate
b. Daren runs slowly at first and gradually increases his speed.
c. Daren runs fast and then gradually decreases his speed.
d. Daren runs very fast and reaches the finish line early.
e. After falling, Daren runs at a constant rate.

20. Each graph sketch illustrates a relationship between two quantities. In each case, describe a situation that is illustrated by the graph



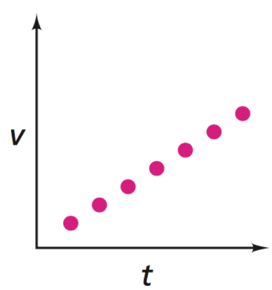




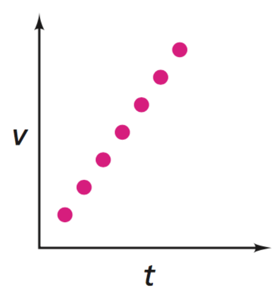


21. Denise and Sarah both observe a leaking faucet. Each of them makes a graph of the data that they collect. What might have caused their graph to look different?

Denises G raph



Takashi's Graph

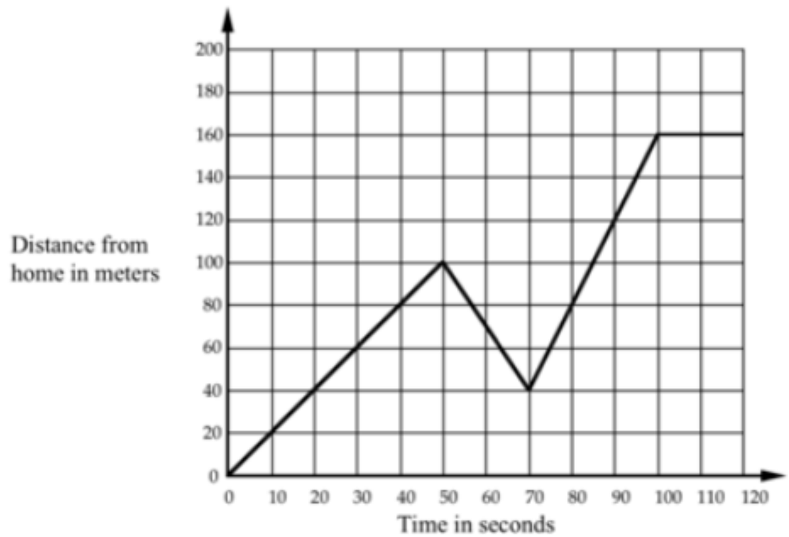


22. How would you describe what a ratio is? What images come to mind?
23. How would you describe what a rate is? What images come to mind?
24. How would you describe what a rate of change is? What images come to mind?

APPENDIX C: SECOND INTERVIEW PROTOCOL

1. Can you please define the word slope?
2. What images comes to your mind when I say slope?
3. What do you need to generate the equation of a line?
4. What are parallel lines? Perpendicular lines? Tangent lines?
5. Why are the slopes of parallel lines equivalent? Why are the slopes of perpendicular line negative reciprocals?
6. In the first interview, a teacher stated that the slope of a line is *m*. How would you have responded if a student in your class gave this answer.

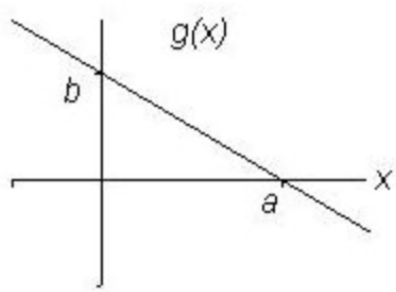
7. Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph below shows his journey on one particular day.



a. Describe what may have happened. Please include details like how fast he walked.
b. Are all sections of the graph realistic? Fully explain your answer

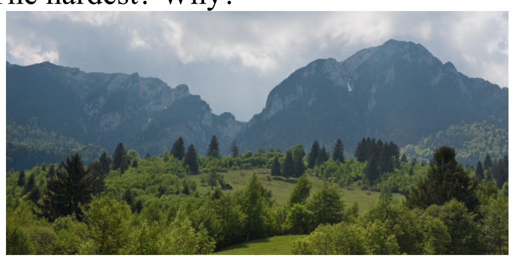
8. Anjelita receives some money as a birthday gift. She saves the money and adds more to it each week. She adds the same amount each week. After five weeks, she has saved $175. After eight weeks, she has saved $190. How much does Anjelita save each week? How much money did she receive for her birthday?

9. Given the following graph, what can you conclude about the slope? Can you draw a line with a smaller slope? A larger slope?

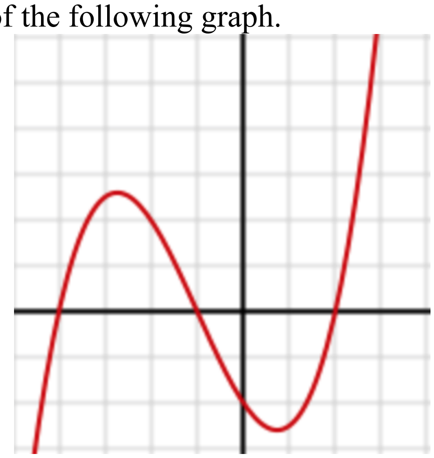


10. The angle, or pitch, of a roof is calculated by the number of inches it rises vertically for every 12 inches it extends horizontally. How would you examine how to calculate the pitch to a friend that was not strong in mathematics?

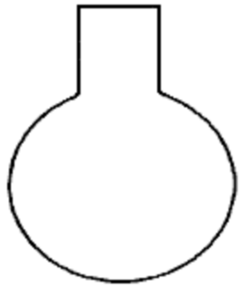
11. If you were walking across the top of this mountain range, where would it be the easiest to walk? The hardest? Why?



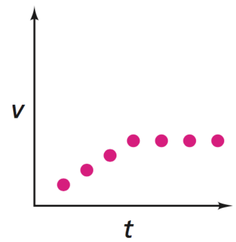
12. Describe the slope of the following graph.



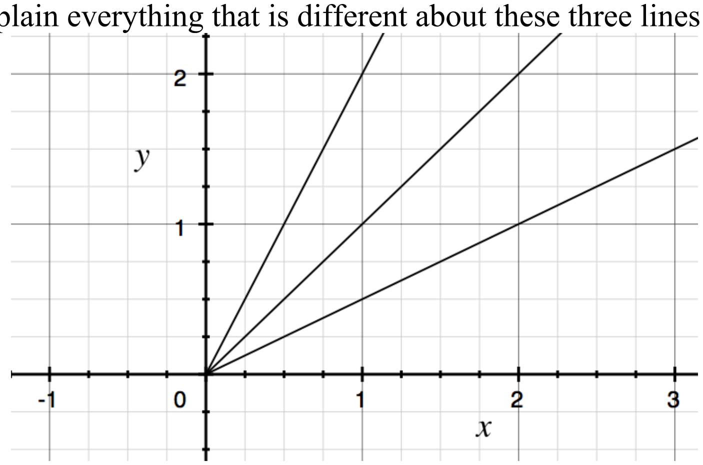
13. Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.



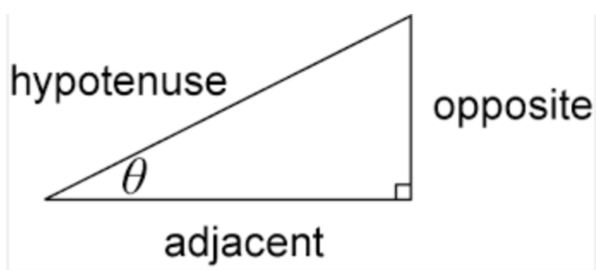
14. Given the following graph for a leaky faucet, please describe what happened.



15. Please explain everything that is different about these three lines



16. How does changing the measure of theta change the slope of the hypotenuse?



APPENDIX D: FIRST ROUND INTERVIEWS

Angela

Began with small talk. Paraphrasing open answers. See was extremely talkative.

I: How long have you been a math teacher? 5:12
Angela: 29 years
I: Why did you teach middle school?
Angela: I think because that was the hardest for me. I loved elementary even though I changed schools. (Discussed her previous schools). It was that awkward, even back in the 70s like that. They are still kids. They want to be grown. I think I felt that way.
I: Why math?
Angela: My math teacher. (Names teacher and talks about him). I always liked math but it was them.
I: What classes did you take?
Angela: I took calculus in high school but nothing that high in college (Paraphrasing)
I: What is your certification?
Angela: I was certified 4 - 9. They don't even do that anymore.
I: Did you earn your National Boards?
Angela: Masters?
I: Yes, did you earn your Masters degree?
Angela: No.
I: When you hear the word slope, what comes to mind? 19:13
Angela: I hate to say it, but rise over run. But you know it is the slant of the line, well I teach it going up the mountain is positive. Skiing down the mountain is negative. I tell the kids about the slant something is.
I: So why do you say you “hate to say rise over run”?
Angela: Because that sounds just like a math teacher’s way to say it. Because when I ask my boys about slope, they say “rise over run”. Then I ask my husband, the accountant, Terry - what do you think slope is? And he said, “well, the degree that something is elevated.” Maybe someone on the street would say, the grade, maybe the slant of a rant, that a wheelchair goes up; the elevation of it.
I: (explains question #1 - 3 tables - which is a linear function?)
Angela: Can I graph them?
I: Absolutely, that’s what the paper is there for.
Angela: *Starts graphing table 1* Well, that one should be.
I: So what process did you use to determine if they were linear functions?
Angela: I graphed them. But I also tried to figure out some sort of pattern.
I: So what was the pattern then?
Angela: Squared. 12 = 1, 22 = 4…(Table A) This one (indicates Table B) goes up by two - multiplied by two. (Now pointing to Table C) This one is times by 3 - no couldn’t be - times by negative 3.

I: So, A = squared, B = times by 2, C = times by negative 3? So all of those are linear functions?
Angela: After I graphed them, I thought they would be.
I: (explains question #2 - two graphs = same line, or different lines?
Angela: They're not the same line.
I: Why do you say they're not the same line?
Angela: Because I'm looking at the slope.
I: What about the slope makes you think that they are not the same line?
Angela: Well, they're both crossing through (0,0) at the origin. But this one here (indicates graph on left and then begins to talk through the problem by saying things like "up one over two" but does not reach a conclusion.
I: So what is your initial impression - same line or different?
Angela: Different.
I: And what stands out about them being different?
Angela: Because of the position the line is...this one is more steep (points to graph on left), where this one is almost parallel to the x-axis I guess.
I: So could you find an argument about how they could be the same line?
Angela: No. I'd love to hear how they could be.
I: So in your mind there is absolutely no way that they could be the same line, just different?
Angela: I'm trying to think of an ordered pair on this one (graph on left) that would be the same. But when I do that...it's not going to be over here on this one (graph on right). Because if you go to the right one and up two (graph on left) and to the right one and up two; it's not gonna be on there (graph on right). So I'm trying to find an ordered pair that's going to be on both; besides (0,0) there's not going to be anything else. 25:24
I: So you see one point in common between the two graphs, but outside of that they're different graphs?
Angela: Yes.
I: Both going through the origin?
Angela: Yes.
I: Excellent (I explain the matching shapes and water problem) 26:00 - 26:38
Angela: (Thinks until 27:36 moving them around). I guess it is that. Now, the reasoning is going to be really weak.
I: What is your reasons?
Angela: Again I am looking at the shape of that, so to me, the water, because of the circular so I matched that one to that one (Matches line to cylinder).
I: What do you mean because of the circular?
Angela: The shape of it. The amount of time that it took to fill. It is coming in like that (Tracing the line). The shape of it. Now, these two, they took longer. (Pauses) I guess this one is just a opening, it is just going to come straight up because of the shape of it. (matched cone with curve and fishbowl with quad) (Talks about peeling back cone). I know that the base is circular.
It is large (the base of the cone) and gets smaller. This is the same size, so to me it is going to fill up like a straight line (cylinder) so to me this one it is going to fill up like that.

I: Thanks. Let me keep these together. (Explain 30 degree angle one) How would you respond?
Angela: Well a straight line is 180 degrees and there is a right angle that is 90. (Pauses) Well is is acute, if I am looking at it that way. I could see that. They are vertical (30 degree and its vertical) So that is 30 (labels the vertical angle). (Mumbles to herself "How did she know it was 30) (Moves paper around but does not talk). Well, I definitely see that if you extend that up, I can see that if this was 90 (The right angle in Quadrant 1). Then you know, here I am again, it is not, I thought it would be 45 but it has to be a little less than 45 but I don't know why she would say it was 30 without having to measure it. I could not look at it and see that it was 30.
I: So would you express slope in terms of an angle?
Angela: Say it again.
I: (I reword and repeat the question) 32:40
Angela: No, I have never seen slope expressed this way. I would want them to explain how they came up with 30 degrees. I have never been taught this or seen it done with a degree. That is being honest. I would be curious how that even came up.
I: So in the beginning, I asked you to explain slope and you said that you did not want to say rise over run since that is a math teacher answer. You mentioned your husband and saying that it involved an angle. What did you mean?
Angela: The slant that it was. How steep it was.
I: (I wrap up this problem and we engage in some talk) (I give her the four graphs task: Positive 2) 34:34
Angela: (Talks to herself) (Focuses on b) (trying to create points).
I: Why are you focusing on b?
Angela: Because that is where it crosses on the y-intercept. (Circles the y-intercept on b)
I: Okay. How about AA, could it have a slope of positive 2?
Angela: Well, it is coming through the origin. (Pauses) You have to get another ordered pair. Down. You did two. (really struggling)
I: So if you were guessing the slope, what would be your guess for A?
Angela: Well, it has to be negative because it is going downward.
I: So could a have a slope of positive two? 36:00
Angela: No.
I: Okay, why not?
Angela: Cause it is slanting down. It has a negative slope. It is like skiing down the mountain. This is a negative slope (A), This is a positive slope (Points to B), this one either, is straight across, which is horizontal (talking about C)
I: So could C have a slope of positive 2?
Angela: No.
I: What would you guess the slope of C is?
Angela: No slope.
I: What is the numeric value for no slope?
Angela: No slope. (Draws a 0 with a line through it implying undefined)
I: What about D?
Angela: It is like an x squared (Writes x^2). We don't deal a lot with these ones. I always tell the kids when they see the squared to know it is a parabola.
I: So is an x squared graph, could have it have a slope of positive two?

Angela: (Trying to create points on the graph) So, no then.
I: So D, no? A you said was negative. C you said was a zero slope. What about B?
Angela: Well because it is going upwards. That is a positive slope. It could be positive.
I: So out of the four, could any have a slope of positive two?
Angela: Only B
I: Why?
Angela: Because it is slanting upward.
I: (I explain the sign)
Angela: Six degrees
I: Not six degrees. Six percent
Angela: Six percent. Oh my gosh.
I: What do you think that they mean? 38:49
Angela: Oh my gosh. I am going to say that it a big hill for some reason. The picture makes you think, it is like holy cow. I mean you don't have a pray to get down there
I: So, what do you think 6% should look like, if you had to draw the picture? How could you draw it accurate.
Angela: Maybe six percent is not that steep. I mean, if you saw 50% you would think you man.
I: Would 50% be steeper or not at steep?
Angela: Well, if you think that 100% and they are only saying six percent, was this near this (Indicating the sign and the mountain in the background).
I: I think they are just
Angela: Maybe when I first saw it, I thought, that it was really bad because of the rocks. If they were looking at it like that cause a lot of the stuff is based off 100%.
I: So what would 100% be? If they were basing it off 100, what do you think that picture would look like?
Angela: Oh my gosh. It would be, I guess just straight (Using her hand to indicate vertical)
I: Straight up and down?
Angela: Straight up and down.
I: Okay, so what would 50% be?
Angela: Oh, it couldn't be that, that would be straight across. That would be stupid. It couldn't be that.
I: What couldn’t be?
Angela: 50%, if I said that it was completely straight, then I thought 50% would be that (indicated a horizontal line with her hand). And that would not even be a hill. You actually found this (Pointing to the picture)?
I: (We discuss how I found it)
Angela: Again, I would think that whoever did this was thinking 100. And they did not even think of 50 being half a hill. Six percent to me is not very steep. It is a little bit but not a lot.
I: So lets say my hormonal or my run was 100, what would the rise have to be to get 6 percent? 41:37
Angela: Oh my gosh. Not very much.
I: Okay. So as the percent increases, what happens to the hill?
Angela: It gets larger.

I: What do you mean it gets larger? The line gets?
Angela: The larger the number, the more slant it would be.
I: The larger the number, the more slant, on the way to 100 which would be?
Angela: Straight up and down.
I: So what do you think horizontal would be?
Angela: (Indicates with her arm a horizontal slope)
I: So zero would be like this (Horizontal) and 100 would be like this (vertical)?
Angela: Yes. I think it sounds crazy but yeah. This is different. I guess (Hands back problem) (Notices an algebra 1 book and talks about it being hers in high school) (Flipping through book) 43:19
I: What happens to that line as A gets bigger? (She writes y=ax+c and then y = 2x+c and y = 4x+ c)
Angela: That would be your slope (pointing to the a value). The degree that it slants.
I: So as A gets bigger, does the degree that is slants get?
Angela: Bigger.
I: Steeper or less steep?
Angela: Okay, so if you did (Starts to plot points and tries to create coordinates) (Up two over one, Up four over one). Yeah the slope is getting larger, the line is getting steeper.
I: What happens if A gets smaller?
Angela: Okay, lets make it a negative. Negative x plus c and negative three x plus c. Down one over. Then if you three and it is negative three then it is (constructing graphs and talking to herself) Negative one and over one. (writes 1,1) (Talking through the problem.) mx+c. -3x+c Okay, so as A got smaller the slope got steeper (comparing two lines that she drew) 47:14
I: So, as a gets bigger, the slope gets steeper?
Angela: Yes.
I: And as A gets smaller, the slope gets steeper as well?
Angela: Yes. Yes. Yes. If I did it right.
I: What if I change the c value? If c gets bigger, what happens to the line? 47:34
Angela: Ok, so that’s where we cross (drew lines on her graph). I have to draw it. So if they cross here at one, they’ll cross here at two. (Draws two coordinates planes with (0,1) and (0,2) on the other). As c gets larger, the slope gets steeper.
I: Ok. What about if c gets smaller?
Angela: If C gets smaller, it is less steeper.
I: (explains question dealing with staircase and ease of walking up) 48:51
Angela: B.
I: Ok, why?
Angela: Again, I’m looking at the rise over run (Talking about B). The rise is not very tall.
I: So, now that you have B being number 1, can you rank the remaining steps as 2 - 5?
Angela: This is the easiest (B) and this is the hardest (C)
I: Okay, why did you say that C was the hardest?
Angela: Again, you are way up here to step (Tracing the vertical of the first step). It is a whole lot higher. Now, of these three. (Brings over E to examine the height of each first step). I still see that as harder (E)
I: What were you looking at with C and E?

Angela: First cause I jumped at it, I was thinking that maybe. The rise here (C) compared to this one (E). That is what I am looking at. Now, these, being right handed, these are easy to walk up. (Talking about the stairs facing to the right as opposed to the others from the left). (Talks about that for a moment) This would be second (D) Again, the rise. (Debates between A and C for the third hardest) That one is still here (4th - C) to me. I looked at that. (Circled the rise on the second step of A) compared. That is the way that I see it. (B, D, A, E and C)
I: So you have B first, followed D, then A, then E and C is last. (She talks more about the staircases facing different directions.) (I explain the Darren problem) 52:16
Angela: Constant. (Matches 1 with A)
I: So you got 1 and A, how did you match those?
Angela: I could immediately see it. Cause it, I know the slope will be over one up one. Over two up two. Just from, it is perfect coming through the origin. That is what I see. It is consistent. For me, I would have gone with that one, if it would have been the fifth one (instead of the first one). Just because of the perfect slope like. I like things like this.
I: Good.
Angela: (Talks about this not having numbers and people using algebra in everyday life. Talks about the book again) After falling, Darren runs at a constant rate. After falling (Moving pencil to D). That one is E (labels 4).
I: Why?
Angela: I will probably do five next. The straight. I saw that he run (Traced horizontal portion of 4) and then he fell. I will be honest, I saw the word constant. I knew it was either this one (4) or this one (5). I may have to go back. I should have read them all first (Talks about telling students to read them all first) (Reads other ones out loud). I don't know. Now I am thinking that this one could be D.
I: Which one? 55:16
Angela: Well 1. This one (1) is still a constant to me. I am thinking the slope is, but then you say that he runs really fast and reaches (tracing e with her finger) but I don't see where these curve ones. That is what I have to figure out. Darren runs slowly at first, slowly (Traces start of 3) Slowly. I guess downward. That is it. You got the slope (Draws a coordinate plane with a negative sloped line) He is going slow, then increases cause the slope is going upward (Traces 3). This must be B. (Labels 3 B). He runs fast and then slows down (Traces 2). This is going up. Like if I drew a thing like that (Draws coordinate plane over 2). He is running, running, running then slowly down (Traces 2 but continues trace to make a downward opening parabola). That is slanting down.
I: Okay. He is running, running, running and then it is going to dip back down?
Angela: That is when he decreases. That would be C (Labels 2 C). This one though (Points to 5). He runs very fast and reaches the finish line early. Let me go back to 4 to see. After falling, (Long pause). (Continues to debate to herself). He runs slowly at first and gradually decreases (Traces 5). That could be D (Labels 5, immediately erases). Oh heck. This one could be B (Labels 5 B). Now, I am doubting that that one (3) is B. And I thought these ones were just the opposites of each other (2 and 3). I fell as confident in that one as I can be (A and 1).
I: One and A you are positive about?
Angela: Yeah. Now, this one I am even doubting (4 and E). After falling. I am seeing it. No. (Draws coordinate plane on 4). Down. (Pause). I feel pretty good about that one (4

and E). But the other three, holy cow. He runs slowly at first and gradually decreases his speed. That one could be that and that could be that he fell (Tracing 3). Oh no. I am in the store and there are ten items. (Continues to read them out loud). If I say that one, he runs really fast and reaches the finish line early (1) then Darren running at a constant rate, I have nowhere to put it. So now, I am not even sure about, after reading them, which I should have done first. Darn. Ah no. This is hard. Cause I am the one on the test. I start doubting myself. Erasing. (She talks more about this). Darren runs fast and gradually decreases. That is the only one that could be (Puts check mark by 2). Darren runs at a constant. Now, I am not sure that that is A. But if it is not A, then (Pauses) After falling, then at a constant rate (Traces 4). I will go with that (4). (Labels 4 E). I am not going to change A or E. (Reads B) I am going to stay with B (for 3). All right. That is what (Circles horizontal segment on 5), I don't know why it would say that he is finishing early. Well, if it had to match up with some. But this would be my least feeling like I would be right. That would have to be based on process of elimination. I feel the least confident about that one (5).

I: So still most confident about 1 or 4? 61:01

Angela: I don't know. Two is like he runs slowly, he runs slowly, cause none of the others I could even get. I never even questioned this one (2) cause I did not see any other picture that looked like he was running slowly (Traces 2) and then increased it. So, I guess that is my most confident (2 with C). I thought it was going to be this one (1) until I looked down and read. See, he has no obstacles. He is just going to run until he gets there. It is the early part. (Doubting between 1 and 5). Constant. It said. Both of those. This said he fell. Fell.

I: So, on 4, where do you think he fell?

Angela: Here, I guess. (Points to the cusp). What why is he running that way? (traces horizontal) instead of that way (traces slanted) That is what I… This one is after falling. I don't know why I am looking at, I think I am looking at that line (Slanted on 4) with that(1).

I: So when you see the line, what are you thinking?

Angela: constant.

I: So number 1 and 4, you are thinking constant rates?

Angela: Yes.

I: Okay. Good job. (I explain the hunger problem) 62:58

Angela: Well, you wake up or whatever and you are hungry. The slope, the line is higher. Then you are not hungry, it dropped. It went downward (tracing the second segment). Gradually during this time of day, it went back up (tracing third segment). Drops, your not. It goes back up, so your really hungry. Eating. Then the end of the end of the day. Again, I am looking at rise over run. The higher, the rise. When it drops, the hunger drops. As it goes up, I am getting more hungry. Then I ate (Tracing three to four). Now, I am not hungry at all. Then I ate and I am not hungry. As the line goes up, the hunger gets more. As the line drops, the hunger level drops. I can see that one

I: (I explain the movie theater one) 64:33

Angela: The more customer in a theater, the more money you will have in the register.

I: What about the origin?

Angela: You have money in the drawer. You might have zero customers but you have to make change. Like when you do game duty, you would have money in the register.

I: Is that regression a constant?
Angela: If I go back to the other, they went through (0,0). This one does not. After that it does. Well,… I mean without getting into the price of the kids and the adults, it would not be cause depending on what they people were and stuff.
I: So looking at that graph, are the kids and adults charged the same or different rates?
Angela: (Pauses) If it was running straight through that origin, I would definitely say yes without even thinking about it. It would be the same. But because it is not. well. if I go back and think that there was already money in the register, it is already going to be off like that (Not through the origin) but it looks like it is.
I: It looks like it is what?
Angela: Constant
I: SO are they charging the same or different rates?
Angela: Same. The same.
I: (Number of Drinks) 67:25
Angela: As the day goes on, there are less drinks in the vending machine. A lot of them buy them here (Tracing horizontal portion of the graph). Throughout the day, there are less drinks (Tracing dotted portion of the graph). The further along in the day, the less drinks there are (Traces whole graph).
I: What about that last piece of the graph? 67:54
Angela: Oh gosh. He fills it back up I guess. At the end of the day.
I: Are people buying drinks during this time? (Trace horizontal section)
Angela: This must be before they get to work (Traces horizontal start) This is before anybody gets there. There must be something wrong though. Oh, it is already filled up from the other day. Oh god. (Pause). Hmmm. (pause). That is easy for me to see, that as the day goes by (tracing dotted portion). Then all of a sudden, they are filling it back up. This is the same as that (Two horizontal pieces). So if I said that, are people buying drinks there (First horizontal section). But it is not dropping, the slope is not going downward. That means there are no drinks coming out of the machine.
I: So, initially you said that they were buying drinks during the horizontal part. Are you still thinking that? 69:15
Angela: No body is buying drinks now. I will be honest with you. I did not see that at first. Then when you did that (Points to second horizontal line). Well, there is no way that did not even make sense.
I: So, no one buys drinks, people buy drinks, they fill it back up and no one buys drinks, is that what you are saying?
Angela: I don't know. Well, it is not like they are back refilling it again. Cause if this is the (trails off)
I: So a little torn on the horizontal part but certain on the downward part?
Angela: I am certain of that (Points to downward portion of the graph)
I: What is happening on the downward part?
Angela: The number of cans in the machine, there are fewer cans in the machine.
I: Okay. Cool. Thank you. (I explain the POW problem) 70:31
Angela: Well as the days, as the x-axis gets larger, the y axis gets larger. 1, 2, 4, 8, whatever it is (tracing the x-axis) the kids are turning them in (tracing the points)
I: Are they turning them in at a constant rate?

Angela: No. Again, I am looking at it like (Lays pencils over points). It would be a straight line. Their slope is not the same.
I: (I explain water faucet problem) 71:40
Angela: They are both going through the origin. If you laid a ruler over the points, they would both be straight lines.
I: What might explain that they are slightly different?
Angela: (pauses) (Traces staircases on first two points of each graph)
I: What are you thinking?
Angela: Maybe what they are comparing. Did they tell them after five minutes to go and look or do you have no idea?
I: No idea.
Angela: It might be, how they, these aren't the same. She might have been looking every five minutes (Denise) and might have been two inches and she (Takashi) might have been looking one minute. The t and v, they weren't told that they had to be the same.
I: Okay. Could those be the same graph?
Angela: They could. I just talked myself, depending on the scale for t and v, they could be.
I: So if two students turned those into you, full credit?
Angela: Well, if I did not tell them the (Counts points). If I just told them that they had to have seven ordered pairs of information and I did not tell them what, the rise and run, had to be, then yeah. That is interesting there (Hands me paper).
I: Okay, I have three follow up questions? Do you teach ratio?
Angela: Yes, I do.
I: How would you describe a ratio?
Angela: A comparison between two different variable things. Students to teachers or families to pets in the home. Whatever you are comparing.
I: Do they have to be different? Teachers to students or can they be the same teachers to teachers?
Angela: They can be the same but you would have to explain the difference. Like teachers to teachers but you would have to say male teachers to female teachers. Say colors to colors but you would have to say your favorite color to least favorite. They would have to be different.
I: When you hear ratio, what mental images come to mind?
Angela: Honestly, teachers to class size. Cause I have watched it go up. One teacher to thirty kids. 35 kids to one teacher. That is honestly the first thing that I think about.
I: What about the word rate? How would you describe it?
Angela: I usually go straight from percent. In my mind, I am thinking like, what rate does something increase. I am thinking like, the percent of it. I don’t know why.
I: Okay, so if you were writing on the board. Ratio is a comparison between two things. Rate is, what?
Angela: You could bring it back to slope. What is the rate of increase in the, the x increases versus the y. I think it might depend on what I was teaching. I can not think of a good definition for rate.
I: Okay, no problem. What images comes to mind?
Angela: Ahhh. Money. Percent. Increase or Decrease. I mean that this is where the curriculum is drilled in my head. Percent. You are buying something. The interest rate.

The rate goes up or down. I guess bringing in money. I am thinking. Interest rate. Interest percent. I guess that is what I think about.
I: Okay. Last definition. How would you define rate of change?
Angela: Rate of change. (Pauses) How fast (Pauses) How fast something is either increasing or decreasing.
I: Do you think of any images? 79:00
Angela: Well, if you do it with money. As the rate goes up, the higher the rate is, the less would, the more you would pay, the less the rate is, the less you would pay. It comes back to slope. Rise over run. The larger the slope, the slant of the line. The negative slope it is downing downward. The slope is positive, the line is going up. I never taught about it that way.
I: Thank you.

Brianna

Interview begins with small talk. Including discussing that she is going to begin her PhD in Urban Education at UNCC.

I: What is the highest math class that you have taken?
Brianna: I went back an got my masters in math education.
I: So you have taken calculus?
Brianna: Yes. I actually taught AP Calculus for a year. I have pretty much taught everything between seventh and twelfth grade expect for discrete. I have never taught discrete.
I: Neither have I. I have taught everything expect discrete and statistics. Those are my two missing pieces.
Brianna: I have never taught statistics either. I never taught pre calculus in a school but I used to work at UNCC pre college program, so I taught Pre Calculus there. I so kinda of taught everything but discrete.
I: So you are certified six - twelve then?
Brianna: Yes, I started out certified 9 - 12 and I then I added the middle school certification.
I: Based on your background, when you hear the word slope, what comes to mind?
Brianna: Rate of change. Rise over run. Steepness of a line. Positive, negative, undefined. Zero. All types of things. 8:01
I: What mental images come to mind?
Brianna: A rollercoaster
I: Why a rollercoaster?
Brianna: It starts out at a zero slope. I always use this example with the kids. As it creeps up, cause it is going up higher it is positive. Right before it drops and sits at the top, I tell them that that instant pause of time, right before you go down the hill, is that zero slope. Then it goes back down, so I kinda of go through those different slopes with them. All of them expect for undefined slope. Then they will tell me about the drop zone, where it takes you up and drops you straight down and I use that as the undefined slope.
I: That is awesome. Do they relate to that example?
Brianna: Yes, because they love roller parks.
I: Do you use that example at the middle school level?
Brianna: Yes.
I: Awesome. The first task is to identify if any of the three tables were produced from linear functions. Please explain your rationale.
Brianna: (Picks up pencil and works on A) B and C (Examining y values)
I: How did you determine that?
Brianna: I saw the squares pattern (Pointing at A). So I automatically knew that was not linear.
I: How did you figure out B and C?
Brianna: I did the, I used the power of slope formula in my head. So, I saw the change in the y values was adding two and all of the xs were the same, one, and that is how I figured out that one. I did the same thing for C. The change in all of y values was minus

three and the change in all of the x values was adding one. So, that is how I figured out that one.
I: So you used change in y over change in x.
Brianna: Yes. 10:11
I: Great start! So last year, I gave this line to my students. One of the students argued that these two lines were the same while another firmly believed that they were different. What might each of these students be thinking?
Brianna: They are similar because they both have a positive slope.
I: What makes the slopes positive?
Brianna: Because they rise if you look at the graph from left to right. They increase as you move from left to right. As the x values move from left to right, the y values increase. Graph #1 (Label the graph on the left #1) has a greater slope than Graph 2 (One on the right). This makes them different. That is how I could see both perspectives.
I: How do you know that there is a different slope?
Brianna: Because this change here (Creating a triangle on Graph 1). This triangle here is bigger than the triangle here (Graph 2). So because this line (Graph 1) is stepper or closer to my y axis. This is what I tell my kids, it makes it steeper than this line which is closer to the x axis, it makes this slope less.
I: If you were answering the question, would they be the same or different lines?
Brianna: I would tell you they are different.
I: They are different?
Brianna: Yes, because their slopes are different.
I: How are you making that determination?
Brianna: The steepness of the lines.
I: Thank you. For this task, I have three different shapes: a cone, a fishbowl and a cylinder. Each are emptied and will be filled with water.
Brianna: Okay.
I: Here are three graphs. Please match them. They are time versus height.
Brianna: (Thinks and organizes her cut out)
I: What are you thinking?
Brianna: I am thinking that the cylinder fills up at the same rate since it is uniform. So I am thinking the first one (the line) is the cylinder.
I: What do you mean uniformed?
Brianna: So, the height is the same from the bottom to the top. It is not a proportion. The base is the same on all of them in my mind but the height is the same, uniform all the way around. So, it will fill evenly. So it is not going to change or do anything like that.
I: Okay.
Brianna (Pauses) So then I have this. I think it is going to fill faster at the bottom then slow as it gets to the top since it is narrowing. And then this one widens in the middle, so I feel it is going to fill up slowly then widen then narrow but not narrow as much (Matched quadratic with fishbowl). It is wide so it is going to continue to fill but it would overfill. That is kinda of my thought.
I: Your good?
Brianna: Yes. (Took away cutouts in order).

I: So prior to this study, I a pilot study. She told me that the slope of this line is a thirty degree angle. How would you respond if a student or colleague stated the slope in terms of an angle? 15:39
Brianna: I would first tell them that slope is not an angle. That slope is the measurement of how steep something, a line, how steep it is. So not necessary its measurement or how wide the angle is, but basically maybe how long it is. More like the hypotenuse of this triangle (Traces the line in Q1) versus the width of the angle. So I would probably try to get them anyway from slope and angle because it does not deal with angles. But it can but it is kinda of a stretch. I would probably try to get rid of the concept of angles because that is a whole different ballgame. and try to focus them on how long it reaches (traces the x axis part of the triangle) and how high it goes (Traces the y axis part of the triangle). Using that to measure how far that line goes. That is what I would do.
I: Okay. Thank you. So, on this task, I have four separate graphs. My question for you is does anyone have a slope of positive two.
Brianna: B could.
I: Why B?
Brianna: Because it is the only one going in the positive direction.
I: What is the positive direction?
Brianna: When the y increases the values as the x goes from left to right or smaller to larger. That is the only one that could. I mean, if you are going to calculus, D could at a particular point like if I was trying to find the slope of the tangent line at a particular point. It could have a slope of positive two on the side (point to Q1) depending on the point that we are talking about. But not necessarily at every point.
I: So B does at every point?
Brianna: Yes
I: And if I am understanding you correctly, D could but at a particular point?
Brianna: Yes. At a minimum of one point. Just depending on where you are located when you are looking at x between positive and negative infinity.
I: Nice job. What would you say the slope of A is?
Brianna: It is negative.
I: What about C?
Brianna: It is zero.
I: How do you define a zero slope to your students?
Brianna: It is always horizontal. It is flat. It is where they walk. There is no incline. It is not the easiest part of your bike ride, if I use that analogy. There is no change in elevation or altitude when you are walking. I also talk to them about running. When you go to train, you might want to train in Charlotte because it is very hilly. So that way, if they go to Denver, they can try to adjust to the altitude changes. They can, the land, the way the land is formed there. Then if they run somewhere easy It is easier to run where it is flat because they ran on hills. I try to use those types of examples.
I: Wonderful. Well speaking on inclines, when I was searching for questions, I came across this picture and I was intrigued. What do you think they mean by six percent?
Brianna: The incline of the slope of this hill is six percent. Off the ground. So, that means the height would be six, and like, if I was talking about numbers, it would be six over 100. So that would be feet, miles or whatever the units are. The height of it would be six but the horizontal bit would be 100. So that would be six percent for me, six over 100.

I: 6 over 100?
Brianna: Yes.
I: What about fifty percent? What would that look like?
Brianna: 50 feet to 100 feet. 50 feet high to 100 feet long. SO I would expect the incline to be a lot steeper.
I: What about 100?
Brianna: A hundred percent incline would probably be a vertical line.
I: A vertical line?
Brianna: Yes. I think it would be impossible to that.
I: Could you go over a 100?
Brianna: You could but it would swing back the other direction.
I: Swing back?
Brianna: Yes, the line would start to become negative. But I think that you would not be able to do it until it got far on the other side that it would be six percent in the other direction. It makes me think of a unit circle, if you go over 100 percent.
I: So it would be in the next quadrant over?
Brianna: Yes.
I: Say you are given the equation $y = ax+c$.
Brianna: Okay. (writes it down).
I: What happens to the line as a gets bigger?
Brianna: The line gets steeper.
I: What about as a gets smaller?
Brianna: The line to me gets more flat. Closer to horizontal (Uses hands to show a line with a slope of zero). As long as you are not talking about going into negatives.
I: Why, what happens then?
Brianna: Then it kinda of changes directions. If a was two, it would be steep but it would be increasing as it goes from left to right. But if I flipped it to negative two, which means a getting smaller, it would be the same line, it would just flip directions. So in stead of it increasing as it goes left to right, it would decrease as it goes left to right. So, it really depends on if a is negative.
I: Good. What happens if C gets bigger?
Brianna: It just shifts the line of the graph up or has a vertical translation versus if c gets smaller, it has a vertical translation down. So its starting position would be higher or lower based on the value of c.
I: Okay, so if I change c, does the impact the slope?
Brianna: No, it just impacts the position of your starting point, but the slope stays the same. If it was positive, it is still going to increase as go from left to right with the same grade. The line is going to look exactly the same, it is just the position of the line will be in a different spot.
I: Awesome. Very good. So on this problem, I have five staircases. No right or wrong answers, just your opinion. Which staircase will be the easiest to use? Please rank them from the easiest to the hardest and explain. (I lay out the five staircases).
Brianna: (Immediately selects B). Thats number one.
I: Why did you select that one so quickly?
Brianna: The y-intercept, for me, which is the starting point, which starts not very high and then slowly, because the stairs are wide and are not very high would make this a

flatter slope (Tracing the points of each stair). Umm, D would be number two because it has a higher starting climb but the stairs are still wide. The slope is a little bigger (tracing the points of each stair) but not that much bigger. The steps are wide but not as wide. (Pauses). This would be three (Writes 3 on A). Similar heights in starting (points to A and D). The steps are more narrow (Tracing first step) so you would have to do a little bit more to climb up it. The slope is bigger (Draws line connecting points of the stairs). This one is four. (Selects E). It is a little bigger than 3.
I: What do you mean a little bigger?
Brianna: The starting point right here is a little bigger (Referencing the heigh of the first step). It is more higher than the steps here are a little bit higher to get up. (Draws a line on E). It may actually be about the same but the starting point is a little higher so that is why I put it here. So, then the last one is C (Writes 5). (Pauses)
I: Why did you put C at the end?
Brianna: Two reasons. This has the largest starting point (Tracing vertical part of first step). So getting up this would be hard. It also has the largest incline between steps (tracing vertical parts of each step) making that slope the largest (Draws line) so that would be reasons why.
I: Very cool. Nice job. (I take them up in order). Your are doing awesome.
Brianna: I am trying.
I: You are succeeding. So one these five graphs. One of the graphs match up to one of the scenarios. Please match each and provide a reason why (I explain it in more detail).
Brianna: (Reads problems). Okay. One is A.
I: How?
Brianna. Because he is running at a constant rate. Nothing is changing. It is not saying that he is slowing down or speeds up. He is just running at a constant rate. So the constant graph is not changing. Ahhh. I think E is four.
I: Why do you think that?
Brianna: It says after falling, he runs at a constant rate. So to me, this bottom piece is he is on the ground (tracing the horizontal portion of D) like he fell then he starts to pick up his constant rate again. SO he is not moving at all at this point (Points to the cusp). So he falls at this point (points at the cusp) then he starts to pick up his pace again (Traces increasing portion of D). Thats it. Ahhhh. What a minute. Let me think. 28:21
I: Sure.
Brianna: (Takes about 20 seconds). I think 2 is D.
I: Two is D?
Brianna: Yes.
I: Why do you link two and D together?
Brianna: Umm. It says he runs really fast and gets to the finish line early. To me it just seems kinda of abrupt like he is running fast and boom, he is done (traces 2). So to me, it is like he running increases and stops. He automatically stops. It is not anything gradual about that. (Pauses). Ummm.
I: What are you thinking?
Brianna: (Reading). B might be three.
I: Okay.
Brianna: He runs slowly at first and then he increases his speed. So, that is exponential in my mind. He starts out kinda of slowly and starts to pick it up. It does not say what he

does after that though. (talking to herself). I think 5 is C. He runs fast… Yeah. To me he is running fast here (traces first part of 5), then is gradually decreases (traces second part of 5) because it is not an abrupt stop. Kinda o like he is slowly down or cooling down. Yeah, I will go with that.
I: So what did you think were the easiest to match.
Brianna: A and 1 because that was just common sense and umm D.
I: D?
Brianna: Yes, I think the two ones that threw me were like 4 and 5 because I over analysis stuff. So, I was like, wait is that. So I think 4 and 5. The other 3 I think I knew.
I: Awesome. Nice job. 31:00. So the next ask gives you a sketch and asked you to describe each of the situations. Here is the first one. Hunger level versus time of day.
Brianna: So I start of in the morning, here where I wake up (points to the start) kinda of hungry. So during this time, I do what I have to do and my hungry increases. I eat breakfast right about here (Points to first cups). Then I am no longer hungry, I am pretty set. Hungry starts to increase here (first bottom cusp) and I have lunch right about here (second upper cusp). Then my hungry decreases as day goes on. I am good. I am really full (pointing at second bottom cusp). So, I feel like I ate a larger lunch than breakfast. My hungry level is really low. As time goes on, I think that I waited longer, but depending on what I ate may make me hungry faster. Maybe something that make me, not full, but does not last as long. Then I here and I have dinner. Then after that I may have a little water or something. Then I am not hungry at all (pointing at the last bottom cusp). Then I start to get hungry again but I think that I might be sleeping and then it starts all over the next day.
I: When is he the most hungry? or she
Brianna: At dinner time?
I: Least hungry? Most filled?
Brianna: Right after lunch at this peak at the bottom. (Second lower cusp)
I: Here is the right on. POW is a homework assignment. (Explain more)
Brianna: Assignment the first day. No one does it cause they just got it. So you have a few, a few students, that say that they have to get this out the way cause I don't want to deal with it, so that graph is kind of low in the beginning. So, as you get closer to the due date, the urgency kicks in. At the end right before it is due, everyone is cramming, so that is why it is the highest.
I: Do they same number of student turn it in each day.
Brianna: No, it is exponential. The number of students turning it in is growing.
I: Good job. (I give and explain the next problem: Drinks)
Brianna: So, the beginning of the day first thing in the morning, unless your (Names a teacher) you don't have a soda in the morning. So this graph, this would be like zero to 9 , 10 am (Tracing horizontal portion of the graph). Then as this progress down, between 10 and 7pm (labels x axis) meaning as the day goes one, people are getting drinks. So people have brought them causes the number drinks to go down. This causes the number of drinks in the machine to decrease over time. It looks like it is pretty steady so the number of drinks has decreased over time.
I: On yours, what happens after 7? 35:56
Brianna: People probably no longer at work. So it stay at the level. Depending on what building your are working in. And so you start it the next day. And it look like, right here

(point at top horizontal line) it looks like the next day. So maybe somebody filled it up at night and you started over again.
I: How would you describe its decrease?
Brianna: I would say that it had a negative slope. But it actually looks like a piece-wise function in a way because of this gap (Bottom of negative to top horizontal). I would say that it was a constant slope or a zero slope here (Pointing to beginning of graph) meaning nothing is changing. Here is a negative because people are buying the drinks causes the number of the drinks in the machine to decrease. Then the machine is filled back up and goes through this process again.
I: Very good. Last one. This is number of customers in a movie theater verses amount of money in a register.
Brianna: This is when or right before the movie theater opens at time zero. They have a specified amount in their register. Then the end of the night is when they close and how much money is in there then. More people come to the movie theatre as time goes on, the slope is constantly increasing. Because even if they don't come at the same time, the money is going to constantly increase because they are increasing, so you are going to have a positive slope.
I: At time zero, what is going on?
Brianna: Normally, in my mind, they prep the register with a certain amount of money to start. So, I worked in retail and we would start the day with 50, 75 dollars. A lot of ones, maybe one twenty. That way when people come you can make change. So that is the starting amount in the register.
I: How do you explain that it looks linear?
Brianna: Because the number of customer coming to the movie theater is not going to be stacked or staying still, it will increase. Every time that I add a new customer, the money is going to increase by the same amount, the ticket price. But if we were talking about adults and children, this graph would probably not look as consistent. So you would have to be only taking about adults or just children. Cause if you were talking about both, that would vary your slope, but this is linear.
I: Nice job. One more graph. (I explain leaky faucet problem). Why are they different?
Brianna: The units of time. So if I observe the graph in seconds and Denise observed in minutes, I feel like the graph would be different. There is 60 seconds in a minute, so, you have more seconds in a minute, so I would be taking more data points over that certain amount of time. So if we were both supposed to take five data points, if this one is seconds, which I think this one would be (Labeling the second graph) and this one would be in minutes (Labeling minutes on the first graph) I would see my dots closer together because I measuring them in a shorter time frame, so I would get more data points in that time frame because I have more points to measure. If she is doing minutes, she may not see as many of them during that time frame. Yeah. Wait, is it revered? Hold on. 40:03 (takes 15 seconds). It might be flipped.
I: What might be flipped?
Brianna: Which one is the seconds and which one is the minutes. But the time frame to me is what is different, the unit of measure for time. Cause one would give you more data points, more data points could. This is a little confused on the graph. More data points could make it so I could see more dots in the timeframe then if I used a different unit. So, the time is different.

I: What if two student turned this in to you, labeled and everything. Correct, incorrect?
Brianna: But they labeled them with their units of measure and the scale was correct, I would give them full credit.
I: Even though the graphs look different?
Brianna: If I did not specify the units of time, then yes, I would give them full credit. They are both accurate.
I: So that is the last task but I have a couple follow up questions.
Brianna: Okay.
I: My first follow up question, How do you define ratio?
Brianna: A comparison between two things or items.
I: Such as?
Brianna: Boys to girls. Pens to pencils. A comparison?
I: What image do you see? Multiple?
Brianna: Probably multiple, I always use boys to girls. It is attainable to them. They can count them. Not necessarily after that. I do that but I don't necessary go further than that. Not in my mind. It is just kinda of like I am comparing two things.
I: Do the two things have to be the same or can they be different?
Brianna: They can be different. 42:58
I: Can the be the same?
Brianna: They can be the same or they can be different?
I: Give me an example where they are the same.
Brianna: The ratio of girls in Mr. Hoffman's class to the ratio in Ms. Brianna's class. I am still looking at girls to girls, they are just in different spots. So you may or may not have more girls. It is the same concept, but different locations and different spots.
I: Define the word rate.
Brianna: It is kinda of like a ratio but it really does involves a unit of time or a unit if measure like money or time. Like, it makes me think of unit rate or like a rate like a speed. So like I am going to try to tell kids to look for the unit of time or the unit of measure. Something is happening over a specific unit of time. Those are the most common ones that they will see.
I: Most common?
Brianna: Maybe something with weight, like pounds or something like that. So, I will give them like a grocery stores. Like buying fruit, price per pound or the speed of a car. Car is going 60 miles in an hour. That would figure out how fast it is going. I tell them to look for the word per. A lot of times it is something per something. Miles per hour. Price per pound. Causes them to divided and figure out what it cost per unit.
I: You are saying unit rate. What do you mean?
Brianna: You are talking about how much you can do for one. For how much money does it cost for one pound. For driving, how much could I do in one hour. I focus on the uni which means one. For example, unicycle, unicorn. That means one. So that they understand the one and they realize that their denominator has to be one.
I: Very cool. What images do you think of when you hear the term rate?
Brianna: Car. Driving. Speed limit signs. Race car driving.
I: What about rate of change?
Brianna: So, using proportions, and ratios and rates with slope, that since they are comparing how the y are changing, the vertical change, with the horizontal change, so

that is rate of change. It is two different things with two different units. You are comparing how one is changing to how the other one is the one is changing and you get a fraction. So it is like a ratio of those two differences. Like if a graph has time, distance and time, you are comparing how the distance is changing to how the time is changing. That is what I consider to be a rate of changes. so using the two things together.
I: Would you say rate and rate of change are different?
Brianna: Not necessary. I think that rate takes you to rate of change or vise versus. I just don't. There not two different terms. I think that it is helpful to understand rate and slope easier using rate of change.
I: When do most students get exposed to rate, ratio?
Brianna: Okay, so rate and ratios are taught in sixth and seventh grade, I would assume. I know I do it in seventh. I think they come with some knowledge. Rate of change will supposed to try those two concepts together in common core.
I: Thank you. Discussing more about teaching and her plans to go to UNCC in the fall.

Carrie

(Begin with discussion of why I am collecting data and the purpose of this study)

Carrie: It is interesting that you are doing all of this because that was a huge concern right at the start of common core. We were redoing everything. Not necessarily slope but we were looking at it more from the algebra perspective that so many algebra II concepts were coming down and your typical middle certified teacher has not been trained to teach most of those concepts.
I: (Continue discussion)
I: How long have you taught math?
Carrie: 15 years.
I: All at the middle school level?
Carrie: Yes.
I: What is your certification?
Carrie: Certification is 6 - 9 Math and 6 - 9 Social Studies
I: Do you teach social studies?
Carrie: No, that was my first year (Laughs)
I: (Discussion ensures and I mention that when I taught middle school they wanted me to teach SS) Why math?
Carrie: I don't know if it was so much a decision, as it was circumstance. That was what was open, so I took it. Math was always my favorite. So, as a student, that is definitely where I preferred to be. My interests lie a little more on the AG spectrum (Discusses her data to back this claim up)
I: Why middle school?
Carrie: Middle school for me is a personal thing. My dad was diagnosed with cancer when I was in sixth grade. So really to me, those were the teachers that were there to really catch me. It was just my mom and I. And he battled literally from my sixth grade year to my tenth grade year. That is just an area to me where I feel kinda of compelled to give back to. So, middle school.
I: That is a powerful reason. Have you pursued your masters?
Carrie: No, but I have my national boards. For me I came into teaching lateral entry. I majored in psychology with a goal of becoming a guidance counselor. (Discusses this in more detail including talking about getting married and moving from WV to NC)
I: What is the highest math class that you have taken?
Carrie: I guess a third level calculus. Something like that.
I: So you have taken calculus?
Carrie: Yes. I was AG going through school so I was on the pretty pushed path. So when I started WVU, I just jumped into some engineering classes even though I was going there as an education major but those were the maths that were appealing. (Engaged in a brief conversation about engineering as a major and where she graduated from)
I: When you hear the word slope, what do you think of? What images come to mind?
Carrie: All images. I start thinking about my notebook and the formula, rise over run, the general mathematical concepts of it, then I start thinking about the pictures that I show to

my students like the wheelchair ramps, and the ski slopes, and all the different visual images. So that is what comes to mind.
I: What would be your definition of slope?
Carrie: The steepness of the line.
I: Would that be your response to a student?
Carrie: For a general concept, yes. But I guess it depends what I am trying to accomplish with that student. So general introduction, I am going to stay steepness of a line because that is what they can visually grasp onto to before they start dealing with the numbers. So they need to have a general concept before they can apply it. Am I on the right direction?
I: You are doing great. So, here is the first task. (Explain table problem)
Carrie: For me, I am looking for a constant slope. So you are looking for your rise in your range over your domain. So here (A) you have an increase of 3, then 5, then 7 with an increase of 1. So you have separate slopes there, so there not constant. Then you have a constant rise of 2 (B) with a constant run of one. So that is constant, that is linear. And here (C) you have a slope of negative there over positive one, so that is constant as well.
I: So B and C are linear?
Carrie: B and C are linear, A is not.
I: Excellent. (Explain two lines problem)
Carrie: The yes they are the same line simply could be going on the fact that they are both positive and you really don't know the intervals of your x and your y axis. So you can’t really calculate your slope. The person that no was looking at simply as that is steeper (Pointing to the steeper graph) and the other one is flatter.
I: If I gave you this problem, same or different?
Carrie: You don't have enough information.
I: Why don’t have enough information?
Carrie: You don't know your domain and range values.
I: So you would not be able to answer?
Carrie: No. (Laughs) Well, I guess I did answer.
I: So the answer is there is not enough information.
Carrie: Yes, there is not enough information.
I: What did you mean there was positive?
Carrie: The slope is positive.
I: How do you know the slope is positive?
Carrie: Because the range is increasing as your range is increasing.
I: (Explain matching container problem) 12:18
Carrie: Okay, so for the cone as the water enters, it is going to take a lot for the height to increase because it is filling a larger area (circles the base) at the bottom versus as it rises it is smaller area. So, the height will increase at an increasing rate (Matches with the quadratic). The cylinder, you have a constant circular base going all the way up the height so that is going to be a constant (matches with the line). And then you fishbowl slash wanna be sphere, has a smaller base and it is increasing and then decreasing again as you rise through that one.
I: So the cylinder you matched up with the?
Carrie: Constant
I: Why?

Carrie: Because your circular base, all of your bases, I believe the definition of a cylinder is a stack of cylinder bases, so that is going to be constant area that the water has to fill as it moves up the height.
I: Awesome. Good job.
Carrie: Are you sure?
I: Yes. (I take up the problem to maintain her matches). (Explain angle problem)
Carrie: How would I respond?
I: Yes.
Carrie: Umm, I mean, I guess, I don't know how I would respond to them (Laughs). Umm, if it is a 30 degree angle then it has a 30 degree incline so basically that would be a ratio happening
I: What ratio happening?
Carrie: A 3 to 10. A 3 to 10, if it is going up by 30 percent.
I: Where is the 30 percent coming form?
Carrie: The 30 percent would be coming from a rise of 3 and a run of 10. But I would have to play with it more.
I: You can play with it a little bit.
Carrie: (Laughs) I'm good.
I: If a student gave you that answer, would that be acceptable?
Carrie: It depends again on what the goal is. If you are trying to focus on umm, I mean literally trying to focus on the slope of the line most of the time a lot of the things that we have to do isn't just with the slope where they could give it to us in any form. A lot of times, we are having to write the equation or we are having to predict the next value and 30 percent is not going to give us what we need for that.
I: Follow up question: What happens to that line as 30 goes to 40?
Carrie: It gets steeper.
I: And what happens as it goes to 20?
Carrie: It gets flatter.
I: What would be the angle if it was perfectly flat?
Carrie: Perfectly flat is zero.
I: Good (She handles me the graph) So you would not recommend a student articulating the slope with degrees.
Carrie: I wouldn't. Not at the level that I teach. I am not saying that that would not be beneficial later.
I: (Explain Slope of 2 problem)
Carrie: A could positive two would be B.
I: Why B?
Carrie: Because it has a positive slope. It is the only one that has a constant positive slope. Cause there is a period in here (D) where the slope is positive two. Potentially. The slope is not constant.
I: SO how could D have a slope somewhere of positive two?
Carrie: It could be right here. (Pointing to a point on the parabola to the right of zero). From that x to that x might be positive two. That would be the only part.
I: How many times would D have a slope of positive two?
Carrie: Once (timid answer)
I: What about B?

Carrie: The whole time.
I: Why did you dismiss A?
Carrie: It is negative.
I: Why did you dismiss C?
Carrie: It is zero. (Slides paper back) I feel like I am taking some sort of crassly intelligence test. What does this inkblot mean to you? Oh here is a picture.
I: SO I found this picture on the internet, what do you think six percent means?
Carrie: The hill has a six percent grade. 18:37
I: What does a six percent grade mean?
Carrie: A rise or decline of 6 for every 10, I mean every 100 sorry.
I: 6 to every 100?
Carrie: Yes.
I: What would a 50 percent look like?
Carrie: A 50 would be one to two.
I: Would it be steeper?
Carrie: It would be a hole lot steeper (Laughs)
I: How about 100%?
Carrie: A 100 percent the truck should not be driving because it would fall off.
I: Why would it fall off?
Carrie: (Laughs) 100% would technically not, almost not exist. It seems like it would be what we consider no slope, in a theoretical aspect.
I: So vertical?
Carrie: (NODS) Yes.
I: Could I go over a 100%?
Carrie: Technically, theoretically you could. But that would just be going downhill in another positive so…(stops talking)
I: What do you mean downhill?
Carrie: If I am viewing this as 50 and this (Straight up and down is a 100) then if I am going to go over 100 then I am going to be coming down.
I: Okay, so you would be going from decreasing, to 100 percent being straight up and down to increasing as you went over a 100
Carrie: If I am thinking about it the right way. So hopefully yes. I don't know. That is a question that I would like to process. I don't know why.
I: What do you mean process?
Carrie: I don't know. I guess think about some difference ways. I mean I am only getting what is coming to my mind first. Normally I like to look at things from a couple good ways before I give an answer.
I: That is fine. You can take your time.
Carrie: What else do you need from this question?
I: So we said, just paraphrasing: 6W% was six over 100. 50% percent you reduced to 1/2. 100% was incredibly close to vertical, if not vertical. And over 100 went from either increasing to decreasing or decreasing to increasing. Correct?
Carrie: Yes.
I: Awesome. That is what I need. I just what to make sure I understand what you are saying. (Give her y=ax+c). What happens as A gets bigger?
Carrie: As A increases, your line gets steeper.

I: What happens as A decreases?
Carrie: As A decreases, your line gets flatter.
I: So A gets bigger, steeper and steeper and steeper. As A gets smaller, flatter and flatter and flatter?
Carrie: Yes. Well that is assuming that A stays positive.
I: How would that change if A was negative?
Carrie: Well, If A goes negative, as it increases, the line gets steeper. As it decreases, it gets flatter. It is just backwards.
I: What happens as C gets bigger?
Carrie: Your y intercept increases. So you cross the y-intercept at a higher point. Or if it is a situation, you have a higher start value, or your are higher up the mountain, or you are higher up the driving board. Whatever your case may be.
I: What if C gets lower?
Carrie: Then you drop it.
I: What impact does changing C have on the slope?
Carrie: None.
I: So changes A changes my slope but changing C does not?
Carrie: Correct. (Laughs)
I: (Five staircase problem explained) 23:04
Carrie: (Laughs) (Looks visually overwhelmed) Well, again I feel like I need more information but not mathematical information. This has the lowest slope.
I: B
Carrie: Yes, B. So for most people this might be the easiest to climb. E is your one to one. It looks like.
I: What do you mean 1 to 1?
Carrie: Your rise and your run are the same.
I: Okay.
Carrie: A lot of people actually prefer that kind of staircase. This one is the steepest, so it would be the hardest to climb regardless.
I: How are you determine it is the steepest?
Carrie: Umm, the rise is greater than the run. And If I was going to put them in order of steepness, I guess technically, I am going that route if I am putting them in order of steepness.
I: How are you determine steepness?
Carrie: That is simply just rise over run. So, I am comparing the height of the step with the length of the step. Cause if you are looking at this in an actual staircase, you many not have to take a big step, but you will have to take a second step, before you even get to the next step (Referring to B) which is why most people prefer E because it is just one step up at a time. Umm, so again back to your question, which one is the easiest to climb, I don't know. Are you going mathematically here? (Brief pause) I am probably going to say that B is the easiest because you are not having to exert as much force to lift your weight to the next level but I am going to feel like this (E) one is going to be the most comfortable. SO I would go that way.
I: So, your lowest is B, followed E because you think it would be the most comfortable,
Carrie: YES
I: then D, then A and then C. Why do you think C is the hardest?

Carrie: C is your highest step up.
I: Do you mean the first step or all three?
Carrie: All three.
I: Let me keep these in order.
Carrie: There is golden ratio for stairs (We discuss stairs for a few moments)
I: (Explain Darren problem)
Carrie: Okay, so Darren is running at a constant rate, so that means that he is not going to slow down or speed up. I am assuming that we are dealing with distance and time so that is going to be A. Do you want me to write it down?
I: Yes, please. What about A made you gravitate to 1?
Carrie: Constant. (Reads B). So that is going to give me 3 cause that is slowly at first and then a gradually increase. Then C runs fast then decrease his speed is going to give me C. So he is running fast to start, so you have the higher slope and gradually decreases, so it is starting to taper off. Darren runs very fast and reaches the finish line early, that is #5. He is running really fast, so he has a pretty steep slope and then he is done, so he is not going anywhere, he has already hit the finish line at that point. And then after falling, so he is on the ground here (traces over the horizontal portion of 4) That is your fall (Starting point) and runs at a constant rate. So that is E. (ALL CORRECT) So for number 4, where does he fall? This is his time that he is only the ground (Points to the horizontal portion).
I: Good job. Nice explanation. Thank you. (Give her hungry problem) End video 1
Carrie: Is there a zero? I need to know if this (starting point of graph) is like complete? like your completely not hungry?
I: Lets say down is is not hungry and up here is hungry.
Carrie: So, time of day. We will pretend like this person, Jane Doe, is waking up and she is relatively hungry but she is not eating breakfast right away so her hungry level increases. When her hunger peaks, she maybe eats a small breakfast or a snack so she is not as hungry but she is not completely full either. Then she continues about her day and gets way more hungry and then she eats a much bigger lunch so she is much more satisfied. And then it starts to tamper off and maybe she around here she gets more active (Inflection point between lunch and dinner) and she is burning more calories and her hunger picks back up a little faster. So she is really hungry at dinner time. Maybe she is out to a restaurant and she is eating an appetizer so she is still hunger for dinner but is getting it under control. Then she eats dinner and right before evening activities she starts to get a little hunger but she is good.
I: Why did you make the distinction that she ate a small breakfast but a big lunch?
Carrie: She did not satisfy her hunger as much at her first meal.
I: How did you determine that?
Carrie: Because her hunger level did not drop as far.
I: Why did you think that she had a big lunch?
Carrie: I guess she should not say big, maybe more satisfying cause she was able to drop her hunger level to its lowest point.
I: How did you know it was the lowest point?
Carrie: Based on the y-axis.
I: When is she the most hunger?
Carrie: Before the third meal.

I: Last question: When is her hunger level rising the most?
Carrie: About here (Positive slope right before dinner) At least it is increasing faster. Her hunger level is rising much faster per time of day.
I: Good job. (I explain vending machines)
Carrie: Okay, well first it is segmented because you can not have partial cans. You have to buy the whole can. All or nothing. So it is either full or at its normal level and then, time of day, each one of these symbolized the purchases of either one or two cans. Looks like some of them are a little longer so that would be when two cans were bought. But everyone else is buying one. So you have your gaps when the vending machine is not in business. So you have a couple of moments when no one is buying. At this point after this last purchase, the road rep comes in and restocks it. 3:34
I: Good job. (Movie)
Carrie: When that, I am going to assume that it is not dotted for any reason. So that is just basic constant. Though we are doing to go with the fact that this theater charges the same price for everyone no matter how old they are. The more customers in the movie theater the money they have taken in the register. And they have not gotten very fall into the graph because they did not take into consideration exiting.
I: So what about the graph made you think that they were charging the same price?
Carrie: Constant slope. If it were not constant, it would be steeper as the adults were going in and less steep as the kids or as seniors were coming in cause they would not be charging as much per ticket. That would mean that the increase in the cash register would not be increasing at the same amount.
I: Very good, How come that graph does not start at zero, zero?
Carrie: Cause there is a bank to start with. Gotta be able to make change. (Laughs and handles back graph)
I: (I explain the POW one)
Carrie: Well, it pretend that that is zero and we assume that everyone did their homework (Labels start and end of x-axis). I know that that is kind of dreamland in some places. This is number of days since it was assigned. We are going to pretend like we live in a perfect world and this was the due date. (makes joke about teaching middle school). You got your overachievers down here who like to get things done then as the due date approaches more and more students turn it in. Typically, you are going to get bigger jumps at the due date because people are trying to turn it in.
I: Can we assume that it is increasing the same each day?
Carrie: No.
I: Why?
Carrie: Because you are non linear. So you have a smaller increase down here.
I: (Explain last image)
Carrie: They just did not draw their scales for the x and y axis the same. If their data points are exactly the same, I mean if this were, I don't know, If that were (10,2) and that were (10,2) (Labels the last point on each graph) then it means that her, Denise's y scale is just different. Your scale is just different. How their intervals. It will still show the same data, if labels correctly.
I: Those could be the same line?
Carrie: Yes.
I: It is okay that the lines don't look the same?

Carrie: Yes, this is were you get into misleading statistics.
I: How so?
Carrie: Because if I am the renter and you are the landlord and I have a big problem with the leaky faucet, I am going to show this graph (the steeper). I am going to make it appear like this leaky faucet is a much bigger deal. If I am the landlord and want you to hold off, then I am going to show you this one because visually it is not displaying the same level of crisis that this one seems to be.
I: What about the right one is showing more of a crisis?
Carrie: It is increasing rapidly based on the scale.
I: Awesome. So how would you define ratio? 9:56
Carrie: (Laughs and repeats question) It is a comparison between two quantities. That is very general.
I: When you hear ratio, what do you think of?
Carrie: I guess it depends where I am going. I am thinking that if I am talking about ratio that I am comparing something like similar figures, unit rate, I am going somewhere but I am basically comparing two quantities.
I: You mentioned unit rate, lets take it back first. How would you define rate?
Carrie: A rate is a comparison of two quantities in which your units cannot be converted. So miles per gallon, you cannot convert miles into gallons. They are different units of measure.
I: Would a ratio have the same units of measure?
Carrie: Technically, yes. But a ratio is that general umbrella. It is kinda of like the square and the rectangle. A rate is a ratio but not all ratios are rates necessary.
I: What images come to mind when you hear rate?
Carrie: Rates, I am thinking speeds, I am thinking prices, comparing growth rate, decay, that sort of things.
I: What about the term rate of change? 11:36
Carrie: Well, I always get frustrated with that one cause that is the definition: it is the rate at which something changes. So, that one you are looking at the change in the value over a period of time typically.
I: DO you have different images that come to mind when you hear rate of change?
Carrie: Not necessary. They tend to go hand in hand. When you jump into unit rates then you are talking about rate of change. Percent of tax and decrease.
I: You keep saying unit rate, what is your definition of unit rate?
Carrie: Basically, your amount per one of your basic unit. SO depending on what you are comparing. If ti is how many apples per dollar or whatever then your unit rate is your dollar. How many apples can you get per dollar. How much per apple. It depends on your situation. So maybe it is how much money per one apple instead of how many apples per one dollar. So you have to define your parameters and know what it is you are trying to focus on.
I: Last question: If a student says what is slope, what is your response?
Carrie: I am still going with the steepness of the line, the rise over the run, how much does the y, the rise, increase or decrease, compared to the the x.
I: Awesome. Thats it. Thank you.

Deborah

I: So how long have you been a math teacher?
Deborah: It has been ten years
I: All at the middle school level?
Deborah: Middle school but also language arts
I: How long did you teach language arts?
Deborah: It was with the math. I had one class with the math.
I: Why did you decide to teach middle school?
Deborah: Ummm, because I was not certified to teach high school, to be honest with you. The older the better, I think. I do not have a degree in education.
I: You don't? Are you lateral entry?
Deborah: Yes. But, I did get a masters in education. Yes, so I have an undergrad in business, an MBA and a masters in education.
I: Why math?
Deborah: Because it is so easy.
I: That is not a answer you normally hear.
Deborah: Well, I have a family of engineers. I probably should have went into engineering. Math just comes very easy to me but I also love language arts because of the creative side of it.
I: What courses and grades are you licensed to teach?
Deborah: Up to ninth. Sixth - nine. 6 - 9. I am also certified in New Jersey. I taught you in New Jersey.
I: You have?
Deborah: Yes, for three years.
I: Is that included in the ten?
Deborah: Yes, three there and seven in NC.
I: Did you purse your National Boards?
Deborah: Not yet. I am doing that next year.
I: Good luck.
Deborah: (Discusses a present project)
I: What is the highest math class that you have taken?
Deborah: In college, I guess calculus 2 and a few geometries. I went back when I did my masters and picked up a few extra math courses. (Talks about NJ).
I: Very cool. So when you hear the word slope, what comes to your mind?
Deborah: Steepness.
I: What do you mean?
Deborah: How I relate it to the kids, well I was a skier, so snow skiing. How steep is the slope? Are you on a blue, a green, or a black diamond? A lot of them could ski, so they could understand that relationship. So how tough the ski slope is? But also rate of change if we are talking mathematically. And since I used to run a business, I talk to them a lot about how everything relates to business. So to me, it is the rate of change also. That is what we talked a lot about this year. More so the rate of change than the slope. How is it changing?
I: Why more this year?
Deborah: Common Core curriculum.

I: Okay. So, my first task. Here are three tables. Were any produced by a linear function? 5:33
Deborah: From a linear function?
I: Yes.
Deborah: (Looking at tables) Well not A.
I: Why not A?
Deborah: Well, the first thing that I look for is a pattern. This is not going up equally. Here (B) this is going up equally. So that is the first thing that I would do is look for a pattern to see if it is going to be linear. Here (C) you have to think more about it. That is also I would say is linear. But then I would also have the kids find the slope and graph it to confirm that it is linear. Is it a straight line. But what we were trying to do is first eliminate if it is not a pattern. Then if you want to confirm that it is linear, graph it. So I would eliminate this right away (A) and then if I was a student and still wasn't sure, I would graph it.
I: How are you looking for a pattern?
Deborah: Xs going up by one and ys going up by two (referencing B). As long as they go up by the same rate, so rate of change is the same. The rate is the same. For this one (A), it (the xs) are going up by one then the ys go up by three and then five and then seven. So as soon as that was different, I eliminated that.
I: Very good. One for one.
Deborah: Good. I am not in school mode. My brain is not on there.
I: I bet. So for this one, I gave these graphs to two students. One argued that they were different while one argued that they were the same, what might each student have been thinking?
Deborah: I think because, This to me (Left) looks like a blown up of this (Right graph). Someone might think that it is just distorted cause this y (on the left) is much smaller than this y (right). Looking at it visually, I might think that it was blown up, so this line looks a little different than this one by placement. That is why I might think it was the same. I might think that it was different because this line (Left) appears to be much steeper than this line (right), so therefore they would have different slopes. So different lines.
I: What is your thought? Same or different?
Deborah: To me personally, I would say different but I would say how can you compared it if you were not on an equal graph. That is what I would say. That is the skeptic in me.
I: That is very good. (Next, I explain the three shapes and graphs problem) 8:19
Deborah: Is that the rate that you are pouring it in at?
I: No, it is the water inside the containers is rising.
Deborah: Got it. Okay, now this is a little tricky to me.
I: Why?
Deborah: Because we have only dealt with the linear equations, to be honest with you, and I have not dealt a lot with the, it is just something that I have not worked with. The other type of equations. So if I was to look at this I would put (pauses) I think I would do it just the way it is and might just be my perspective on it. But I don't think that it is mathematically the way that I am looking at it another way.
I: Okay, so how did you look at it? Lets start with the first one, you matched the cylinder with the linear function.

Deborah: Because everything is proportion. It is round, it is going to go up the same amount. So to me the rate of change would be like that (tracing the line). You know cause it is like this (moves arms like a cylinder). This I feel like would fill up the bottom more and be less at the top so the rate of change at the top will be less because there is less water to fill up there (Matches cubic with cone). And then this one, (Matches quadratic with fishbowl) you are going to be filling up the bottom lower and then it is going to rally spread out at the top. (Shrugs shoulders indicating I don't really know if I am correct)
I: Okay, so we matched up the line with the cylinder, the exponential or quadratic with the fishbowl and the more turn one with the cone.
Deborah: Yes.
I: So for this one, I presented a line to a fellow teacher and she expressed the slope as a thirty degree angle. How would you respond?
Deborah: Well you can present the slope as a degree but we have not done that at all. Just being honest, I am not up to date. But cant you use sine to present the angle? And then the slope of the line would be from here to here (X-axis to line) versus the slope from here to here (point to point on the graph). So that is the difference that you are representing, but it will still represent the same thing.
I: So, if a student gave that to you, would you mark it correct? incorrect?
Deborah: I think that 30 degrees could represent the slope. If you are relating it to the x axis, then sure. Because if it is over here (pointing higher in quadrant 1) then it might be steeper, like a 45 degree angle. So you would have a steeper slope.
I: So as the angle increases, what happens to the slope?
Deborah: It becomes steeper, higher. A larger slope, a higher slope. (Laughs) Is that okay?
I: Yes. Like I said, there are no right or wrong answers.
Deborah: IT just that some of this we have not gotten into, I have not done it in a long time.
I: No problem. So far you are doing great. Here I have four graphs. Could any have a slope of positive two?
Deborah: No (A), No (C), No (D), positive two (Examining B). I would say no. Even if I am just looking at this as rise over run, it (B) is just not proportional. To me this looks like a slope of 1/2. Up two, over 4. If i am just putting it in my own scale, it is not up two over one. So I would say none of them.
I: Okay, none of them. What would you say the slope of A is?
Deborah: Negative one.
I: Why negative?
Deborah: Because it is going down left to right.
I: B, you said the slope was?
Deborah: Positive one half.
I: What about C?
Deborah: Zero.
I: Why zero?
Deborah: Because it is a vertical line. Now that is assuming that my eye is correct (motioning a horizontal line), it does not go up a hair. But to me it looks like it is parallel to the x, so if it is parallel to the x it is a vertical line and has a slope of zero.
I: So this is a vertical line?

Deborah: This one here, horizontal. Sorry. 13:58
I: Just making sure.
Deborah: You know what I meant to say (Laughing). Because your eye can be off, unless it states that it is vertical to x, I can not be positive. It could be the hairest up or down. To me it looks horizontal.
I: What about D?
Deborah: D is not linear, so I don't know. It is not going to have a slope.
I: So A negative, B positive slope, C zero slope and D no slope?
Deborah: No slope cause I have not used quadratics in a long time. I don't remember all of that stuff to be honest with you.
I: Okay.
Deborah: Good
I: Yes.
Deborah: Whew! (Wipes brow)
I: (explain six percent)
Deborah: We talk about this a lot.
I: You do? Awesome. What do you think they mean by six percent?
Deborah: It is the grade of the road. We talk a lot about the percentage grade of a national highway is based on the fact that you have to have the road accessible to the army. That is why it can not be a certain percentage because if it was too steep, tanks could not get over.
I: Wow. (We talk about more about this) How do you think they got 6%?
Deborah: It is exactly how it looks here (Pointing to the picture) It is the rise over run of the distance of how far you are going versus the height of how far you are going. If you are going to do down a thousand feet, they would do it that way. I always make the kids put it as a percentage. Like a road, if we were talking about a road. Like a ramp.
I: Would the six be the numerator or denominator of the fraction?
Deborah: Neither. It is the numerator divided by the denominator to get the six percent.
I: What would happen if it went to 50%?
Deborah: (Laughs) You would not be able to drive it. It would be way too steep (Did no calculations). If it went to fifty percent, it would be you know (Holds her arm very steep).
I: What about 100%?
Deborah: A 100 percent grade?
I: Yes.
Deborah: So that would be like, what would you get, would you get this (holds arm straight up and down). Your not going to tell me. So six percent, seven percent will be steeper. Fifty percent will be steeper. A hundred percent is going to be a straight line.
I: 100% is going to be a straight line?
Deborah: Yes, I would think. You are not going to tell me the truth.
I: No, I am not. Could you go over a 100?
Deborah: I would think not. If we are talking about a road, no because you would be going backwards. I mean, I think.
I: What do you mean backwards?
Deborah: The car would roll backwards (Laughs) But I don't know if I am understanding that correctly. That is a good question. Yeah, a good question.
I: Thanks. So, if I have $y = ax + c$, what happens as a gets bigger?

Deborah: The line becomes steeper.
I: What if A get smaller?
Deborah: The line becomes less steep. And I am assuming, we are talking about positives. If we are talking negatives, it still the steepness. Talking absolute values or whatever.
I: What happens if I do go negative?
Deborah: (Her phone rings). (I repeat question) It still the steepness of the line is going to change. It is still going to stay negative but as the absolute values of it becomes larger, the line will become steeper.
I: Awesome. What if I change the C value?
Deborah: It just means where the line hits the y axis. It is where is starts. It starts at a different place. As C becomes higher, you go higher up on the y axis.
I: What if c becomes smaller?
Deborah: You go lower, on the y axis. If c was 100, you would be up here and if c was 50 you would down here.
I: Does that impact the slope?
Deborah: No. Whew! (Laughs)
I: This is another matching problem. (I explain the Darren problem) 19:58
Deborah: (Reads silently) Okay, I am going to go backwards. So after falling down, he runs at a constant rate. That is four.
I: How did you know that was 4?
Deborah: Because when you fall you are not moving. You are on the ground then he runs at a constant rate, so it is a constant.
I: Where did he fall?
Deborah: On the ground.
I: (Laughs) With respect to the graph?
Deborah: Right here (Points at the start of graph). Because he was not moving.
I: When does he get up?
Deborah: Right here (Points at cusp).
I: Perfect.
Deborah: Darren runs very fast and reaches the finish line early. So here gets here (Points to the cusp) and gets to the finish line and he can just rest until time is up or whatever. So he got there early so that is 5. Umm. Darren finishes running at a constant rate is 1. And these are the two that go with these two (2 and 3). He runs slowly at first and gradually increases his speed, that is 3. So he is going slowly at first then increases his speed (Traces B). Darren runs fast and then gradually decreases his speed is 2.
I: Good job!
Deborah: Thank you.
I: (Explain staircases)
Deborah: Easiest to climb versus most difficult to climb?
I: Yes.
Deborah: Well the steepest will be the most difficult.
I: How are you going to determine the steepest?
Deborah: The number of steps versus the height versus the width. So this is the steepest, it will be the hardest to climb, E. Umm, gosh. Oh wait this one is pretty steep too (C). Hold on. That is like a rock climbing thing. Okay, lets say C, hardest. So lets go C then E.

Because they are less steps, but just as much height. It looks like it to me. So C then E. And then A. And then, I would say D and then B.
I: Why is B the easiest?
Deborah: Because it is a lower amount to step up on. And you have more time to get this length. It is like hiking, the steeper it is, the harder it is. The steeper it is in the shorter amount of time. Like if the elevation was 1000 feet in one mile. That is going to take a little effort. (Laughs)
I: I agree. (She hands paper back). The next one is being able to explain a scenario. Hunger level versus time of day. How would explain that day.
Deborah: So I am assuming that this is when you wake up in the morning (Pointing to the start of the graph). You are basically hungry and you shower and then are really hungry (points at top cusp) and you eat breakfast. Then once you eat breakfast you are not hungry and once the day goes on, you get hungrier as time goes on. This is lunch (Points to the second upper cusp). You ate a big lunch and because you did, you are really not hungry. Then as the day goes on, it takes a shorter amount of time for you to get hungrier from lunch to dinner than it did from breakfast to lunch. So this might be till 5, lets say (Points to third upper cusp) and then you are hungry and then you are a little more hungry, so you have ice cream before you go to bed (Points to change in decreasing slope after dinner) and you get hungry as you are sleeping.
I: Good. (Explain POW one)
Deborah: Reads graph. As the number of days increased, the percentage of student completing the assignment increased. At an exponential rate.
I: Why exponential?
Deborah: Because it is going up like this. (Tracing the line). It is definitely positive. It could be (Draws an imagery linear function). It slows down here (traces upper portion of graph). This was faster a faster rate (Tracing lower portion of the graph) and then it slows down.
I: So in the beginning it was a faster rate, and then it slows down towards the end?
Deborah: No, no, no. It increases towards the end. I'm sorry. This goes way up. This is the percentage, this is the number of days. So lets say this is ten days and this is ninety percent (Points to highest point). This is one day and ten percent completed (points to first point). So as the days increase, the percentage increases.
I: Does it increase linearly?
Deborah: If you connect the dots, it is not a linear function. But you could have a linear regression through the points. If you look at as a scatter plot, you could have a line of best fit, which is linear, to describe the trend of the data.
I: Would you choose a linear function for this regression?
Deborah: I think you could. I think you would still interrupt it the same way. End Video one.
I: (Give soft drink problem)
Deborah: So it was filled last night. This is the morning when most people are not drinking soft drinks. And then as we reach 11, people are starting to buy drinks. As the day goes on, more people buy drinks until 4 when people start to go home from work and we have a minimal amount of sodas left. And it has to be refilled (Tracing top horizontal right graph) during the night and continues the next day.
I: Excellent. (Customers)

Deborah: So the cash register with the person receives it has lets say 50 dollars in the till. So they start off with 50 dollars and then it is a constant rate because the ticket prices are equal for everyone. They don't have a children and adult rate. They have 6 dollars per ticket for this matinee or whatever. So as the number of customers goes up, so to does the money in the register.
I: How did you determine that the prices were equal?
Deborah: Because it is constant. To me because it is going up at a constant rate, it is like six dollars per ticket. So if I have ten customers versus two customers (Points to two different points on the graph). I would have an equal amount. The slope is the same. That is how I determined it.
I: (Explained water faucet problem) 2:22
Deborah: Was it the same faucet?
I: Yes.
Deborah: This person (T) it looks like they took less time (Tracing along the x axis) to look at it than this person (D). Assuming their time is the same scale. It could just be scale, first of all. Their scales could have been different and that is what their graphs are different. So that could be just that. First thing, it does not show a scale for either one of them. Her (T) scale might be going up by two where her (D) scale might be going up by one. So then it could show a different graph but the same numbers, perhaps. Ummm, thats would be the first thing that I think. Cause if they are looking at the same faucet then they should be getting the same values. They have just done their scales differently. That is what I would interpret it, assuming they are both correct.
I: Assuming they both correct, would you give each student full credit?
Deborah: If they are accurate based on the scale. I am always telling my students to look at the scale. I would have to look at their scales, but if they did not put a scale, it would be based on what I told them to do. If the scales are the same, this person (T) is interrupting that the faucet is leaking at a faster rate than this person (D). This person (T) also feels like there is more volume versus this person (D), if the scales are the same. This person (T) looked at it as a less amount of time then this person (D) did.
I: Nice job! So, I have some final questions. First, how would you define ratio.
Deborah: A comparison of one thing to another. I have not taught ratio in a while. So, like 4 to 2. Something like that. A ratio, how much of this compared to that. 4:49
I: So when you hear the word ratio what comes to mind?
Deborah: A comparison, to be honest. I guess food shopping. Sometimes, like how much for this price compared to this price.
I: Any mental images?
Deborah: For ratio? Not really. Keep going, ask another question.
I: Okay. How about rate?
Deborah: Umm, when I think of rate, I think of unit rate more than rate. Like bring it down to a unit rate of one. That is where I find, I like to bring it down to a unit rate of one. A comparison. Rate of change, I think of too. But rate, how does it cost for one ounce, for one day, for one mile. That is what I think of when it comes to rate.
I: So if a student was writing down definitions and for ratio they wrote a comparison between two things
Deborah: (NODS)
I: what would you have them write for rate?

Deborah: You are testing me here. Unit rate will be bringing it down to one unit. For one ounce, for one day. Rate versus the ratio is the comparison of things that are the same versus the comparison of things that are different. As far as unit of measure.
I: What do you mean a comparison of things that are the same or different?
Deborah: I have to remember this. Ratio is the comparison of two different things, rate is a comparison of, no. Rate is the two different ones which means ratio is the same. I am sorry that I am not up to date of these.
I: No, you are doing amazing. I am just trying to get at it.
Deborah: I know it, it is just not in my head right now.
I: What do you mean by two different things?
Deborah: Like, miles per hour. Feet per second. Like that kind of thing where you are comparing two units of measure. Things like that.
I: What about the same? When you spoke of ratio being the same?
Deborah: When I think about ratio, it is like four blue to three black. Like 4 to 3. Like you would put a red here and a blue here or something. I know I am messing this up. I just have not done this in a while.
I: Your fine. I wrote a paper and math scholars don't even agree on the definitions.
Deborah: Woo, Woo (Excited response). And it is confusing. It confuses me. Sometimes I go back and look at it to make sure I have it correctly. What we deal a lot with is unit rate. Okay, we deal a to with if this is four red beads. Red and black bead question and setting up the equation, this is the red and this is the black. You know, what is the ratio of this to this. Setting up the equation for it. The terminology sometimes throws me off.
I: Okay, I have one more. You said rate of change, How would you define rate of change?
Deborah: Umm, we define rate of change, like with pizza and the pizza costs ten bucks and the toppings are \$1.50. How many are you going to buy is your rate of change. If you are going to buy three times a \$1.50 plus the ten dollars. The rate of change is how many toppings are you going to have. Things like that. We try to really bring it back to things, like in a food store. Things that they bought. Rate of change is the cost of something How much is it going to cost each time that I do it. So that is what rate of change is. 8:56
I: Any mental pictures come to mind when you hear rate of change?
Deborah: Pizza. Oh gosh, I have to think about all the different things. we talked about, oh gosh, I have to remember all of the examples. It is like a per unit thing. That is what I think of. It is when you get it down to that unit rate. How much is it per the unit? It is 20 cents per mile. 20 cents is my rate of change. Things like that. Gas prices. Ticket prices per theater. Baseball prices. I cant think right now.
I: That was great. So in conclusion, when a student says what is slope, how do you respond? 9:37
Deborah: I guess it would be what they asked me for slope in. If it is y=mx+b, what is slope? Well slope is the rate of change of the x. It is how many xs do I have sometimes I would say to them. If they are looking at a graph when they ask what is the slope, it is the steepness of the line. It how steep, it is how much it is changing. If I look at it as two points, it is the rise over the run of those two points or it is the change in y over the change in the x of those two points. So when they ask what slope is, it depends on what we are looking at. If we are looking at a word problem, well then it is the rate of change. If we are looking at a graph, it could be rise over run or the difference between the two

points, change in y over change in x. And if it is an equation, it is the coefficient of x. So that would be my answer.

Elizabeth

Elizabeth: Speaking about recent events in her school year and trouble with students' parents.
Transition to interview questioning.
I: So what's the highest math class you've ever taken?
Elizabeth: Uh, geometry, I guess, in college.
I: But no Calc, no PreCalc, no?
Elizabeth: No.
I: Ok, cool.
Elizabeth: I actually had to take geometry twice. The first time I took it in the summer in college, but you know how you just (?), man, boy I got it - did great after that. First time I flunked, had to retake it.
I: Yeah, I had to retake Calc in college, cause my calc teacher in high school was our 8th grade computer science teacher and was the only one in the district who could come, who they could find to do it. And then I learned everything on the calculator and when I went to Penn State we weren't allowed to use a calculator. So like I took the beginning and I was like, "I gotta start over". And like I didn't understand - my teacher's name was Huey Ho (?). So he used to say "If someone asks you who your teacher is, you say correct". *laughter* That's like the only thing I remember from that class.
Elizabeth: Statistics, actually I did take statistics. When you said the name, I thought of this man from Slovenia. He was very honest. You know the reason why I passed that class was because I showed up to every class.
I: There ya go.
Elizabeth: He was upfront with me and told me to my face. I had absolutely no idea what I was doing in there, but I showed up - I'm willing to learn!
I: That's how I was in my stats class. I made 100 on the first test, I think a 50 and then I got a one out of six on the final and there was no partial credit.
I: Um, so when you hear the word slope, what comes to your mind?
Elizabeth: Well, I think of a map, and of landscaping.
I: What do you mean on landscaping?
Elizabeth: The slope of the land, how you need to plant things, or how you're going to have to move the soil around.
I: Ok, so the first question that I have is, these right here are three tables. So, the question is - are any of these produced from a line, or a linear function? Yes, no, and how do you know?
Elizabeth: It could be
I: OK
Elizabeth: A linear function because all the x times f all equal the answer. Or you could read off the graph and show the table going out. (pointing at Table A) Like when x includes of 1, how it affects f.
7:55
I: So like Table A, is that a result of a linear function?
Elizabeth: I was thinking C was.
I: You were thinking C was. Why did you think C was?
Elizabeth: Because of the negative numbers.

I: Because, what do you mean the negative numbers?
Elizabeth: Uh, -2 = 6
I: Ok, and then -1 =
Elizabeth: 3, positive 3
I: Ok, what about Table B? Would you say Table B was a linear function?
Elizabeth: Yeah, it could be. When you multiply, everything is working out. Even in the correct answer, everything is taken by 2's.
I: Everything is taken by 2's?
Elizabeth: Yeah.
I: What about Table A?
Elizabeth: It's consecutive; it's more of a line.
I: Table A is more of a line?
Elizabeth: Yeah.
I: OK. Very cool.
Elizabeth: Whether it's right or wrong, I don't know.
I: It's ok - again - there's no right or wrong answer. You're going to see all of, like, this question here...there's really no right or wrong answer to this question whatsoever, it's really just getting your thought process. There's no right or wrong answers to these questions whatsoever; I designed them specifically that way.
I: And so, this one is a question that I basically gave two of my students last year, and the question simply was, "are those the same line - yes or no?" One of my students argued that the answer was yes. Another student said, "No, you're wrong. The answer is no, they are not the same line."
I: What do you think the students were thinking? How could they be, how could they not be the same line?
Elizabeth: Because they're both zero for x. Um, you would think possibly, because they're both starting at the same point, but the y number is different when you do the graphing, so to me they're different numbers.
I: Ok, what about, what characteristics...
Elizabeth: ...yeah, and…
I: ...ok, I'm sorry…
Elizabeth: yeah, and the negative. They're both...I mean, they're both in the same quadrants at 3, but, uh, the y numbers are definitely different. (pointing to lower left quadrant in graph on the right)
I: What do you mean the y numbers are definitely different?
Elizabeth: Um, like in quadrant 3 over here (pointing to lower left quadrant in graph on the left) the number's going to be, say 6 - where the line's ending, and up here (pointing to the upper right quadrant in graph on the left) it might be like 4 or something. Over here, (point to lower left quadrant in graph on the right) when you're looking at the y, you might have 6 over here, but it's gonna to be a, uh, 6 over here. You know what I mean? Like the numbers aren't (gestures with hands)...like, if I were to put numbers on my line, I would do it differently…
I: Yeah, you can write on that all you want.
Elizabeth: Yeah, I'm a writer. Yeah, you would be at a different ending point.
I: OK
Elizabeth: So, to me the numbers are not the same.

I: Ok, so what would your conclusion be - are those the same lines or are they different lines?
Elizabeth: They are different lines.
10:35
I: They are different lines? And what argument would you give to correctly say that those are different lines?
Elizabeth: Where the endpoints are...where they end. (Interviewer interrupts and Elizabeth cuts back in) Based on the numbers...where they align with the numbers.
I: Ok. (waits while Elizabeth makes hash marks on the graph.) Ok?
Elizabeth: Yeah. Yeah, we're fine. I'm just looking at it.
I: No, no, that's fine.
Elizabeth: Coordinate plane. Whenever you letter C, quadrant 1, 2, 3, 4, that's all I see now.
I: There ya go.
Elizabeth: Now, you'll have to give this to Jill too; make her think.
I: Give this to...oh, I already have. She was my initial subject/guinea pig.
I: Ok, so on this one, there's three different shapes. You got a little fish bowl. We got a cylinder. And we got a cone, and it's gonna basically be with the top cut off.
Elizabeth: Ok…(Elizabeth draws a line on cone to "cut" the top off.)
I: And now what we're going to do is assume that each of these shapes are empty and they're going to fill up from the same source at the same rate. And we're gonna put this figure underneath the source and let it fill up with water.
Elizabeth: Ok
I: So here I have three different graphs that represent time on the x axis versus the height of the water. And the question is, which of these three graphs match each of these three shapes. So, how is the water - the height of the water - filling each container?
12:10
Elizabeth: (picks up all three graphs) Ok, let's see.
I: And just think out loud please.
Elizabeth: Ok, I'm looking at the shapes and I'm thinking that the bowl here, because it's a rounder base, that it's going to start out, the height is gonna go up slower cause it's rounder on the sides. So, if that makes sense the way I explained it.
Elizabeth: Looking at this cylinder, I'm thinking it's going to go straight up. It's gonna slowly increase cause it's the same width across so that the water level's gonna become a continual. So it's gonna be going across the same, so it's going to be steady.
I: Ok, what do you mean by continual?
Elizabeth: Um, water, as the waters permit the same, it's gonna raise...the height's gonna be a continual rising.
I: Ok.
Elizabeth: Because it's the same width across. It doesn't have to expand out anywhere.
I: Excellent.
Elizabeth: This one, I'm thinking the triangle, because you've got the wider base and then the height is going to increase it gets narrower; it goes up.
I: What do you mean the height is going to increase?

Elizabeth: It goes up. So my base here I need more water to fill the bottom up. So here's my bottom, and my height's not increasing until this bottom...this larger area is filled up. And then as I come up...my height's gonna increase.
I: Ok.
Elizabeth: Did that make sense?
I: Absolutely. Yeah. Perfectly. Perfectly. Good job.
Pause as Interviewer organizes materials.

13:35
I: Ok, so this was kinda interesting. I had to do a study before this one, which, to make sure it was good. So,
I made a teacher this line right here, and I said "what is the slope?" and they said, "well, the slope's a 30 degree angle". And I was like, wha...ok. So I put the 30 degree angle in and they said, "yeah, that's how I would define the slope - based on that angle". How would you respond if a student, or a fellow teacher came to you and said that the slope of that line is a 30 degree angle?
Elizabeth: I'd say "what's that?" I don't know how to teach slope; I've never taught it.
I: Ok. That's fine.
Elizabeth: I honestly don't know. I would just think that you got a right angle...I don't know. I mean, cause if you do a 30 degree angle here, don't you have to take the rest of your angles to equal 180 around?
I: Um, 360 around.
Elizabeth: 360? I don't know. (I: no worries) Don't have to teach that in 6th grade, sorry.
I: This has actually been the most difficult question because most people have said the exact same thing.

14:26
Elizabeth: The slope to me would be what is going down so I wouldn't take it as 30 there.
attempts to fill in angles around the line. Interviewer helps her record the angle amounts correctly.
Elizabeth: I just need to figure this out. There's always a math way to solve stuff I just didn't know how to do it. Um, but I would not say it's a 30 degree slope. Because to me the slope is more of the angle down. This (gestures to the 30 degree angle) I'm thinking is more of a right angle when I'm looking at that. So I would not be able to defend it to a student to tell them they were wrong. I'd say, "see Mr. S or Miss C on that."
I: So on this one right here, there are four graphs. The question is: could A, B, C, or D - any of those four - have a slope of positive 2?
Elizabeth: (looks over all graphs and circles letter b) I would think, for me, slope is going up or down, but because it's up above on the (?) line, I think it's b as a positive 2.
I: So what about A? Could A be a slope of positive 2?
Elizabeth: Well, I'm looking at it and I'm thinking no because I'm going from the top down to the bottom looking at the arrow, so I'm thinking this one's negative. If I look at slope as a piece of land, this is going from my home to the bottom. So if I'm looking at it, I'm visualizing it as my landscaping.
I: What about C?
Elizabeth: C to me is just flat; it's flat across.

I: Could you put a number to that flat slope?
Elizabeth: It would be negative first of all, because they're underneath the line (gestures to x axis). I'm looking at it as a coordinate plane, and they're both in the negative section.
I: What about D then?
Elizabeth: Um, D to me, there's a gulley in here; like a hole. To me, that's the reason you would not have a slope that is positive in here. Because it's equal up here (gestures to the top of the parabola where the arrows are) so it's not increasing or being positive...if that makes sense.
I: Absolutely. So here, when I was searching the internet, I found this picture. (truck on hill with slope).
17:26
Elizabeth: 6% grade
I: Correct. What do you think they mean by a 6% grade?
Elizabeth: The steepness is going to increase as the vehicle is going downhill and their speed is going to increase.
I: So what do you mean the steepness is going...
Elizabeth: Um, it's more...instead of...uh...it going up it's more of a dramatic down. Going downward.
I: So what do you think...if that's 6%, how do you think they came up with 6%?
Elizabeth: They probably measured the distance from another spot. So even if they went up the road, say, 100 yards, and measure they can see where it suddenly drops faster. And then they make the road smoother; just adjusted it.
I: So what do you think a 50% drop would look like?
Elizabeth: It would be more dramatic. It would be almost like coming off the edge of a cliff; when you go down.
I: What about 100% drop?
Elizabeth: It would be straight down.
I: 100% would be straight down?
Elizabeth: Yes. That's how I look at it.

18:28
I: Ok, so this will be right up your alley right here. So we have five staircases: A, B, C, D, and E. So, the question I have for you: which would be the easiest to walk up? - and then rank them from the easiest to the most difficult.
Elizabeth: Ok, the easiest to walk up would be staircase A.
I: And why would you put staircase A as the easiest to walk up?
Elizabeth: Not just because there's four steps, but because of the width of the steps. So that would be the easiest. The hardest to walk up, would probably be C.
I: Ok - why C?
Elizabeth: Because of the width of the steps and the height going up. You're taking a larger step up each time you go up. Staircase D would be the second easiest. Probably two steps up, but it's not a high step up, and then two steps up. E would be next. You're higher up. B, to me, is easier to trip on since they're such small steps. In fitting you won't have your balance as well. But that's just how I'm reasoning it out.
I: So, you got A, then D, then E, then B, then C.
Elizabeth: Yeah.

I: Ok. (discussion about dissertation topic) So, this question right here - we've got a guy who finishes a race. It's his last five minutes, so one of these graphs (1-5) matches up with one of these scenarios (a - e). A he finishes at a constant rate. B runs slowly at first then gradually increases his speed. C runs fast first then decreases his speed. D runs very fast and reaches the finish line early. And then E, after falling, Darren runs at a constant rate. So…
Elizabeth: Which graph aligns with all those?
I: Correct.
21:19
I: (after long pause) What are you thinking?
Elizabeth: Ok, let's see. Ok I'm thinking it's #2.
I: What's number 2?
Elizabeth: His speed. Because first he runs at a constant rate, then he runs slowly and increases his speed.
I: Oh - I'm sorry. I did a poor job. Basically question a itself matches up with one of these five. (discussion ensues about Elizabeth's misunderstanding of the question asked.) So letter a: Darren runs at a constant rate...does that describe 1, 2, 3, or 4?
Elizabeth: It would be 1.
I: How did you figure that out so quickly?
Elizabeth: Because it's at a constant rate and from endpoint to endpoint it's just straight up. So it's constant. (rereads letter b out loud). I'm thinking that that would be number 3. Because it's curving up slowly and gets faster. (rereads letter c out loud) I'm thinking that's number 5. (rereads letter d out loud) I think that's number 2. (rereads letter e out loud) That's number 4.
I: Ok, how did you get e and 4 together?
Elizabeth: Cause I'm looking at after falling he runs at a constant speed and it goes up which means his speed was constantly running up. So I'm putting right here (indicates the vertex of the angle) is where he fell. That's how I'm looking at it.
I: Thank you. So the next question is basically four different graphs. We'll start with this one first. So this right here is the time of day versus someone's hunger level. Can you explain that graph based on time of day versus hunger level?
Elizabeth: So you can look at it as you have three meals there, and as their hunger level increased (numbered these 1, 2, and 3 to represent the 3 meals of the day), at the peaks, that's when they would eat. And looking at time of day, their hunger level reaches a peak and then it decreases and it's pretty consistent cause it happens three times during the day. At the peak where hunger level reaches at lunch time...so if he ate then it drops until dinner, then it increases again like at dinnertime. I'm looking at it like meals.
I: So when is he the most hungry?
Elizabeth: After he's finished lunch he gets the hungriest right before dinner.
I: When is he the most full?
Elizabeth: After he eats dinner.
I: So (points to x-axis) this is the number of days since POW (name of the resource question is taken from) homework was assigned. And this (points to y-axis) is the percent of those that have been completed. So we're moving closer to the due date (points along the dotted line of graph) - what is that graph indicating as the days since it was assigned moves - increases basically?

Elizabeth: The closer it gets to the deadline the more...is it each person completing it or is it a group?
I: I believe it's a group.
Elizabeth: So, as it gets closer to the assignment due date, the more people are completing it.
I: Is it the same percentage of students each day that are turning it in?
Elizabeth: Actually it looks like it was a higher percent at the end turning it in. Because there's a bigger gap between the dots.
I: Does that seem how it would be if you gave an assignment?
Elizabeth: Yep. Pretty much. Last minute crunch. *laughter*
I: More students are more motivated on those last couple of days?
Elizabeth: Right.
I: So on this one we have time of day versus number of drinks in a machine. Can you explain that graph please?
27:24
Elizabeth: So we start out and have x number of drinks in the machine and it means more people are purchasing it during the day (follows the graph down with her pencil). And it shows a lower number at the end of the day. So I guess each line represents the purchase of a soft drink and how many are left by the end of the day.
I: And what about that line at the very, very end? What do you think that means?
Elizabeth: (indicates the solid black line at the far right of graph) Oh, that's the starting point again - to show where you came from.
I: Alright, last one. So we got number of customers in a movie theater versus amount of money in a cash register. Can you explain that graph please?
Elizabeth: The number of paying customers increases the amount of cash.
I: Does it make sense that it doesn't start at the origin or at (0,0)?
Elizabeth: Yeah, because this is the amount they had in the register. They may not have had any in starting. They may have only just taken the cash in; there might not have been any to start with, so it's starting just with the actual cash turned in by customers.
I: Ok, last task and then I have two follow-up questions. So this right here, these are two students. They were observing a leaky faucet. These are the graphs that the two students came up with while observing the same faucet. What might have caused their graphs to be different?
Elizabeth: The numbers they have on their graph.
I: What do you mean the numbers they have?
Elizabeth: Say Takashi did hers by twos (numbers the graph along y-axis), and say Denise did hers by ones (numbers the graph along y-axis) or even halfs. So if they used different numbers, the result would be different. They have the same information...it just depends...on the sequence of the numbers.
I: So if you were grading that task would they both be correct even though the lines look different?
Elizabeth: Yeah, depending on the numbers they put on the side (indicates the y-axis) to justify it. Because Denise might have put 5, 10...(writes numbers on x-axis), and she might have done something different over here but it could be the same information just they mixed the numbers up.
I: So could those be the same lines?

Elizabeth: Yeah. Depending on the numbers being used on the graph.
29:58
I: So, here's a question...and this one you may...because you never taught slope. The equation of a line is given as...have you heard the expression $y = mx+b$?
Elizabeth: yes.
I: Do you know what m does?
Elizabeth: no.
I: Do you know what b does?
Elizabeth: no.
I: So in sixth grade, there's no slope at all?
Elizabeth: no. They no longer learn about circumference; they no longer learn about pi.
I: So then how about ratio? Do you teach ratio in sixth grade?
Elizabeth: yes.
I: Ok, so how would you define ratio?
Elizabeth: It's comparing two numbers. So you might have three different outfits...ok how to explain it...flowers in a garden...we have six pink flowers and two red flowers. You would say it's six to two, the ratio. Six red to two pink, or whatever the numbers are.
I: Do they have to be different? Like you said red and blue flowers?
Elizabeth: Usually ratios compare two different things.
I: Do they have to be both flowers?
Elizabeth: No. It can be like triangles to circles. Different shapes. You can compare two different things and you're looking for the number that comes first, is the first thing asked about in the sentence and then the second number is the second.
I: So what images come to mind when you hear the word ratio?
Elizabeth: Colons.
I: What?
Elizabeth: A picture of a colon.
(dialect made it different for interviewer to understand word used by Elizabeth. Conversation occurs where they both are understanding and laughing about the misunderstanding.)
31:47
Elizabeth: The colon that you write or the words "to"...and there's another way to write ratio…and then sometimes just illustrating it out.
I: What do you mean by illustrating?
Elizabeth: On one of my tests I had some different shapes and I said what's the triangles to the circles. And for some kids you could see where they had to draw it out to see, ok there's five circles here and so they drew five circles. And then they put how many triangles. Sometimes they would illustrate it and count out their pictures.
I: What about in sixth grade, do you do rates?
Elizabeth: yes.
I: How would you define rate?
32:35
Elizabeth: When we did rates I took it based on cost of items. The example was...most of them have dogs or cats, so if you were buying dog food, which is cheaper, is it better to buy a 30 pound bag or a 20 pound bag? If it tells me it's \$16 for the 20 pound and \$30 for the 30 pounds, how would you divide it in? The rate is how much each part costs. And

also you can use ounces for different things. You know, is it cheaper to buy the Big Gulp for so many ounces or is it a better deal to buy smaller ounces. So if you break it down per ounce you see that it's only going to cost me 16 cents per ounce versus the other one you thinking being a better deal but it's really costing you 22 cents an ounce.
I: So if you wrote on the board: ratio - what would you write beside it so your students knew what you meant by a ratio?
Elizabeth: How to compare the two numbers.
I: And then underneath it you have rate. What would…
Elizabeth: Unit rate is comparing the cost of something. Each ounce, or each pound, or each part of it. Which is the better deal.
I: What about the term "rate of change"?
Elizabeth: I don't know that.
I: When you hear the words rate of change, does that invoke something different than the word rate or do you think those are...?
Elizabeth: They're probably together. Maybe I was supposed to teach them and just never did. (dialogue about curriculum guides)
I: Just a final closing question: Define slope one more time.
Elizabeth: Slope is the angle or change from the top to the bottom; the distance in between.
I: So from this point here (view unseen) to this point here?
Elizabeth: Right
I: Ok, and then what do you mean by the angle?
Elizabeth: The angle as far as how fast it comes down. Yeah, sometimes it's more of a rolling slope. I just think of it as landscaping.
I: So can a rolling hill have a slope?
Elizabeth: Yes. It can go down.
I: Ok. Awesome

Jackson

Interview Transcript - Jackson 8th grade teacher

I: Thank you for agreeing to participate and for filling out the Google sheet in advance of this interview.
Jackson: Your welcome.
I: I did have one other question. Why did you decide earn your six - nine certification.
Jackson: Well, I came down from Pennsylvania. So, I was certified K-6. I was teaching sixth grade math, reading and writing. Sixth grade went to the other school. So to stay at seventh grade, I had to get my middle school certification. I was four credits away from math, English or History and I picked math because it was my favorite subject. So, I figured I would just take the math.
I: Very cool, thank you. So, when you hear the word slope, what do comes to your mind?
Jackson: A rate. A rate of change. That is what we talk about a lot in my classes. A lot of people think it is all about numbers, I subtract this from that. It is miles per hour. Something we see everyday. Price you pay for a vegetable. Anything at a certain rate, a cost, an amount. That is what I think it is. I try to teach that to the kids so that it makes more sense to them.
I: What other images comes to your mind?
Jackson: We talk about how roofs have slopes, water is positive or negative. We talk about how planes have to descend. We talk about how the slope of the road has to be so that water can run off. Slope is all around us.
I: Thank you. So, my first task is involves three tables (I place the three tables in front of him). Are anything of these tables the result of a linear function and why?
Jackson: Well, I look at, since your xs are going up constantly, I look at that slope as going up on Table B. It is going up by the twos. I look at that as a good slope. And if you look at Table C, it is going down. So that would be a negative slope. Table A, umm, it is going up at a rate of five each time.
I: Okay, so all three of them are linear?
Jackson: Yeah.
I: And what method are you using?
Jackson: (Laughs). I am just looking at the xs. Making sure that is a constant and then looking at my ys. And making sure that is a constant also.
I: What do you mean by constant?
Jackson: It is going up by the same amount each time. It is not a change. Like going up by three, then by seven then down by four then up three. So, I am just making sure it is a constant.
I: Okay, thank you. So, last year, when I was still teaching. I gave my students these two graphs equations. Are those two the same? One argued that the were they same line while another student argued they were different. How would you respond?
Jackson: Well, it depends on the type of question. Are they the same line? They both positive. You could say it that way. You could say one has a higher rate and the other has a lower rate of change. They could be different in that way. So it just depends on how you pose the question. You can see on the first one, it is a steeper rate of change and on the other one, it is a more lower rate of change.

I: What do you mean by steeper rate of change?
Jackson: It is going to down up higher. You might have a big difference from your first one to your second one. That is the way that I looked at it. The other one is not as steep as a line. If we are talking about water runoff on a roof, I don't know it I would want that first one. Coming off, it is going to kill my foundation. The other one might be a little bit better depending how much I need. But it depends how we are looking at it.
I: So what conclusion would you make?
Jackson: I think they are both right.
I: They are both right?
Jackson: Yes. As long as they could support their answers. That is what I would say.
I: Why would the one that said they were different be correct?
Jackson: Because of the type of slope that they are. One has a more of a steeper slope. They are both going up, so you could look at it that way. (Hands question back to me)
I: Very cool.
Jackson: Thank you.
I: On this next one, we have three geometric shapes.
Jackson: Great. I have not done geometry in forever.
I: So we have a cone, a cylinder and a fish bowl. Right now, they are completely empty. We will fill each with water from the same source. Your task is to match the shapes with the graph of the time versus the height of the water in each source. Therefore, I have three different time versus height graphs.
Jackson: Okay. (Matches fish bowl correctly and has an error on the other two)
I: Why did you match them that way?
Jackson: I looked at the shape of the bowl (Points to fish bowl). If I am filling this up here, it might fill up here, then go up and around (Tracing the outside of the fishbowl). This one is going to fill up the bottom then gradually come up (Pointing at the cylinder). This one is going to keep going at a different rate (The cone).
I: What do you mean by a different rate?
Jackson: Well, it is going to fill up the bottom more, it will be bigger on the bottom and then smaller at the top. It is going to take a longer time here (pointing at the bottom of the cone). Filling up the bottom portion than the top portion.
I: And what about the middle one (the cylinder)?
Jackson: It will be at a constant rate. So let me do it that way (Flips the two graphs. Correct now). Thank you.
I: Why do you just change two of the graphs?
Jackson: Because it is the same (Moves hands straight up and down to demonstrate the sides of the cylinder). There is no differentiation in the sides. They are the same. The level is the same. It is going to keep filling up at a constant rate each time.
I: And what about the fishbowl? How is it going to fill up?
Jackson: The fishbowl is going to fill up the bottom then fill up the sides (moving hands out wider modeling the shaped up the fishbowl). Then eventually to the top.
I: Okay. Cool. Let me grab these in this order. (I take up the cutouts). So during a previous interval, I gave a teacher a line and asked her to define the slope. Her response was a thirty-degree angle. Therefore, I have labeled this line with a thirty-degree angle. How would you respond to a peer or student if they stated that the slope of the line was an angle?

Jackson: I would say that would be incorrect. The only reason that I would say that is cause on the other side, you have a different degree angle, as far as I know. So, I don't think an angle is going to determine the constant rate of this line. Plus, looking at the thirty here (Points to angle), the angle is going to be steeper up on this side (points to the arrow and comes down to the x-axis). So I don't think it is going to determine the constant rate of this line.
I: What do you mean steeper on the other side?
Jackson: There is a bigger gap between your Xs.(Points again to the arrow and comes down to the x-axis).
I: Oh okay. So the angle as you move left to right is getting bigger?
Jackson: Yes.
I: So what would you say to the teacher?
Jackson: I would say to look at it again and think why it is not.
I: So, what would be an appropriate answer to what is the slope of that line?
Jackson: I look at maybe a rate of two, maybe three.
I: What did you pick a positive number?
Jackson: The line is going up from left to right (Hands paper back).
I: Okay, thank you. So this next one, there are four different graphs. Please explain whether any of these functions have a slope of two? Positive two.
Jackson: B possibly could.
I: Why B?
Jackson: Because that is the only positive line showing. A is a negative. C is a flat line. D is dealing with something different.
I: So neither A, C or D could have a slope of two?
Jackson: Yeah.
I: Why did you know that A was negative?
Jackson: Because it is going down from left to right.
I: And you said C had a slope of….?
Jackson: We are saying a slope of zero.
I: How did you figure out zero?
Jackson: It is a flat line going from left to right.
I: And D?
Jackson: It is not going at a constant rate. You have some going down and going up. So, that is why I said it was not (Hands me back paper)
I: Okay. So when I was searching the internet, I found this image (I hand him image). What do you think that they mean by a six percent?
Jackson: (Stutters for a moment). That is the grade, of course. It is going down. Maybe, depending on what rate they are using. It is going to decrease six percent every so many feet, miles, whatever they are looking at. That is going to be the rate. It is going to be declining six percent every so often.
I: How do you think someone determine that it was a six percent decrease?
Jackson: Maybe they took a picture of it and overlaid a graph on it. But that is not an actual picture of that road or that mountain. It is not talking about the mountain but the road going down the mountain.
I: Good point. What would a fifty percent look like?
Jackson: A decline of fifty percent?

I: Yes, sir.
Jackson: I think that would be very, very steep. (Used hands to demonstrate a steep slope)
I: What about 100 percent?
Jackson: A 100 percent decline?
I: Yes, sir.
Jackson: Straight up and down (Uses hands to go straight up and down).
I: Okay, could you go over a 100?
Jackson: I don’t think so. I don’t know. That is my honest answer (Laughs).
I: So, again, 100 percent?
Jackson: (Thinks for a moment) I am thinking that it will be straight up and down. Straight down.
I: Thank you. So on this problem, I have five staircases. My question: Which would be the easiest to climb and why. Please provide a rationale.
Jackson: I can climb it or go down?
I: Yes.
Jackson: (Organizes and places in order). I think this one because it is an easy step to go up (Points to B). This one I am looking at it coming down not going up. I don't think it would be too bad of a decrease (Referencing A). This one (D) here, if I was going to climb down it, it is going to be tough for me to go from one to another, I might have to take an extra step (Pointing to the width of the steps). This one here (Pointing to E) I think it is going to be very hard to climb up to get up that high. This one (C) is going to get me confused. I have a big jump here (points to the first step) and another big jump here (points to the second step).
I: So in D, you said you would have to take an extra step, what do you mean?
Jackson: It is a longer distance between each step.
I: Why is B the easiest?
Jackson: It is not much of an incline and I think it would be an easy step each time.
I: So overall, if you were designing steps, what makes it easy to climb?
Jackson: It depends where I want to go. I think this one (A) would be more steps if I was getting up a staircase but I am looking at this, I am also looking at this as a handicap ramp. It I was thinking about it that way, I would want less of an incline.
I: What do you mean “less of an incline?”
Jackson: If I am having trouble getting up stairs, this (B) might be a little bit easier for me. I don’t have to take that first big step or that second big step. Here (A) I have to take that bigger step.
I: Thanks. (I take up the staircases and maintain his order B, A, D, E and C). So we have a runner Darren. These graphs model the last five minutes of a race. Please match the graph to the scenario.
Jackson: 1 is A. (Immediately).
I: How did you figure that out?
Jackson: Well, it says at a constant rate. It is just going up, they are no change.
I: What do you mean no change?
Jackson: Well, he didn't fall, he didn't do anything else. It said at a constant rate, so he is going the same amount each time. So, A shows that. You are looking at (Pauses). I would look at B as I am running slow then increases my speed (Tracing graph of number 3). 3 would be B.

I: What about 3 made you think it was B?
Jackson: I am visualizing my axis here. Is speed, if this is the higher speed, it goes slower, slower then increases (Tracing 3). He goes fast and gradually increases his speed (Reading problem). I would say 2 (Writes C by 2). (Reads D). I would say maybe 4. (Reads E). (Traces 5) while reading E again).
I: So you said D was 4. Correct?
Jackson: Yes. Cause he is sprinting very fast here (Tracing horizontal line portion of 4). Then it made him reach a speed of very fast as he is going up (Tracing the linear portion). (Hands paper back to me)
I: Excellent. Thank you. So for this task, I have four scenarios. They are each a relationship between two quantities. Please describe each scenario. (I read him the first time of day versus hunger level)
Jackson: Okay. I am looking at here like time of day is going up. He wakes up a groggy and wants to eat (Traces first line segment). And then he is getting hungry and takes a snack and it is going down. Then it comes towards lunch time and he is going to get his meal. Drops right down because he just ate. Right around that lunch time (point to second cusp). Then you are finishing out your work day and right around dinner time you are going to have your snack again. Then, why does it go up (pointing towards last portion of graph) depending on the time. You ate around 4 or 5 o'clock and it is now ten, you might want to get a quick snack.
I: When is he the most hungry?
Jackson: Later in the day.
I: Do you have a specific time?
Jackson: Maybe around 6 o'clock. Five six o'clock.
I: Where would that be on the graph?
Jackson: Here. (Points to third high cusp).
I: When is he the least hungry?
Jackson: Here (Points to the lowest point). I would say around 2 o'clock.
I: What about the graph made you gravitate towards those?
Jackson: Just the lowest point of the hunger level and the highest point.
I: Thank you. Here the next one (I explain the graph). How would describe it.
Jackson: I look at it as a person that is procrastinating.
I: What do you mean?
Jackson: Number of Days. This looks like one day since it was assigned and very few people have completed it (Points to first point). As it gets closer to the due date, if that was the due date, more people are getting it done.
I: So does it progress as at a constant rate?
Jackson: I would not say a constant right off the bat (Traces the points). I would not say at a constant. NO
I: Why?
Jackson: Just from looking at the curvature of the line.
I: Curvature of the line?
Jackson: I look at a constant of being the same each time. It looks like it starts to get constant around here (Points to the upper right portion of the scatter plot). It is going up gradually then all of a sudden it goes up even more (tracing graph) (Hands back graph)

I: So on this one, this is a vending machine. Time of day versus Number of drinks. How would explain this graph?
Jackson: It is definitely going down (tracing downward slope portion). It is a negative type of graph. It starts out in the very beginning, not too many people get drinks, maybe they just had breakfast (tracing horizontal portion). As the day goes on, more people get drinks.
I: What about towards the end of the day?
Jackson: Maybe they restocked it. (Hands back graph)
I: Here the last one. How would describe that one?
Jackson: Going at a constant rate. Mostly because each person is paying the same amount.
I: What about the very beginning?
Jackson: The very beginning is the actual price for per ticket to start. If it is five bucks or whatever per ticket. They will charge the same amount each time. Or it might be the money that is already in the cash register.
I: Why would they already have money in the cash register?
Jackson: For change.
I: Was that a positive or…?
Jackson: (Interrupts) Positive.
I: Why?
Jackson: Two reasons. One, it is going up. Two, if I am selling a ticket, I should be making a profit or I should not be open (Laughs).
I: (Laughs) Very true. So the last task, look at these two graphs. These two students observed the same leaking faucet and asked to record the same scenario. They submit these graphs to you. What might have caused the graphs to vary?
Jackson: I looks like their, if I am looking at it, t is standing for time?
I: Yes
Jackson: Each person validates time different.
I: What do you mean validates time different?
Jackson: Maybe someone is in a hurry, they are thinking that is really coming down. That is why he is choosing this one here (Points to Takashi's graph). This one (Denise's) maybe is thinking that time is not too important to him. That is the way I look at it. He could sit there all day and it does not matter to him.
I: Thank you. When you hear the term ratio, what do you think of?
Jackson: I think of, you know I go back to, I teach my kids a lot of that when I start to introduce slope. Your ratio is a type of a slope. Doing unit rates back in elementary school. How much price per pound? I tell them you have done slope all of the time. That is how I look at ratio as a rate of change. How much you go up over a certain amount of time. How much you go down over a certain amount of time. 4 to 6 ratio, you know, might be down four every six feet.
I: How would you define it to your students?
Jackson: I would say it is comparing things. It is a rate of change.
I: So a ratio is a comparison and a rate of change?
Jackson: Yes.
I: Okay, so what would be your definition of rate of change?
Jackson: Ahhh, the change of a certain thing over a period of time.

I: What about the term rate? How would define a rate?
Jackson: Rate is what we pay, cost. It does not always have to be cost. That is a good question. It is hard to define rate. I would say what we are doing. It could be how fast you are going. Add line 24:52
I: Are rate and rate of change the same thing?
Jackson: I would say yes. Yes.
I: Why?
Jackson: Just if I would say, I have my rate is 4 dollars an hour. I think that would be the same thing as a rate of change.
I: When you hear the word ratio, what images come to mind?
Jackson: I think of money. Like 3 out of 4 dollars. Or baking. Like 3 parts to one or four parts to one.
I: What about rate. What images come to mind?
Jackson: Money. (Laughs). What is the going rate for something.
I: Anything final thoughts?
Jackson: I think it all intertwines. It all comes together. I think students focus on y/x. It is not all that. You see it everyday. I think once they see it in the real world, they have a better understanding.
I: How do you introduce it?
Jackson: I really show them different videos about lines, ski slopes, roofs, miles per hour signs, price per something like a grocery ad. Those are all slopes. They are different ways to find slopes. We do talk about to find it on a graph but also how you find it in the real world.
I: How do you introduce the formula?
Jackson: Change in y over change in x
I: What if I asked students? What is slope?
Jackson: Rise over run (Laughs)
I: Do you tell them that?
Jackson: No, I don't. I say you might have heard it before, but I don't. I tell them change in y over change in x.
I: So you gravitate more to change in y over change in x?
Jackson: Yes. I tell them their other teachers are not wrong.
I: DO you have students say the slope is m?
Jackson: Yes
I: How do you respond?
Jackson: I say that M is a letter (Laughs). A symbol.
I: If I gave you $y = ax + c$, what happens as a gets bigger?
Jackson: If a gets bigger, it will have a steeper slope.
I: What do you mean by steeper?
Jackson: It will go up at a higher rate. If is one, it will do up more gradually than 6. It will shoot up more quickly.
I: What if a gets smaller?
Jackson: It will , less change in the line.
I: What if c gets bigger?
Jackson: That does not have an affect. It changes the starting point.
I: Can you explain?

Jackson: If I say my cell phone bill starts at 20 dollars, the line will start at 20. My line is not going to change from there.
I: What if c gets smaller?
Jackson: Same thing. It will start lower. My line is not going to change.
I: What do you mean my line would change?
Jackson: The steepness will not change.
I: So, changing c has no impact on the slope?
Jackson: No.
I: DO you think most of your students understand that?
Jackson: They better. (Laughs)
Ends with some small talk.

Liam

I: Good afternoon. Thank you for agreeing to participate in this study. We will begin with some background questions. Why did you decided to teach math?
Liam: It is something that I have always been interested in. I was pretty good at. I like passing knowledge along. I was always more math than language arts. I like pre-algebra and algebra.
I: Why did you pick middle school?
Liam: I figure that is a good age. Children either go in two directions. Up or down depending on how the perceive school and what they really want to get out of it and if they think it is important. Some of them do and some of them just need a little push.
I: So this is your first year, what did you teach?
Liam: I taught pre-algebra which is eighth grade math and Math 1.
I: Which did you like better?
Liam: (Hesitation) Whewwwww. I guess pre-algebra. But it did not matter either way. Some of the stuff for math 1, you know, when I taught it the next day, I had to look back until I remembered it. Some of the stuff in pre-algebra as well. Rotations and translations. Stuff like that. When you look at it for a second, it makes sense. But I hadn't look at it in ten years. You're like I can't remember which one this is.
I: I understand completely. What are you certified in?
Liam: AT UNCC, it is just middle grades. 6 - 9. Just mathematics.
2:19
I: Will this be your masters or undergraduate?
Liam: Well, I did my undergrad at APP but I didn't do math. So I had to go back and take all of the math courses and the teaching courses as well at UNC-Charlotte. They have a two phrase, ahhh, graduate program. Some one you get done with phrase 1, you are licensed. You just need four or five more classes to get your masters. So, I am almost done with phrase 1.
I: Nice. That is awesome man.
Liam: It is a lot, as you know.
I: So have you ever taken calculus before?
Liam: I took AP Calc my senior year of high school and I took a business calculus class at Appalachian. But it has been a long time. Then I took a trig course when I went back to school down in Charleston. So about two years ago. Pre-Calc with trig. But other than that. I was signed up to take a calc course over at CPCC. But whenever they offered that job to me as a mathematics teacher, I kinda of figured that would be more beneficial than trying to take the class. I was the only class that I would have had to drop.
I: So then the business calculus class is the highest course that you have taken?
Liam: Aaaa, Yeah. I reckon. Business calculus was the only math course that you had to take for the business school. So that is the one that I took. It was all online.
3:59
It was a lot of formulas and stuff. It did not go into a lot of derivatives and stuff like that. Online. Not exactly spreadsheets but a lot of stuff like typing equations.
I: That makes sense for a business major. With respect to slope, when you hear the word "slope" what do you think? What comes to mind?

Liam: Something that the eighth graders really need to work on. Steepness. Slope. you know. Rise over Run. Steepness. Change, Delta y over delta x. That is mainly what we talk about in the eighth grade. Trying to get those kids to figure it out. Some of them, Yeah, you can go over the same question ten times and they still will still put x over y or something like that.
I: How do you typically introduce it then?
Liam: Aaaa. Well. positive and negatives. You know Climbing up the mountain versus sliding down the mountain. So that would be a negative slope over a positive slope. Trying to take two points and just count. Just count. Up and over. That sometimes gets lost cause you know kids will look like this is x. Then when we are trying to plot points, they go back to slope and plot y,x instead of x,y. It is kinda of frustrating, tough to grasp at first. But just tell them, you know, you have a linear equation, it is the steepness or how far it is rise over run. Two points on that line.
I: Okay, here is the first task. I have three different tables. How would you figure out which of the tables was produced by a linear function?
Liam: You would look at the x and the f(x) which is the y and you would do y2 minus y1 over x2 minus x1. For each point and make sure they are consistent. Which would then be a linear equation.
I: So, is Table A is that a linear equation?
Liam: (Thinks for three seconds) No, it is not
I: Why not? How did you make the determination?
Liam: So I did 4 minus 1 which is three over two minus one. So the difference in that one is three. The difference in the next one is five. So it is not.
I: How about Table B?
Liam: Table B is a linear function. The slope is two over one the entire time.
I: And what about Table C?
Liam: (Evaluating the points using change in y over change in x) It is a linear equation. It is a negative slope. Down three over 1.
I: Okay. Excellent. Perfect. For this question, I gave to my students last year when I was still in the classroom. One of my students, Christine, believed that the two lines have different slopes, but Ahmad disagreed. What might each student be thinking?
Liam: Well, they don't look the same which basically starts out that they can't be the same line. But, as Christine says, if you are not given the (Struggles to find the word he wants to use, Draws ticks marks on the x axis) on the line, the, different intervals, instead of one, two, three, it could have been more spread out. So technically, it could have been the same slope, if given values on the x and y axis. But it does not look like the same slope.
I: What about the graphs make you say it does not look like the same slope?
Liam: Well it is two different lines, but the steepness of the two. Graph 1 or the graph on the left is much steeper than the one on the right. They almost look like it would be half of the other one. It goes up about half as fast.
I: So if you asked your students this question, what do you think they would say? Are these the same equations?
Liam: I can't image that they would. Though they do have the same y-intercept so she may have been looking at the y-intercept and not the actual line. They would look at the line itself and see that one goes up quicker than the other. Unless it was like a trick

question. But no they do not look alike. That is what they would probably see to differentiate.
I: So on this problem, we have three different shapes (I lay out the three shapes). A fish bowl, a cone and a cylinder. Each of these containers are empty and will be filled with water from the same source. Here are three functions that show the time versus the height. Can you match the graph to the container?
Liam: (Lays out the cards). (Picks up the straight line and matches it to the cylinder). I feel like those two go together.
I: Why do you believe that those two go together?
Liam: Cause as you fill it, it will go up the same amount, the height and time. The shape of the object is flat on the bottom and straight on the outsides. So as you pour it, it goes up the same. It goes up the same amount, it is consistent. Ahhh. (Pauses to examine). Due to the shape, I am not real sure, but these two would go together (Matches the fishbowl to the cubic)
I: Why do you think that one goes with the fishbowl?
Liam: Cause the shape is a little different. It (the fishbowl) goes out and back up. This one I would say you put together (Matches cone with right side of parabola). Cause (pause). Height versus time. Versus time. (Pause)
I: What are you thinking?
Liam: Aaaaa. It is going to go up at a decreasing rate (Pointing at the parabola). Cause after you get the bottom filled, you don't have a lot of space to fill towards the top. I am not sure that is the right one. But I will go with that anyways. I don't know if it is right.
I: Well, I told you were right, how would you justify it? Assume that you are correct. Focus on the cone. Assume you are correct.
Liam: Cause as you get towards the top, you are going to have less space. So the time is going to decrease as the height gets closer to the top. I am not explaining it right. Cause there is less space to fill. The top. Time kinda of levels off. (Stops talking).
I: Good job. You match them perfectly. Let me take these images back. So when I was interviewing early, the teacher stated that the slope of this line is thirty degrees. I asked what was the slope and she stated a thirty degree angle. How would you respond?
Liam: The angle is thirty degrees. That means you have sixty degrees left (Writes 60 between line and the y axis. So you would almost go up one. It is not one half. It is not a thirty degree slope because it is not essentially a fraction. It would be three over nine which is one third. So the whole things is ninety degrees (Points to the right angle in the first quadrant). Thirty out of ninety is a third. So up one over...three. Thirty percent, well. Thirty percent of ninety is thirty. But I don't think the slope is thirty percent. Just cause the angle is thirty percent does not mean that the slope is thirty percent. You are looking for slope. Actually the slope would be...I still say it would be one third since thirty is one third of ninety.
I: So, in your opinion, is that a valid way to articulate the slope?
Liam: I don't know how. Maybe. I guess. The whole thing is ninety. If you take thirty out of ninety, it is one third. So the slope would be up one over three. (Pauses) 15:30
I: Okay, thank you. So we have four graphs here (I provide him with question 15). Could any of those of four graphs have a slope of positive two? A, B, C or D?
Liam: A can't cause it has a negative slope. C is a zero slope because it is a horizontal line. B could be a slope of two. (Pauses). D is non linear. I guess B is the only one that

could possible have a slope of two but it looks more like a slope of one half the way that it is angled but you don't know the points on the graph. So the only one it could possibly be is B.
I: So B could be a slope of two depending on the scale. What about A made you say no?
Liam: Cause A is a negative slope.
I: How did you know that?
Liam: Because it is sliding down. If it said negative two (pointing to the 2 in the question). If I made up a point, say negative three, three (pointing to the top arrow on graph A) and the point zero, zero (pointing at the origin) and subtracted it out, it would be a negative slope. There is no way it could be a positive slope of two. Then like I said C is a zero slope because any horizontal line is a zero slope. D is a nonlinear equation.
I: So D could not have a slope of two?
Liam: No.
I: So B, good. (Staring What are you thinking?
Liam: Thinking if D can. Technically, nah. Nevermind.
I: You are focusing on D. What are you thinking?
Liam: I can't remember if technically D can have a slope. They still go up and over. It is a nonlinear line. So I would say it could not have a slope of two.
I: I have a picture for you. What do you think they mean six percent?
Liam: You are going down the mountain. It is a six percent out of one hundred, the grade of steepness. So six percent steepness as you go down the mountain. So if it was ninety degrees, it would be going straight down. Six percent. So it is going down. So, I guess for every one foot over it is going down six feet. (Drawing on paper). So it is a slope. So I guess percents could be a slope.
I: What about fifty percent? What would that look like?
Liam: It is going down a lot steeper than just six percent. (Uses his arm to show steeper). More steep than the six percent. The higher the grade, the steeper the road.
I: The higher the grade, the steeper the road. So what about one hundred percent. What would that be?
Liam: I am getting confused her. I can't remember. It is going ninety degrees, so straight down. (Draws a ninety degree angle) (Draws another ninety degree angle) (Draw a triangle). I think I am confusing myself. Six percent grade. So for one 100 feet you go left, horizontally, you go down six feet. six over one hundred. Six percent. So if it was fifty degrees, for every one hundred, it would go down fifty feet. So that would be a fifty percent steepness. Maybe I should not drive on mountains. I am trying to figure this out. For every 100 feet, you go down six. That gives you six percent. Rise 6 Run 100, gives you 6 over 100 which is six percent.
I: So what would 100 look like?
Liam: One hundred percent is straight down.
I: Straight down?
Liam: Vertical.
I: So what do you think fifty percent would be?
Liam: Halfway. So if you were going over 100 feet, you would be going down 50 feet. Yeah. For every hundred feet you are going down fifty. I guess. That would be fifty percent because fifty over one hundred is fifty percent.
I: So for every hundred over, you go down fifty?

Liam: (Nods). Yeah.
I: And one hundred would be straight up and down?
Liam: Yes.
I: Would it be possible to go over a hundred?
Liam: I don't think so. If my reasoning, if what I saying, then no. (laughs). What I am saying might not be correct. (Pauses).
I: Still thinking? Talk out loud.
Liam: I am just thinking of one hundred over one hundred for some reason. One foot down every foot. Yeah, I guess so.
I: What if I told you the percentage formula is 100 times by rise over run?
Liam: Yeah, Yeah. So for every six feet you went down, so six over one hundred is 0.06 times 100 is six percent.
I: So we have the formula y = ax+c. What happens as A gets bigger?
Liam: The line gets steeper. (Uses his arm to show steeper).
I: What happens as A gets smaller?
Liam: It keeps on becoming more and more horizontal instead of more and more vertical. Until you get to a zero slope which is horizontal. The less steep it gets. If you have a negative slope, it is the same thing with a negative slope.
I: What if I change the C value? In y = ax + c, what if C gets bigger?
Liam: It just changes the y-intercept. It goes up or down the y-axis. Your starting point on the y-axis.
I: Does changing C impact my slope?
Liam: No (Shakes head)
I: So if I have 3x+4 and 3x+14,
Liam: They are just parallel lines.
I: How do you know?
Liam: Cause the slope is exactly the same and the y-intercept is different.
I: Awesome man.
Liam: So we have five staircases? In order, which would be the easiest the climb? Think out the climb. Then rank them.
Liam: What are we talking about? The most consistent? Where you don't have to step up as high?
I: That is entirely your reasoning. There is no right or wrong answers. Just your thoughts.
Liam: (Laughs) I think D, I would say, would be the easiest to climb just because it is only three steps, it is spread out so you can put your whole foot down and go up consistently. Followed closely by A just because A looks like it is the same distance up and over. (Pauses). Cause a set of steps is normally seven by eleven and that almost looks like it would be seven by eleven.
I: Okay
Liam: I would say E would maybe come third just because it is consistent but it is a little bit steeper.
I: What do you mean by consistent?
Liam: Consistent. Rise over run is about the same. It looks fairly simple to walk up. I say the last two just depends on how much you want to step up or step over. I will go with C as my fourth highest cause you step up a lot and it is less steps. I would say B is not the most difficult but the most tedious to walk up very little amounts. Maybe if you don't

walk very well it might be easy because it does not rise a lot. It runs more than it rises. I would just think it was annoying if I had to walk up all those kinds of steps (Pointing to B). But. So I say the first three are just more consistent. Well, maybe not D. D looks like E but with just less steps. So I would say, let me change my order. Let me go A then E cause they go up the same amount that they go over so that would be easiest to climb straight up without having to go too far over or step up much higher. Not level, but it goes up a consistent rate.
I: Can you write your numbers on the back?
Liam: Yes.
I: Thanks.
Liam: I guess it depends on how you want to look at it.
I: So on this question, we have five graphs modeling Darren as he finishes a race? Please match each graph to the scenario.
Liam: So this is like increases at a decreasing rate, increases at an increasing rate?
I: Absolutely. Which one corresponds to A?
Liam: I think A is number 1. Cause he runs at a constant rate. So that is a constant slope of one over one. Constant. So him running slowly at first then picking up his speed is 3.
I: How did you figure that out so fast?
Liam: Cause x is always distance (tracing the x axis on graph 3) and the y axis would be time. So he kinda of starts by not going very far. He is kinda of staying at a slower rate and gradually increases his speed. Distance over time goes up more at the end as he gradually increases his speed.
Liam: (Thinks about C) (Muttering to himself) 2 would be C cause he runs real fast at the start and the slope is going up exponentially. Decreases his speed towards the end. He gets all the way up here and it levels off. 2 is C. I guess 5 is D. That is the only one that I did not know about. He gets to the finish line (Points towards the cusp on #5). So as time goes, he does not move anywhere. The distance goes not go up or down. He gets to the finish line and just stays there. As time goes, he is not moving. He is standing still. The distance goes not go up or down. So this one (#4) would be E. So he falls at the beginning, as time goes, he does not move in terms of distance, then he runs at a constant rate like number 1. It is consistent.
I: That was fantastic.
Liam: I really like these problems.
I: You are welcome to use of these problem. So for this one, I am going to give you four graphs to describe. Time of day versus hunger level, how would you describe what is happening in this picture.
Liam: Basically, when you hit these top three points is when you are eating. So your hunger level goes down. Your level rises until you eat, then it goes down. Then it build up until you have your next meal and then drops. So I guess he had a really big meal for the second meal cause it dropped drastically. Then it went up again until he eat his third meal and then went back down again. So every time it peaked, whenever you have a vertex at the top, a maximum, is when he ate so he hunger level decreased. (Tracing the graph). His hunger level went up then he ate, I guess you could call it his breakfast. He ate a little bit for breakfast cause it went down, not as much as the others. They it went back up as he got hungry. He ate his middle meal, I guess lunch. It dropped a lot so I guess it must have been pretty significant cause his hunger level went way down. By the

time dinner came around, his hunger level was back up above where it was for his first two meals.
I: Okay. Awesome. Here is the next one. POW is a homework assignment.
Liam: So the number of days goes up as does the percent completed. They are finishing one assignment at a time then assignments start racking up. As time goes on, a higher percentage of POWs is completed. So the correlation goes up. So you are completing more as the days go up.
I: So you are completing more as the days go up?
Liam: Yes, as x increases, y increases.
I: For the next one, we have time of day versus number of drinks in a vending machine.
Liam: So you the the machine as your y axis and time of day as your x. You start out with the machine full. As the day goes on, people continue to get drinks. Since it is not all the way at the bottom, there are still drinks in it. But, it gets refilled to where all of the drinks are back in the machine. So as the time of day, gets larger the number of drinks goes down. Which would make sense. You start out with a lot of drinks and end up with a smaller amount before the machine is refilled.
I: For this one, number of customers in a movie theatre versus the amount of money in a register.
Liam: Same thing. Well, this one actually goes up at a constant rate. They start with x amount of money in a register. This one does not start at zero. So obviously, you have some kind of money in the register and they are increasing at a constant rate. So every single person that comes in pays the same amount, so the register increases in money. As x goes up, y goes up. The difference is the y intercept is a little higher. Obviously, if you are running a business, you want money in the register to begin with.
I: Nice. Great job. Perfect on all of them. So for this task, Denise and Sarah both observe a leaking faucet. Each of them makes a graph of the data that they collect. What might have caused their graph to look different?
Liam: Their time must be different. One of them may have gone over every two minutes as opposed to Denise's that goes up every minute. Hold on. Let me make sure I am saying it right. So hers looks like it is going up at a higher rate but they are seeing the same amount of water. They are using different, I am blanking on the word, different intervals. So there intervals must be different since they are using the same exact data. So her y axis intervals must be a little bit smaller than Sarah. Her looks like there is not on as much water on second graph but it is the same amount. It is just different intervals on the y axis and x axis on the second graph.
I: I just have a couple final questions. The word ratio, how would you define the word ratio.
Liam: A ratio compares two fractions with an equal sign between. So you have a fraction equal to a fraction. It is used in measurements. Kinda of like when you are trying to convert two different measurements like miles to feet. Something like that, you use ratios. Or a lot of the examples that we use in class are like a person stands, you know. A person and a shadow. A building and a shadow. You can set a ratio to determine the unknown variable. They are basically equivalent fractions and using it involves, it you are solving for a numerators or denominators, you solve for the variable by cross-multiplying and dividing.
I: What images come to mind when you think ratio?

Liam: The example that we do in class with the shadow and the tree. When you are trying to see the ratio of, trying to find the height or the length. The height of the person or the tree and the length of the shadow. I don't know why, but that is always what comes to mind. We go over it a lot in class.
I: How would you define rate?
Liam: Rate is a constant at which. You know. Distance equals rate times time. It is how fast something is going or rate of change is the same thing as slope. Delta y over delta x. It is how quickly something is moving. Rate of change. You know, change in y over change in x. Like miles per hour. So miles over hour.
I: So you said rate of change, so would you say rate and rate of change are the same or different.
Liam: The same.
I: The same?
Liam: Yes, the same.
I: So what image do you think of when you hear rate or rate of change?
Liam: How fast someone is going. All of the questions about rate of change are change in y over change in x. You go fifty miles, it takes an hour. Miles per hour. How fast are they going? Rate of change. Change in y over change in x. So it would be change in y, the distance, over change in x, the time. So it would be fifty miles per one hour. So fifty miles per hour.
I: What is your comfortable level with ratio, rate and rate of change?
Liam: That is all we do. That is what we do all year. Ratio is more seventh grade but that they bring it back in eighth. A lot of them don't quite, they know how to solve it, some of the do. The hardest part of any word problem is trying to figure out what they are asking. We talked all about slope. So it is still fresh. Then we do these kinds of of graphs, y-intercepts, trying to intercept. Seeing what is saying.
I: I appreciate it. Thank you so much.

Luke

Interview began with small talk about his teaching and teaching slope. The following is paraphrasing the answers to my questions.

I: How long as you taught?
Luke: This is going to be my tenth year. Four in eighth, three in seventh and last three in sixth.
I: All at the same school?
Luke: Yes. (Talked about his previous job)
Why did you decide to teach math?
Luke: I was an accountant and started my own business. I was working way too much and never saw my children. I would leave and they were sleeping and come home and they were sleeping. I took six years off and a Principal friend called and asked me to sub. I did and loved it. I wanted to teach math but at the middle school level. I did not think that I wanted to go back and learn the high school curriculum again. Plus, I enjoy that age. Plus, no one will do that age.
I: What are you licensed to teach?
Luke: 6 - 9
I: Have you earned your National Boards?
Luke: No, never tried
I: Have you earned your Masters?
Luke: No, I was lateral entry and never.
I: What is the highest math class you took?
Luke: I took calculus in high school. In 1979.

Begin actual transcript.

I: What comes to mind when I say slope?
Luke: Steepness of a line. That is what I think of. This way, that way (Moving his arm to show positive, negative, zero slopes). Steepness of a line. That is what I try to tell the kids. Slope. Steepness of a line. Then we have to figure out how to solve it. We have to know that slope means it is going to go up or down.
I: What images come to mind when you hear the word slope?
Luke: A straight line.
I: A straight line?
Luke: A straight line. Because, with your rate of change… We don't really get a whole lot into rate of change. If it is a straight line, it is constant (Moves hand to demonstrate a straight line). If it is not a straight line, it is not a constant. You have to know those different variations of it.
I: So, I have a couple of tasks.
Luke: Oh great. I will probably fail (Laughs)
I: Absolutely not. Most of them do not have a right or wrong answer. I am mainly interested in the thought process behind what you are doing. Look at those tables. Where any produced from a linear function?
Luke: It is cubed. The first one is cubed.
I: The first one is cubed?

Luke: Yes.
I: Okay.
Luke: Time two. (speaking about B) Times negative three (Speaking about C) 18:08. I think A and B are. I think they all might be.
I: What is leading you to believe that they all might be linear functions?
Luke: I was trying to graph them in my head.
I: You were trying to graph them in your head.
Luke: Yes (Laughing). For this one, for everyone one that I went here, I went up three here. But this one, because it was cubed. No, cubed it wont be. Doesn't that make a parabola? No, so the first one would not be. See I am trying to reeducate myself here. Now, for this one (B) it is cause for every one up, I go up two. So for every one up, I go.
I: So A is not and B is?
Luke: B is. Right. (Muttering Down one, down three) And C is because the rate of change on the f(x) is equal to each one. So is that right?
I: Yes. Absolutely. A is a quadratic and B and C are both linear functions. Good job.
Luke: Took me. I had to think. Squares are not linear functions. Anytime you have a square, it is not. It is going like (Drawing a U in the air) like a parabola.
I: That is right. Very good.
Luke: See I am an old man but it is starting to come back.
I: You still got it.
Luke: Laughs
I: So, last year was my last year in the classroom and I gave this problem to two students. They both presented valid arguments. One believed that the lines were the same and the other believed that the lines were different. What were each student thinking?
Luke: They are straight lines (Laughs)
I: So if you were the teacher, how would you respond?
Luke: The coordinates would be different. They are, it would be a linear equation if you would go to solve it. But if you were looking at the, if you were doing it from a geometric standpoint where you are looking at the angles, the angles of each would be different. But…
I: What angles?
Luke: Here (Pointing at angle between the line and x axis) This angle here with x and y. Those measurements would be different. But if you were looking at the line, they are, it is a straight line. So, I would say the lines are going to be the same if it is going to be a straight line. If you are looking at something, the coordinates would be different. But they are both straight lines
I: What do you mean the coordinates would be different?
Luke: Like if I say this is over one (Drawing on the left graph) and up 2, over two and up 4. This one would be (Right graph) like over one and maybe (up) a quarter.
I: So, overall. Same different? Final conclusion.
Luke: I would say they are the same. They are straight lines. They may be different in looks but they are both straight lines. They both are going to have a constant rate of change.
I: Would the rate of change be the same?
Luke: No, they would be different but the would be straight lines.
I: Okay, so they are both straight lines but the rate of change is different?

Luke: Yes.
I: Okay. Here is three geometric shapes. (I explain problem). So you matched up the cylinder to the linear function, why did you do that?
Luke: Because it is the same measurement all the way down. If I put an ounce in, it is going to be equal all the way up the sides. As I continue to pour it, it is going to go at a constant rate up both sides of the object.
I: How about the other two?
Luke: The other one (Thinks) I put that there and that there.
I: Why?
Luke: Because, if you said this top was going to be cut off, it would be slower filling up the bottom, and then as it went up, the rate of increasing of the volume would go up. (Talking about the cylinder and the quadratic). This one, because it is smaller on the bottom, it would be at a quicker rate then when you hit the wider part where it would slow down, then go quicker as it narrow back up.
I: Awesome. Nice job
Luke: See, I am not as dumb as I look. (Laughs)
I: So you said something earlier that I thought of as well. So I gave this line to a teacher last year and she said that the slope was a thirty degree angle. How would you respond?
Luke: Well, it is supposed to be, it is not really the angle, you can get it but it is supposed to be a, like we said, a rise over run. What is the change in the y variables or the y coordinates versus the x coordinates. It would be more of a decimal or a fraction, not as a percent. It what, I mean. I am trying to reach back in my memory banks here. I would be more apt to say that if it is a ninety degree angle something like this is, but for the slope of a line, I would not say to do it as a percentage. I would say, you know, well I guess you could turn fractions into a percentage and so forth and so on. But when we write it in slope intercept form, we want it in fraction or decimal form. Not as a percentage.
I: Okay, so if a student said the slope is a thirty degree angle, would you accept that?
Luke: I would steer them to convert it into either a fraction or a decimal. 24:56
I: Does it make sense to be a thirty degree angle for an answer for slope?
Luke: I am going to say NO not really. You probably got me, but I don't know.
I: Like I said, there are no right or wrong answers.
Luke: I have never heard slope explained that way. Of course when i went to school it was you learned it this way (Spoke for a bit on this off topic)
I: For this one there are four graphs, could any have a slope of positive two?
Luke: B
I: Why did you gravitate towards B? End of video one for Luke/ 30:00 in downward video
Luke: Because it is increases
I: Why did you dismiss A?
Luke: It is a negative slope.
I: What about C?
Luke: C is a zero slope. It is a flat line
I: What about D?
Luke: D is a parabola. So if you put like a negative two in there, you are going to get the same answer. All of the answers are going to be positive.
I: So with respect to having a slope of positive two, only B?

Luke: Yeah.
I: Cool. I was trying to develop tasks and I was playing on the internet and found this sign. What do you think six percent represents?
Luke: Reminds me of traveling through the mountains of West Virginia.(Laughs)
I: Exactly.
Luke: Six percent grade, or six percent I guess. Yeah, a six percent grade going down. That is what it says on the sign. Yeah, a six percent grade. I guess you proved me wrong on my thirty percent. I guess it can be a slope.
I: You said that you would tell your students to go more to a fraction. So what do you think that six percent implies?
Luke: That, instead of being a flat, that you are going downhill at a six percent decline. I guess decline, incline would be up. Yeah, I don't know. I worked for the state highway department for two years but I worked in a valley.
I: What about 100 percent? What do you think that would be?
Luke: 100 percent would be (Holds arm straight up and down) 31:57 100 percent would be 1 to 1 (Total light bulb moment). So for every one, maybe what you are saying is that you are dropping six inches for every 100 inches you go out. Does that make sense?
I: Yes.
Luke: It is taking me a while.
I: I saw that you initially went to do a vertical line for 100 percent.
Luke: But it is not. A 1 would be over one and up one. So for every inch of fall you would have an inch of horizontal.
I: So could you go over a hundred percent?
Luke: Well, I wouldn’t advise it in a car. (Laughs)
I: But mathematically?
Luke: Yes, I could go over one and up two. So that would be two hundred percent.
I: Very good.
Luke: I am not as dumb as I look.
I: To be honest, you did what everyone has done so far. You went to say 100 percent was vertical, but you fixed it. A hundred percent is a 45 degree angle. Every person I asked this question to in designing the study said 100 was straight up and down (we continue to talk about it)
Luke: Makes a joke about deserving a raise
I: If I gave you, $y = ax+c$, what happens as A gets bigger?
Luke: As A gets bigger, the line gets steeper
I: What happens as A gets smaller?
Luke: The line get flatter.
I: What about if C gets bigger?
Luke: The y variable change. (Writes down $y = ax+c$). As this (c) increases, my y coordinate is going to get larger. As c increases, it is just going to change my y coordinate. My slope is not going to change. The slope of the line is not going to change. (Thinks). It is not going to change, should it?
I: The line is not going to change or the slope? What do you mean it?
Luke: The slope should not change just the coordinates are going to be different.
I: Correct. The slope should not change when you change C. That is correct.
Luke: Just took me a while.

I: No, you are good. (Laughs).
Luke. Kids at (his school) has a chance. Is that what you are saying.
I: They do. (Laughs) They have a really good chance based on how you are explaining this. So on this one, you have five staircases, please rank them in order based on the easiest to hardest to climb.
Luke: Because the slope is less on. The flatter the slope, the easiest it is. Just like if you are going to walk up a hill, if you are going to walk up a mountain, that is a little bit harder. You know, like if it was more gradual, even an old guy like me can walk up a small hill
I: SO how are you figuring out the slope?
Luke: Well, I am looking at these points (ends of stairs) here. Where I am connecting the peaks of each step. So this one is the flattest one (B) so therefore that would be the least going up the hill. The least amount overall over the longest turn, it might be longer, but I could probably handle it since I walk every night, I could probably handle that. Now, this one where you see that it is going up (C), up, it is a little further up and the increase in the last one are significantly higher. Now this one, I may switch these two around (A and E) because it is steeper. No six dozen to one half a dozen to the other. They look about the same.
I: Okay, so B, D
Luke: This one would probably be easiest to climb because you would have more time to catch your breath before you had to go up the next level. (Laughs) (Switches back A and E)
I: So you have B, D, A, E and C?
Luke: Yes.
I: Cool and you basically looked at it by?
Luke: Just the angle. I connected the top parts of each step and that would give me the steepness of what the hillside would be.
I: Thanks. Let me keep these in order. 2:48 on second
I: So we have another matching game. Here is the last five minutes of a race. Please match a graph to a scenario. (I explain it more detail)
Luke: A goes with 1. It is a constant rate of change. Straight line is a constant rate. (Reads the remaining questions out loud). E would be 4.
I: Why would you say that?
Luke: Because he fell. When he fell,he would not be going anywhere (Moves his hand horizontally) and this is a graph over time and once he gets up, her runs at a constant rate. So once again, that would be a straight line.
I: So where would you say he fell?
Luke: He falls here at the beginning and does not get up until here (Points at the cusp). All right. This would be D, he runs very fast.
I: Which one would be D?
Luke: 5. and when he reaches the finish line, he stops (points at the horizontal). (Reads last two). All right, this (C) would be runs slowly at first and increases his speed (traces graph) and this (B) would be runs fast at first and then gradually decreases.
I: Why? 4:40

Luke: Well, if I am going slow, I am not going very far but then as I increase my speed, then I am just saying if this is going to be time (x-axis) versus distance (y-axis) my distance increases as my time.
I: Perfect. Matched them all. On this one, please explain the scenario. (Hand him a graph and explain them each)
Luke: I saw you wake up, eat breakfast (points at first upper cusp). Goes down. Come back you eat lunch. Falls way down. Then as you go through, this is dinner (Last Upper Cusp) it falls down then maybe you get a snack before bed time (Laughs)
I: When is he the hungriest?
Luke: He is hungriest at the peaks. At dinner time.
I: Fullest?
Luke: Right after lunch.
I: Give him the POW one
Luke: I would say these kids are very much like my students (Laughs). When the due date becomes, the increase goes up. When you get very few that get it done early on. You say "Hey, this is due is two weeks, you can turn it any time that you want", they all seem to come in on that Friday (Laughs) or that Thursday. Depending on what the kids schedules is, so you have a bunch of papers to grade over the weekend. Instead we would prefer them to come in so that we could grade a couple at a time.
I: Very true. Is that a constant rate of change?
Luke: No, that is a variable rate of change.
I: Give him vending machine problem and explain
Luke: Number of drinks versus time of day. It looks like as the day goes on, people need their caffeine fix (Laughs). It looks like you wake up in the morning and no one is having a soda. Having a wife that is dependent on Diet Coke, this looks like it might be around that 10:00 (pointing when negative regression starts) or that time when she needs that jolt of caffeine. And then as it goes down, as you go through the day, you are selling them as you go out.
I: What happens towards the end?
Luke: It stops. It looks like everyone went home from work. They then refill it.
I: Good. Last one. Money in a register. How does that graph?
Luke: It looks like they charge the same rate whether you are a student or an adult.
I: How do you know that?
Luke: Cause it is a straight line. Constant rate of change, so if it looks like it is one student say like five bucks, then two students would be ten. Three would be fifteen. Constant rate of change. (Hands it back)
I: Why doesn't it start at zero, zero then?
Luke: (Takes it back from table). Because your entry fee is something, it is not zero. So if I have a customer, I have to start somewhere on the money trial. I am not going to let someone in for zero dollars. If I put the first person in for zero, then the second and third will want in for zero as well. Being a business guy, I would know that (Laughs and hands back paper)
I: One final task, I explain faucet problem. 9:09
Luke: Looks to me like Takashi either saw something different or it was leaking faster because the time is less but the volume is greater.
I: SO if you were grading that, would you think that they were the same graph?

Luke: Depends what you want me to grade. If I observed it, I would have my key (Laughs). Neither one is wrong until you, time and volume, you would think that they would be same. I don't know if the faucet is leaking at a different rate. But you said they were observing the same?
I: Yes. Exact same.
Luke: I wonder if Denise got the variable changed. They got time and volume switched. I don't know. Maybe. I am lost for it. I have not had this many questions given to me in a while. So why would their answers both be correct? Knowing how my kids graph in sixth grade, maybe Denise does not know how to graph or Takashi does not know how to graph because so many times they put 2,3 instead of 3,2. That just might be the solution. I don't know.
I: A couple of follow up questions. Do you teach ratio?
Luke: Yes.
I: What is a ratio? 11:47
Luke: I always go from the definition. Numbers whose relationship is with division. Ratio, numerator to denominator. I teach them the ratio, it looks like we have a number on the top and a number on the bottom, it is division.
I: What images come to mind when you hear ratio? 12:20
Luke: Well, it is like one to two, one over two. Those are the gimmie points on my quiz. Can you write a ratio in three different forms? Ratio is when one thing happens, you know, and once again, this would be a constant. You could get a straight line out of that. That could be your slope.
I: When would someone use a ratio?
Luke: One part sugar to two parts water. There are four cats and six dogs. (Elaborates)
I: What about rate?
Luke: I teach both rate and unit rates. For unit rates, I always tell them the bottom number must be one. Common uses of unit rates: miles per gallon, miles per hour, you know, those type of things. Cost per pound. I have a thing when I do unit rates, I always get the grocery ads and say okay lets figure out the unit rate for this object. (Talks about a student not knowing ounces to pound). For rate, I say that it is anything, say you go 120 miles in two hours. That is a rate, it is not a unit rate. It is a rate. Usually they have two different measurements. One is length and one is time. It does not have to be, anything with two different types of measurements.
I: In that case then, they have to be different?
Luke: Yes.
I: What about rate of change?
Luke: Rate of change? Slope. Slope, that rate of change. You can have a constant rate of change or you can have a variable rate of change. That is one of the things that we always, you know, and in middle school we don’t get into that much in depth. But in general, I always them that if it is a straight line, it is a constant. If it is a wigglely line, it is a variable rate of change.
I: What do you mean by a variable rate of change?
Luke: It does not change at the same rate. Just like when the kid in your example, starts off slow and gets fast. Okay, it varies what happens. This is happening constant, the whole way. Where this is over one and up two and the next point is over one and up two. 1,2. 1,2. Where this one, this could be say over one and up five then then over one and up

three. It would change. I know you have taught high school and would get more in depth with that but middle school is pretty cut and dry. Straight constant, variable swigglely lines.
I: What images come to mind when you hear rate?
Luke: Being an accountant, it is interest rate. (Talks about being an accountant) Also mentions percentage but says speed of a car.

Rachel

I: So how long have you taught math?
Rachel: 18 years
I: At the middle school level?
Rachel: No. I did 4 years at the high school, and then I did a year of middle school (grades 6-9) at Bridgeport, but that was a K-9 building. I was the only math teacher there.
I: So you taught all the kids then?
Rachel: I did not. I taught all the kids 6-9 but K-5 was one teacher for all subjects. So there would be however many, let's say, first grade teachers and they would have their students all day for all subjects. Then the kids would move to sixth grade, upstairs and that was like the middle school level and then 6-9 there I was the math teacher.
I: Oh, that's interesting. So what's your certification in?
Rachel: I'm actually elementary certified and then I had a middle school math extension put on when I was in just regular college and then I went for my Masters in secondary.
I: Ok, so you do have your masters?
Rachel: yes.
I: Do you have your National Boards?
Rachel: I do.
I: So you're as high as you can be basically with respect to...that's awesome. When did you get your National Boards?
Rachel: I just renewed so I'm good until 2023. I just did the Governor's Teachers Network for the state. And then I just found out last week that I got the Presidential Nomination. It's almost like National Boards where you have to send in all these portfolio entries and videos. So I just got selected as one of three to represent North Carolina. So if I win I get to do a week in Washington and meet the President.
I: So what's the highest math class you've ever taken?
Rachel: Oh gosh...probably Calc II but it's been a long long time. And I'm not really math brained. I know that sounds weird, but I'm really not. But my dad was always one of those work hard type of people. And so that was my weak area in high school and I

remember him saying that was were I needed to focus all of your time. So I went to college for social work and I actually ended up tutoring kids. And I was tutoring math and I was like, oh my gosh, I can actually think the way they think. Because I don't really know what they're doing either, you know, like, I don't really get it the easier way. There's people at our school that, they just get it; it's just innate. My brain just doesn't work that way. I have to think about it and process it.
I: So why did you pick middle school?
Rachel: Because when I did high school, by the time they got to me in 9th grade I felt that they already had a pathway. And if they were going to drop out (we were by the Marine Corp base) - so there were a lot of single family homes cause the men were away. So by the time they got to me I felt like they already had their path of whether they would drop out, or if they were just spending their time there until they turned 16, if they were going to a 4-year, if they were going to a 2-year. So for me I wanted to go lower, to hopefully give them the confidence that they could actually make it. So that was my whole idea, that by high school they knew what they were doing and by middle school I thought that I could be more successful with kids that were unsuccessful.
I: So when you hear the word slope, what comes to your mind; what do you think of?
Rachel: Like the steepness of a line. That's like a math term for me; I mean, we do it all the time. If I was talking to kids I would reference that - I would talk about ski slopes or skateboarding or anything like that, but slope to me reminds me of the steepness of a line or a path of some sort.
I: (explains question #1 about tables/linear functions.
Rachel: The first one, no. (I: How do you know?) Because the change in x is the same, but the change in y is not. The second one is yes. (I: Ok, why is it yes?) Because the x and y are consistent; they're changing the same amount. They're references. And the last one is. Because of the same reason. The x values are increasing by the same amount and the y values are decreasing by the same amount.
I: And how did you figure that out?
Rachel: I just subtracted. Did I subtract wrong?
I: No, that's exactly correct. (Now explains question #2 about two graphs - are they same?)
Rachel: Well, I'm going to say that they could be the same graphs because they both have positive slopes and they both go through the origin. The student who said that they could be the same was probably because the x and y's have different increments between the two graphs, so here (gestures to graph on left) the x values might increase by one and the y values by one, and over here (gestures to graph on right) it might be that they are increasing at a lesser value. The student that disagreed and said that these were not the same graph, they were assuming that they had consistent axes on them.
I: (explains question #3 - water flowing in different shaped containers)
8:26
Rachel: Is this a sphere or is it not supposed to be a sphere? (points to the fishbowl)
I: It's more like a fishbowl.
Rachel: I don't know. I need the numbers. Maybe it's just me cause I can't look at the graph without plugging in h and r…(thinks a while and then aligns graphs underneath shapes. - see overhead camera view)
I: So why did you match them up the way that you did?

Rachel: I chose this one because as you fill in the water it's going to fill in consistently. The height is going to rise consistently because it's not like it (the shape) slopes in or out at any point. *pointing/referring to the cylinder shape* This one (points to the fishbowl) it would go kind of consistent as it filled up and then as it got to the top it would have to go faster because it's less wide. And then this one (points to cone shape), the same idea. As you fill up the bottom part it's going to take a little more time to get the height up but as you go towards the cone, as it tapers off at the top, it's going to rise quickly because it's not as fat, as wide.
I: So the linear one is…(Rachel: cylinder), and why did you say that?
Rachel: Because the width is consistent, or the height, the whole way through so it's going to go at a consistent pattern to fill it up.
I: (explains question #4 - slope = 30 degree angle)
10:51
Rachel: I don't know - how would you know it's exactly 30? Like, why is it not 40? I mean, it could be 30. So you're saying the slope of the line is 30? Is that from the x axis?
I: This angle, the angle in here (gestures to the angle marked 30 on the paper) is 30 degrees.
Rachel: I don't know if I agree with that. Like, why could it not be 31 degrees?
I: So if a student told you that it was 30 degrees, what would your response be?
Rachel: I would say "maybe". Like, for me, just looking at it, it looks like it's 45, because it looks like it's half of the 90, but that doesn't mean it has to be. It definitely has an angle, it's an acute angle, so that's correct. I don't know if it's exactly 30 - I think for me I would have to see proof. I would need to measure it. But I wouldn't disagree; I mean, it is what it is - I would have to get out a protractor.
I: But you wouldn't mark it wrong if the student had a protractor and they measured it as a question on a test and they said that the slope was 30 degrees?
Rachel: I don't think I agree with that being the slope. I think it could form a 30 degree angle. Slope is the measurement of the change in y over the change in x, so really that's going from one point to the next point, and I would be going up and over. Cause I think 30% would be like the incline...which I guess is like the steepness. Ummm...I would probably question it. I don't know if I would take off full credit, but I think I would say something like, "the slope measures the steepness so I would need a change in y over change in x." That type of value. Because degrees is what you measure a triangle in, not what you measure slope in. So I wouldn't agree with their answer, but I don't think I would necessarily mark it wrong. I would definitely take off points because it's definitely thinking outside the box, but I think slope should be measured as the steepness of a line based upon the point before...that's kind of a neat thing though for a student to think of. I mean 30% is the angle that it forms, if this is a triangle, 30 degrees, and so, you don't measure slope in degrees.
13:41
I: (explains question #5 - 4 graphs - can a slope of positive 2 exist?)
Rachel: Yes. (I: which one?) B.
I: Why B?
Rachel: Because it's increasing.
I: How do you know it's increasing?
Rachel: Because it's going from the left and rising to the right. It's going up - the slope.

I: How did you dismiss A as having a slope of a positive two?
Rachel: Because A is going downhill. It's headed towards negative.
I: Ok. What about C?
Rachel: C has zero slope.
I: What about D?
Rachel: D is a parabola, so the slope is different from one end to the other.
I: What do you mean the slope is different?
Rachel: Well, if you start on the left and go down it's a decreasing slope which then goes to increasing when you get to the y axis.
I: Ok. So could it have a slope of two.
Rachel: If the increments at a certain point on there it could. I mean, the line…it's not a line! I mean you asked if the line…what was your question again?
I: Does any of those graphs have a slope of a positive 2?
Rachel: At any time?
I: Sure.
Rachel: Like any little…
I: Sure.
Rachel: Maybe on D but definitely B.
I: Definitely on B, but maybe on D. What do you mean maybe?
Rachel: Like at a certain increment, like maybe from here to here (indicates area on graph from where x=0 to halfway up the right portion of the graph.) depending on these little…
I: Ok, so can the slope then change at different points?
Rachel: Oh yeah - depending on the function. Not on a linear function, but on a function it can.
I: So it's possible that D could have a slope of 2?
Rachel: At a certain little spot, yes. It doesn't look like it could, but I'm assuming it can depending on the increments as it's increasing on x and y.
I: So why did you say it doesn't look like it would?
Rachel: Because it looks like it's going up rather quickly. But it could be zoomed all the way in.
I: So you said that it started out as a negative slope for D?
Rachel: Well, it's going downward first, so that's a negative slope and then it increases. A parabola is going to be a U shape anyways.
I: So, final answer then…which of the following could have a slope of a positive 2?
Rachel: B and D maybe.
I: B definitely, D maybe?
Rachel: yes.
I: (explains question #6 - truck with hill grading)
Rachel: The decline. It's like 6 over 100…we did these at one point. It's it like 6 over 100, the percent that the…uh! I did this…Hmmm, I can't remember. I want to say it's like for every 6 feet you go down, you would go over 100 feet. I got asked that a couple of years back. So we looked it up and I think it was a percent, so it goes down 6 over 100, so you go down 6 and over 100, which is neat.
I: So what would 100% look like?
Rachel: A cliff.
I: What do you mean a cliff?

Rachel: *laughter* Straight down. Well, no, if you go…I don't know.
I: Well, I will confirm that your reasoning of 6 over 100 is correct and then you would multiply by 100 to get the percentage. So 6% would be down 6 over 100 and then multiply by 100 to get 6%. So, what about 100%
Rachel: One.
I: What do you mean one?
Rachel: What did you just say? You put 6 over 100. So down 6 and over 100. And you're asking me what 100% would look like? Ok. Down 100 over 100 which is one, times 100 which is 100.
I: Ok. So it'd be a cliff?
Rachel: No.
I: Ok, so what…
Rachel: I don't know.
I: So, if you went down 100 and over 100, what would that look like if I drew it?
Rachel: A triangle. Um…(working out in head difference between down 6 and over 100 versus down 100 and over 100) I don't know. I mean if you go down 100 and over 100 it would look like a right triangle with equilateral angles on the side over here (indicates the two angles opposite the right angle.) - a 45 and a 45.
I: So 100% a cliff or not a cliff?
Rachel: Not a cliff.
I: So what would the angle then be?
Rachel: 45
I: So could you go over 100%?
Rachel: no, well, you could. You could.
I: So I could have a sign that says 200%?
Rachel: So 200% would be you're going down 200 and over 100…yeah, you could.
I: So what happens to the steepness of the line (that's what you said is slope) as the percentage increases?
Rachel: The steepness of a line as it goes…so as it increases, the steepness increases.
I: Good, and is there a limit to what the percentage could be?
Rachel: No.
22:05 - **Timing from here to end is recorded by the overhead camera)
I: (explains question #7 - staircases)
Rachel: Which would be the easiest? B.
I: Why do you say B?
Rachel: Because the stairs are not that steep. The small steps are definitely easier to climb.
I: So what would be the second easiest to climb and then go all the way down to what in your mind would be the hardest to climb.
Rachel: (spends some time arranging the staircases in order: B - D - E - A - C.)
I: Why did you rank them in the order that you did?
Rachel: D because the steps don't look as wide, or high…D steps are pretty consistent, but the reason why I picked D above E is because D's were not that high and then they get a little bit higher each time. In comparison the highest on D is almost close to the highest on E. And then A I put underneath E because this one (indicates staircase A) starts a little bit smaller and looks a little bit harder and then this one (points to C) has

less steps but they're definitely bigger steps, or higher steps so I think they would be harder to get up.
24:20
I: (explains question #8 - equation y=ax + c - what happens to the line as a gets bigger?)
Rachel: It gets more steep.
I: Ok. So what happens as a gets smaller then?
Rachel: It gets less steep unless you're taking about going into negatives.
I: What would matter if I went into negatives?
Rachel: It goes downward. So it goes less steep and once you run into the x axis, it'll start going into a negative which means the steepness is really not…I mean if you get really steep, you'll just be going downhill, just the other way.
I: (explains question #9 - five graphs with 5 descriptors about last 5 minutes of a race)
Rachel: (talks her way through the scenarios) So, gradually increases his speed - that's probably B. Goes fast and then slowly decreases his speed. That's probably C. Runs very fast and reaches the finish line early - that would be D. After falling…A. And then E.
I: So how did you match them up?
Rachel: 1 goes with A, 2 goes with C, 3 goes with B, 4 with E and 5 goes with D.
I: How did you match them up?
Rachel: Because this one is a constant (shows graph 1) so that was A. This one (shows graph 2) goes up quickly and then slows down so that matches up with C because it says that he runs very fast and then decreases his speed. B says that he runs slow at first so he's not going very fast just yet (traces along graph as she reads the scenario and then goes fast. Four is after falling, so right here he's not increasing at all (indicates the flat portion of graph #4); he fell and then he goes up really quick. And then number 5 goes with D because he runs very fast and then he goes to the finish line and he doesn't have to increase.
I: So number 4 - where does he fall?
Rachel: At the very beginning.
I: Ok, so when does he get up?
Rachel: Right here (points to the vertex of the angle)
I: Ok, good. And on number 5 when does he finish the race?
Rachel: Right here (points to the vertex of the angle).
I: Good, perfect. Good job.
28:38
I: (explains question #10 - Time of day vs. Hunger level graph)
Rachel: Ok, so at whatever time this is (points to the very beginning of the graph where x = 0) they're pretty hungry, not extremely hungry, but pretty hungry. Ok, can I make up a story?
I: Sure.
Rachel: So they get in the car and they drive to McDonalds (traces graph upwards from x=0 to the first peak) and they eat a Frosty. Ah, that's Wendy's. And that kinda decreases their hunger quite a bit (traces graph down to the first valley). And then they go to play some basketball and as they're playing basketball they realize that they're starting to get hungrier and hungrier because they're doing all of that exercise (traces graph up to second peak). So now they go home and raid mom's fridge and they make a peanut butter and jelly sandwich and they're not hungry anymore (traces graph down to second valley).

So then they lie around on the couch (traces up a small portion of the graph) and then decide to go on a bike ride, so of course they get hungry more quick again (traces graph up to the third peak) and then they go to their friend's house and have some pizza but they're kinda hanging out and talking at the same time. (Traces the beginning downward portion of the graph) So it takes them a little while to eat their pizza. And then they're done eating their pizza (traces all the way down to the third valley in graph). And then they just hang out outside with their friends and swim for a little bit. (traces up to the endpoint of the graph).
I: That's awesome - see - bringing writing into the curriculum! When are they the most hungry?
Rachel: Right here (shows the top of the third peak on the graph).
I: When are they the least hungry?
Rachel: Here. (points to the second valley on the graph).
30:17
I: Awesome. (explains question #11 - POW homework assignment)
Rachel: So at the beginning not many kids turned it in at the very beginning. And then I'm assuming this must be the due date (indicates the last point drawn on the graph - far right) because it then increases pretty well.
I: Does it increase at a constant rate?
Rachel: No.
I: Ok, how does it increase then?
Rachel: It's a positive correlation but it's definitely not a constant rate. It's almost like…it kinda looks like an exponential, but a half of a parabola is what it reminds me of.
I: So, more kids turned it in at the beginning or more kids turned it in at the end?
Rachel: Oh kids definitely turned it in at the end.
31:21
I: (explains question #12 - drinks in a vending machine)
Rachel: Well, if it's like at the high school, machines aren't on until the school day ends (indicates the solid flat line at beginning) so the drinks in the machine stay the same until 2:15. And then when 2:15 hits the line decreases and then number of soft drinks, as they take drinks out, decreases.
I: Ok, and what's happening at the very end?
Rachel: It doesn't go empty, so maybe at 5:00 or whenever the place gets locked up. Then when the place gets locked up it doesn't decrease anymore it just stays there for the next day.
I: Ok, so what about that little…(and points to the solid flat line at the end of the graph).
Rachel: I'm assuming that the person comes from the soda place and refills it.
32:10
I: (explains question #13 - movie theater cash register)
Rachel: Ok, so it looks like they had some money in the cash register (indicates the portion of the graph where x=0). And maybe there's a line or something until they open because it looks like it increases at a consistent rate. So at the same amount of time they let in the same amount of customers.
I: Ok. Did they charge the customers the same? Did they charge the customers a different rate?
Rachel: No. They charged them the same - it's a constant rate.

32:54
I: (explains question #14 - leaky faucet)
Rachel: Well, it depends on what the axis is. So this one (graph on the left) could be from 1-60 minutes. And this one (graph on the right) would be from 1-30 so it's going up faster. This one (graph on left) she's going at a slower rate, so if she's doing it over 60 minutes then maybe she's (on right) doing it over less time or more time. And so for the volume to increase…this would have to be from 0-2 hours (on right), it would increase faster - or it would look faster. You can skew graphs to look the way you want. So if they're not going up by the same increments then they could be the same graph. It's kind of like the one you showed me beforehand. You would have to look at the increments that they're going up by. It could be the same graph though.
I: So the students turn those graphs in modeling the same thing. Full credit for both students?
Rachel: For my classroom, no because I make them label their axes with increments so that they can't do stuff like that. So for me, I would say "redo" on both. I wouldn't accept that because it doesn't tell you data wise. I mean, yes, I can see that it obviously has a leaky faucet, but who knows if that's happening over days or months. If it's over years then I obviously don't need to replace the faucet, but if it's over a day than I may need to replace it.

33:20 (time recorded here to the end based on front-facing camera)
I: Do you teach rate and ratio - either of those?
Rachel: Yes.
I: So, what is a ratio?
Rachel: A ratio is more of proportions and stuff and that's what 7th really does. We touch on ratios as a comparison of the two. With rate we do more of unit rates and stuff like that where we change it to decimals. We don't do as much of that. We do more of the slope, and then the linear function, and scatter plots and being able to tell the line of best fit. So ratio and rates should be more so in 7th grade at this point in common core. We used to do it all the time before common core.
I: So how would you define a ratio if a student came into class and couldn't remember that day - how would you define it?
Rachel: A comparison of two…well, anything…two things or objects - like two sets of data.
I: Do the two things have to be the same? Can they be different?
Rachel: No. I mean your ratio could be a comparison between apples and oranges.
I: Can it be apples to apples?
Rachel: Sure. Some are green, some are red.
I: Ok, but does there have to be some sort of difference between them?
Rachel: Not necessarily.
I: So when you hear the word ratio, what images come to your mind?
Rachel: I always just think…like it's a comparison of two things so like a comparison of the number of soft drinks to sandwiches that you have. And then we would use ratios to set up to figure out equivalent fractions. To link fractions ideas so they see it as a ratio. I mean, that's really how it should be introduced.
I: So how would you define rate then?

Rachel: Rate is almost like a speed. I guess there's also unit rate. So, something increasing or decreasing. Rate is like a simplified ratio as a decimal. Like miles per hour - like you're comparing the two and then simplify it to be a whole number or a decimal. I don't know if that's truly correct.
I: So, what images come to mind when you hear rate?
Rachel: Like speed or like miles per hour or unit rate. We did a bunch of which is the best buy and went through a bunch of circulars for stores. So for me it's always like money or speed or something competitive in nature.
I: What about rate of change? How would you define rate of change?
Rachel: Slope.
I: What do you mean slope?
Rachel: Like how one axis changes with another. How the increments…like the rate of change is like the steepness of something, I guess you could call it - like the change in y over the change in x. That's pretty much what we teach all the time.
I: How do you teach slope? Do you usually use change in y over change in x?
Rachel: We usually do y2 - y1. The very first day, no. The very first day we show them a skiing scene on you tube and then they can see that that's the slope. Then they can think about what else is there a slope on besides skiing? Because kids have to see it first. Then I asked "how can you measure that?" So I did it with Geoboards and they have to do rubberbands. And then I ask "how do you get from one to the other?" And rather than talking to them about how this is the change in y over the change in x, I try to have them figure out the formula on their own. So then I'm like, "what if we put it on a graph, then what happens?" Typically they can get it, except for inclusion…those kids really think outside of the box just not always correctly. So we use the geoboards to count the positive and negative and then discuss slope and then from there after they have the physical sense of what slope is, then instead of doing rubber bands over and over again we start to say "what's the slope of these two lines?" and then they do it without their rubber bands and so that's where we get the y2 - y1. Then we show them the little triangle, the delta showing them the delta sign means change. So we try to make a smooth transition into high school. For me, I want the kids to know and understand slope and understand the whole process.
I: So, the term rise over run, you use that?
Rachel: Yes. All the time.
I: If I asked a kid to define slope, what do you think they would say?
Rachel: They would either say, rise over run or y2-y1.
I: So more of the algebraic, or formula version?
Rachel: Yea, and then we do the whole positive = hold your hand up, negative = hold your hand down stuff. But they did a whole business project thing linking in with slope, so I'm thinking they would know it's the steepness of the line. I don't know if they would tell you it's the steepness of a line, because that's what I keep telling them. Or they may just say rate of change because I always talk about slope = rate of change.

Sarah

Begin with start talk about our kids

I: How long have you been a math teacher?
Sarah: 16 years. I have taught history and science as well but for the last nine I have taught only math
I: Why math?
Sarah: I am not a reader and I just love math. It is very factual. (Elaborates a bit about the black and white and math)
I: Why sixth grade?
Sarah: Everyone asks that. I don't know. I guess I am insane. Well, my first job was seventh grade and I fell in love with that age. My friends suggested that I teach elementary but I did not want the little ones. I worked summer camps, always worked summer camps and I was coaching softball. So I have worked with all ages. I would never want to teach the younger ones. I just knew this is where I wanted to be. I have taught sixth grade for so many years, I could not image being anywhere else.
I: What math classes have you taught?
Sarah: Honors Pre Algebra and sixth grade math.
I: What topics are taught in sixth grade?
Sarah: Fractions, Ratios, really not fractions but we have to say fractions just to get us prepared for ratios. Like ratios, percents, algebra. That is the topic that really crushes us. It is very abstract for a sixth grader. (Discusses a student). Algebra is biggie, ratios, decimals, and percents.
I: Have you pursued or earned your masters?
Sarah: No
I: How about national boards?
Sarah: No. Prior to having our child, I was planning to. But you know with having kids, it gets a little busy and I put it on hold (Discusses NC and lack of pay raise and how busy she is) 2:00
I: What is the highest math class that you have taken?
Sarah: I guess it would have been Algebra II/Trig.
I: The term, slope, what do you think when you hear it?
Sarah: A coordinate plane, an equation, $y = mx+b$.
I: What mental images comes to mind?
Sarah: Lines. Lines on a coordinate plane. Positive. Negative.
I: What do you mean positive or negative?
Sarah: If it is going up it is positive, if it is going down it is negative. Reading it left to right.
I: So you see a line, positive or negative slopes?
Sarah: Yes.

I: (Explain upcoming task, purpose and first task)
Sarah: SO I get to work it out (Grabs a pencil)
I: Absolutely. Whatever you need to solve it.
Sarah: (Draws coordinate plane and starts to plot the points from A) (Talks about her poor handwriting) A is a function
I: Is it a linear function?
Sarah: Yes. B. Linear. I am am basing this off a function, what I know about a function. If a function then it should be linear. The way that we teach ratio, when teach ratio tables. So instead of teaching this with a function, cause they would not know what f(x) was. (Talks about it being a topic addressed at the end of the year). What most of them look at is the x and just look at that (f(x)) like it is a y, input output, Do you seen a pattern? Patterns are going to make functions and functions are going to make lines. So that is the way that we would look at it with sixth graders. 9:21
I: So what pattern do you see in A?
Sarah: Well, for the numbers, well… I try to get my kids, especially sixth grade, a lot of times they will identify addition, but no no no that is not what we are looking for. So for the first one, we are going to go that we square x, x squared and for B, they would say times two. Then I would say that very few are going to get C because of the negatives. We do not have to teach, I do not know why because it does come up, we teach integers as far as coordinate graphs. We do not teach adding, subtracting, multiplying and dividing. Unfortunately, in a lot of the work that they come across they do, so I give them mini lessons on integers, and the majority of them would not see the relationship that you are actually multiplying by three. No too many of them would see that.
I: Okay, so A squaring. B multiplying by two. C negative three?
AF: C multiplying by Positive 3. Wait, no, it would be positive three, positive three, here it does not matter (0,0). Here you would have to multiply by a negative three. So I think C would throw them for a loop.
I: So, are all there linear functions?
Sarah: I don't think that C would be. (She attempts to plot the points)
I: What is is your goal in trying to graph?
Sarah: If it draws a line (Goes back to graphing) Yes, it is still linear (referencing her graph)
I: So all three are linear functions?
Sarah: To my knowledge, yes.
I: Very good.
Sarah: C would not be friendly to sixth graders.
I: Sixth graders would not like those negatives?
Sarah: No.
I: (I explain the next problem, two lines) How would you respond?
Sarah: Well, I am go based on, hold on, let me write this out. (Writes y = mx+b). I mean based on where the line is and the first line goes, I mean, it is not labeled, you could be using different intervals, so there is a lot that could go into this. I would agree with the student that said it is not the same
I: Why?
Sarah: If I plotted a point on the line, based on, let me think about this
I: Sure 12:34 - 12:49

Sarah: I would think, based on plotting a point and seeing where x is, it x was is a point and y is a point, or is it just showing you part of an equation. I would think based on where the points are, if I were not plot a point, they would be totally different places.
I: So are they the same of different?
Sarah: They are definitely different.
I: Could your provide a rationale for how they would be the same?
Sarah: Well, going back to, they do form a line. They do look positive. Um and the rationale could be depending on the intervals that they used. One could have used much greater intervals. This could be counting by ones (right) and this one could be showing counting by 20s or 50s (left) and it is not going to look at drastic (Uses arm to shoe steep slope). That would be my explanation.
I: What do you mean not as drastic?
Sarah: Well, if I plotted and this was one (Plotting on left graph) two, three and this lines up with 3 (Points to 3,3 on the left graph) and this one, I don't know, by tens, say ten, fifteen, twenty (Labeling right graph) three is going to be way down here (Plotting to a point on the line near the origin) and it will just not look the same. They used different intervals.
I: Very good. (I explain the water and shapes problem) 14:35
Sarah: I think that this one (the line) matching with the cylinder cause there is no change in the shape from the top to the bottom, so it is consistent. I wanna say that this loop (the quadratic function) would match up to the cone cause in the beginning it is not really getting very far (Tracing the quadratic) because it is wider at the bottom as it smaller at the top, the height, it is going to fill up quicker. And then I would probably put this one with the fishbowl because it starts off small and that (points to the bottom of the graph) is showing where it is going pretty quick and then the graph is kinda of leveling out, that is where it (the fishbowl) is getting wider and that is going to take more time and then it gets quick again and that (points to the graph) represents the top.
I: Great job. (I maintain the matches). (I explain the angle problem)
Sarah: Say that one more time
I: (I explain it again)
Sarah: I think that it is more user friendly. We do a little a little bit of geometry and we always save that to the end since it is easy and we never taught kids the formulas. We taught them to build. (Elaborates more on this) It is probably a good idea (Teaching with an angle). It has stuck with them. They have songs in their heads about what is a right angle, what is an obtuse angle, and what is an acute angle. So in the end this is very familiar and I think that a lot of them would think that that was responsible like the measurement on there so I think that it would be user friendly for kids.
I: What would happen if that angle got bigger?
Sarah: The line would tilt. It would get closer to y. It would become more vertical.
I: And happens if the angle decreased?
Sarah: Same thing, it is just going more horizontal.
I: Awesome. Very good. (4 graphs) 18:35
Sarah: I have to think about it.
I: Sure
Sarah: Well, A is negative. There is no change in C.
I: So what is the slope of C then?

Sarah: Well let me plot points to see. (Mutters to herself) I cannot remember how to use the formula. That is what I was trying to figure it out.
I: What formula?
Sarah: y=mx+b. I am just trying. I am missing something.
I: What are you trying to do?
Sarah: Just be able to plug in, my ordered pairs. To what, again, the pattern is. That would be the slope, the pattern. A is negative. C, there is no change, so there is no slope. So it is zero. B would be the only one, you said positive two?
I: Yes
Sarah: B would be the only one that could be positive two cause it is the only one that is going in a positive direction. And then D is a curve so it goes negative and positive, so it cannot be consistent.
I: So if it is not consistent, could it have
Sarah: Hold on let me play. (Muttering to herself) So it can only be B.
I: What did you mean in D when you said it could be positive or negative? Unfortunately 21:15
Sarah: Well, it does end up going up. So the function could be, could be positive but you do not have it labeled so I do not know what the number are.
I: Okay, and you said it was not consistent, what did you mean?
Sarah: Well just because it went down and then up so then there is not a constant pattern.
I: Okay. What about A lead you to believe that it was negative?
Sarah: Because you read graphs left to right, staring up top and goes down
I: Okay. And then one final question, if you just had this right here, how would you use that table to calculate the slope? (I explain that she needs two points to use y=mx+b) Without using y=mx+b, is there a way to use this table to calculate the slope?
Sarah: You would draw a grid and plot the points. That would be the starting, you would plot the points and then you would have to plug your numbers, you still have to understand how to use the formula and plug it in because this is the slope, the formula would give you the slope.
I: What part of this formula is the slope?
Sarah: Well, y is y. m is the slope. I just don't remember how to use it. It is funny that you asked cause the other day a friend asked what is b? I had to stay that I don't remember.
I: B is the y-intercept. (I explain this in more detail)
Sarah: I could not remember b.
I: (Explain traffic picture)
Sarah: I have never seen slope as a percent.
I: What do you think they mean by six percent?
Sarah: (Thinks for about 8 seconds) I have never seen the percent signs. Obliviously, it is cautioning to slow down. Then as far as the slope, 6%? I don't know if that is something like trucks totally understand that knowledge. I don't know what six percent means. I have no idea.
I: Okay, a different question. What would happen if 6 percent increased to 50%?
Sarah: To me, I would probably think that, I am still trying to literally get the angle.
I: What angle?
Sarah: How steep? Does that makes sense? Speed. Like how much of your speed should you increase or decrease. I have no idea. I am a blank on that one.

I: What about 100 percent, would that help you out?
Sarah: No (Shakaes head)
I: Have you heard of the formula rise over run?
Sarah: Yes.
I: Okay. So for that one, the rise is 6ft and the run is 100ft. That is how they got six percent. Then the multiplied it by 100. To make it as a percent. Does that help you at all?
Sarah: Yes, that does.
I: So what happens as it increases to 50?
Sarah: Okay, so know I am thinking proportions (Writes 6/100=3/50). So if you went from 6 to 50, it would get much steeper.
I: How did you draw that conclusion?
Sarah: Because if it is dropping one foot every two, what was it again?
I: So it went from 6 to 100, to 50 to 100.
Sarah: Well I am thinking now about percents, 50 is much larger. It is going to drop more rapidly.
I: So from 6 to 50, steeper?
Sarah: Yeah.
I: What if I went all the way to 100? What would that look like?
Sarah: Well that is not even possible, it would be her (Proves hand to indicate a vertical). Well, 100 over a 100. Now you are making me think backwards. (writes 100/) Wait, that is one to one. (Finished 100/100). But for every…(Pauses). So that would be significantly steeper, if that were even possible. I am just thinking that if it is dropping a foot every, whatever, however this is calculated, what did you say? 6% means 6 ft for every?
I: 100?
Sarah: Every 100. So if it is dropping 6 feet for every 100 feet, this one is dropping one foot for every one foot. Which is…steep.
I: So, 100 percent steeper?
Sarah: Yes.
I: Can you go over 100%?
Sarah: (Pauses) Yes.
I: Why do you say yes? Cause initially you said 100 was straight up and down.
Sarah: Because you can drop two feet every feet.
I: So you can go over 100?
Sarah: Yeah. Wow. I have still never seen (indicating the sign) End front camera video 1
I: (Praised every for going back to formula) (She talks about never seeing it again) (Give her the y=ax+c) What happens as A gets bigger?
Sarah: It is steeper, cause A is the slope.
I: What happens if A gets smaller?
Sara. It levels out (Using her arm to indicate a horizontal line)
I: What if C gets bigger?
Sarah: That is the one that I could not remember (Draws a coordinate plane and attempts to use points from a horizontal line) (Draws new table) (Writes 3 in x value) (Erases 3) (Writes new values) Ask again.
I: (Recapped what she said about A) What happens as C gets larger?
Sarah: (To herself) C is the y-intercept, it is where it crosses, so…it can't get larger. The y intercept is where it crosses the y axis. 1:57

I: Very true.
Sarah: So, if that is the case, you can't, well,
I: Lets say that I had 2x+3 and 2x+30 (writes them down), What happens in the second equation compared to the first.
Sarah: It depends on the x coordinates, cause I am still thinking of plotting points, it is where the line crosses. Obviously, 3 would be close and 30 would be way up here (Draws dots on a graph that she constructed earlier). Depending on the coordinates, I would need to know what y or x1 was to be able to determine. So if I had y or x (Stops talking)
I: So in those two equations, I had two x in both cases. By changing 3 to 30, did I impact the slope?
Sarah: No, because the slope is two.
I: So changing the c value changes the y - intercept but not the slope?
Sarah: No, because m is the slope
I: (I back off) Perfect. M is the slope. Good job. (Give her staircase problem)
Sarah: Easiest to climb? 4:37
I: Yes.
Sarah: (She is taken back that all staircases do not face the same direction) This is back to rise over run.
I: What do you mean rise over run?
Sarah: That is what I am visualizing after you mentioned that earlier. How far do climb, how long. Some of the steps appear to be longer than the others. 5:30 (Continues to move them around). All right.
I: Okay is this the easiest or the hardest?
Sarah: Easiest.
I: So you you ranked them as D, B, A, E and C. How did you determine?
Sarah: Well, you said climbing and I was thinking about being able to get up them. This has less of an incline (B). It is not as big but you have more of them. You only have three here (D) that is why I put him first. But for these three (last 3), I just went off how steep.
I: What do you mean by how steep?
Sarah: These appear to be much steeper (C) than these ones (E).
I: (I recapped her explanation to ensure that I understood). 6:23 (Explain Darren)
Sarah: (She reads all of the problems) All right.
I: Lets start with #1, how did you match it up? You said it was D.
Sarah: I read the constant rate, finishes at a constant rate. This shows the constant (Points to the positive portion of D).
I: So 5 has him finishing at a constant rate?
Sarah: (Nods) Yes
I: And B you matched with which one? 8:20
Sarah: 4. It says Darren runs slowly at first, so it showing real not much change and then gradually increases and this shows gradual enough and drastic. It is consistent (Moves arm to show a straight line). C, Darren runs fast and then gradually decreases his speed so this to me is drastic change and then all of a sudden he starts to decrease (Tracing B as she explains). So it is kind of leveling out.
I: Excellent.
Sarah: D, Darren runs very and reaches the finish line early. To me, it is just a straight shot, so there is a constant speed, so to me he ran fast the entire time (Traces A) and got

to his destination the quickest. And then E after failing, to me this significantly a fail because there is not much of a progress going on there and then Darren runs at a constant rate, so he starts gradually picking it up.
I: So how would you describe the graph of one?
Sarah: It could be a linear equation, cause I saw a line. It is positive. There is a function there. Thats it.
I: Very good. Thank you. (I explain the next task) (Time of Day versus hunger level)
Sarah: I am not a good person for this, I am hungry all of the time. This is different. It depends on people, people are different. I mean everyone should be hungry first thing in the morning,
This is morning time, like for me, I actually eat my breakfast when I get to school around 8. This is me at 10 (Labels first upper cusp). This is me at lunch time, starving (Labels second upper cusp) This would be me around 4, bout ready for dinner, ready to bite my hand off and this would be around 7 (End of graph). So the peaks are obviously when I am hungry, which is the times that I documented.
I: Which peaks, the highest points>
Sarah: Yes
I: How does it go from really high to really low?
Sarah: Well your full (She traces and explains the graph again)
I: When are you the most hungry?
Sarah: Around 4. (Points to the highest part of the graph)
I: Well are you the least hungry?
Sarah: Around 1:30. She points to the lowest part of the graph.
I: Awesome great job (POW problem explained)
Sarah: (Reads the labels) You had a few at first that turned it in. More people wait till the end. So as the days go on, you get a more increased rate cause it is getting closer to the deadline 12:24
I: It that a constant rate of change?
Sarah: In the beginning no, but, when you said constant rate? Cause constant to me is saying the same, consistent. But then you said change. So then, umm, (Pauses)
I: So is that graph consistent?
Sarah: Not in the begin(ning), well close in the beginning. I mean this point (Third point) looks like it is a little off. But it is trying to make a line and then (Tracing the first three points) and then here it is consistent (Traces the final four points and draws a line)
I: Would both of those lines that you drew have the same slope? 13:19
Sarah: No.
I: (Provide her the drinks example)
Sarah: There is plenty in the room because most don't get a drink when they come in first thing. They have had breakfast. They had their orange juice and hopefully most people are not drinking soft drinks. By the end of the day, there is obviously going to be less because people were buying drinks through lunch and a lot of people are going to have them throughout the day like they need a little pick me up (Talks about teachers). Lunch time and then they need one during their planning period in the afternoon. And then in the evening they are gone. (Talks about the high school)
I: What about this part?
AF: This is when they refill the machine.

I: Last one. (I explain movie)
Sarah: Well, when you have customers in a movie theatre, they are going to be buying snacks. So you will have more money in the cash register.
I: Why does it not start at zero, zero?
Sarah: No, company, even a vending machine, any store or company needs to have cash to start with so that they can make change. 15:27
I: And you said buying snacks, are those snacks the same price or different price?
Sarah: (Picks and looks at the graph) No, they are differently not the same price. We know that, we have all have been to the movie theatre. It will not effect the beginning amount of money.
I: Would the price of the snacks effect the rest of the graph?
Sarah: (Long pause) Not directly. It depends on what the people are buying. They tend to buy the popcorn and the sodas.
I: All the healthy stuff. Thanks. (I explain the water faucet)
Sarah: (Picks up pencil) It goes back to what I said in the beginning, one of them may have counted by ones, like one or you know, counting every minute and the other kid could have been counting every five minutes. This could be the one minute kid (Left) and this could the five minute kid (Right). So there graphs are going to look different. So that would be my answer.
I: Prefect explanation. You mentioned earlier that you taught ratio, how would you define it? 17:25
Sarah: Well we constantly drill in their heads that fractions is part of, and ratios are part to. It is either part - part or part - to - whole. We did not allow them to say fractions when talking about ratios and if they said of, we corrected them. By definition, a ratio shows a part to another part or a part to a whole in a numeric value
I: What mental images come mind when you hear ratio? 17:55
Sarah: We always, two apples to three oranges would be a ratio. In the classroom, five boys to every two girls. (Talks about lack of girls in her class this year due to having AIG).
I: What about the term rate, how would you define it? 18:37
Sarah: The word per, I would talk about the use of the word per. Give examples like how fast does a car drive. Like an example would be, how many of you kids have been to the beach. How long does it take? If it is 160 miles and you drive 60mph, how long would it take? we would use our estimation skills. I am a fan of estimation. (We talk about this a bit)
I: What is the similarities and differences of rate and a ratio?
Sarah: A rate can be a ratio. A rate can be written as a ratio but a ratio is just the actual, how do I say this, a rate is the actual units. Like a rate involves the actual units involved like miles per hour. Gallons per dollar. A ratio is just part to part or part to whole. A rate is a part of ratio as far as how we teach. Ratio is the big umbrella, rate falls under it. Rate is like concrete wording and examples as opposed to ratio is just the definition.
I: For your examples, you have been saying miles per gallon. Can rate be miles per miles?
Sarah: No, it has to be two different units.
I: What about rate of change? What is your definition?

Sarah: To me I think about time, how long does it take something to be in a different state, or a distance, or how long it takes something to be from point A to point B. For me rate of change, when I think about teaching my kids, we obviously know what the word change is, most of us get what a rate is, but I don't know how well we would put together what rate of change is. So I would probably have to talk about, umm, more like something that we could do in a science lab, we like to blow up stuff. This would be a good one, I did a hot/cold, we starting talking about heat, conduction, energy and all that stuff, I had hot water and I put red drops in and cold water and I put blue drops in, and we timed them for two minutes to see how long did it take the color to fill up the glass. So we could have timed it and that would have show rate of change. How long it took to fill up the glass. (Continues to talk about this experiment)

I: Final question, if you had a sixth grade student come in and ask what is slope? How do you respond? 22:31

Sarah: It depends on the kid. If it was one of my higher kids, I would write out the formula. If it,

I: What formula?

Sarah: y = mx+B

I: Okay

Sarah: And then we would draw a coordinate graph, probably make up some ordered pairs and then try to figure out all of the pieces of the formula based on the graph.

I: What pieces?

Sarah: If we had the x and y coordinates, we would try to find the slope and the x coordinates. Maybe that would help them understand it a little more.

I: How would you go about finding the slope?

Sarah: Umm, you would have to be given the slope, hold on, (Writes y=mx+b) In algebra, we would try to solve as (Explains how she would struggle to make up points that would work out nicely) We would try to solve for the variable.

I: Okay, so if I had (0,4) and (2,6), and I wanted to solve for the slope, what would I do? Could I do it without using y=mx+b?

Sarah: Well, if you plotted them (Attempts to plot them) So, (pauses) I am trying to remember. I would assume (pauses).

I: What are doing?

Sarah: I am counting. Slope would be measuring how far up and over so the slope should be 2.

I: Two, okay how did you get that?

Sarah: Because your going, I am looking at the y, going up 2 and you are also going over 2. But see that is what I cannot remember, how the slope would be actually figured out. I don't even know if I am write. That would be my guess. But I would not just guess. I would go and do some research. I would talk about how the line looks on the coordinate grid cause we know coordinate grids. (We talk about coordinate grids and I show her how to do use the formula to find slope)

APPENDIX E: SECOND INTERVIEW TRANSCRIPTS

ANGELA

Began with small talk.

I: Define the word slope?
ANGELA: I am automatically thinking how high the slant of it is over the run that it is. How much something tilts or leans.
I: What images do you visualize?
ANGELA: Well, I see a hill because I went to Appalachian and I can remember coming down those mountains and the signs of the side. You could tell the degree or the angle of whether it was going to be a really bad mountain going down. Going up, you knew it would be sucking up a lot of gas. So, I think about a hill. And of course being a math teacher, I think about graphs.
I: Okay, so what do you need to generate the equation of a line?
ANGELA: Where it crosses, the x and y intercept. Ordered pairs, so you would know how to graph.
I: Excellent. What are parallel lines?
ANGELA: Lines where the slopes are the same.
I: What do you mean, the slopes are the same?
ANGELA: Well, if you had an equation, $y = ½ x + 2$ and $y = ½ x – 4$. They would both be slanting the same degree.
I: What about perpendicular lines?
ANGELA: The slope is opposite reciprocal. I think that is right, it has been a while. Yes, I had to stop and think.
I: Do you know why they are opposite reciprocals?
ANGELA: Probably at one time Tim. But no, off the top of my head, no.
I: What is the definition of a tangent line?
ANGELA: At one time I did, but I don't know now.
I: (Explain m problem) **3:06**
ANGELA: Well, it is in the equation $y = mx+b$. M represents slope but not just to say that it is the letter. I mean, I don't know. They learned the formula, which I am glad that they learned. But just to tell me that the slope is m, it is whatever the number in front of x, which is m. So if you had the equation, it is the numerical value in front of x which is m. But that is the formula, but I would never define it as just m. The kids would have to know what the formula is first. I never thought about it like that.
I: (I explain the Tom problem) **4:39**
ANGELA: Well, he is walking at a constant rate. Gets sidetracked, maybe there was something in his path, so he has to turn, backtracks cause it is going downward. Then he is able to consistently walk the same there.
I: During the first increment, how fast is he walking?
ANGELA: Every 10 seconds, 20 meters. There is ten, he is at 20.
I: What would be his meters per second?
ANGELA: Two.

I: What did you use?
ANGELA: Well, you said meters per second, so I reduced it to two.
I: What about where you said he backtracks, how fast was he walking then?
ANGELA: Lets see. Well, 100 meters, then he is at 70 seconds, oh goodness, then I would have to go and add it, he is going backwards. He is back at 70, he is at 40. Wow. (Pauses). I am not really.
I: Okay, how about this question: "When does he reach the bus stop?"
ANGELA: 100 seconds.
I: What happens after 100 seconds?
ANGELA: It flat lines. It goes straight over.
I: Why?
ANGELA: He reaches his destination.
I: Excellent. Is that a realistic graph?
ANGELA: I guess it could be. I don't know. I guess it could be.
I: (I explain number 7)
ANGELA: (Reads the problem aloud). (Repeats problem two more times) Five dollars. Cause the difference between the two is 15. (Pauses) Three goes into 15, so she is saving 5 dollars. I can see the fifteen cause she is saving five dollars a week. How much does she save each week? How much does she receive for her birthday? She at 5 weeks, it is 175, 4 is 170, 3 is 165, 2, is 160, 1 is 155. 150? 155? She started out with 155 for her birthday. She ended with 190. Yes, she got a 155 for her birthday. I just backed it off. No, wait, that is supposed to be 150. She saves 5 dollars each week.
I: Good job. I will trade you. (Give her cubic graph) Can those graphs have slopes? **11:43**
ANGELA: Yes.
I: Okay, how would you describe the slope of that function.
ANGELA: Wow. I know that when you have a parabola like that the slope is a negative then when it goes up it is a positive. But I have never seen one where the graph curved that much. I haven't.
I: Okay. (Gave her bottle problem)
ANGELA: (She could not graph it)
I: (Three lines problem) **15:12**
ANGELA: The slopes. Let me think how to say this. When I first see this, I am trying to get ordered pairs off the top.
I: Why do you want to find ordered pairs first?
ANGELA: It is just easier for me to see. This one is zero, one two, three.
I: What is the first thing you notice about those lines?
ANGELA: They all start at the origin. This one is one, a half, over one, up a half. (Struggling and muttering to herself)
I: Do they have the same slopes?
ANGELA: Well, lets see. That would be. One ordered pair would be (1, ½) and this one would be (2,1) and then I would go find two more ordered pairs and find the slope (Struggling)
I: So you would lean towards change in y over change in x?
ANGELA: Yes, for this one I would
I: Is there another way to explain the difference in their slopes?

ANGELA: Well, I am looking at the steepness of them. This one (slope of two) if you were climbing that mountain, it would be almost going straight up. If I was skiing, I am not a very good skier, so I would have to start with this one (Slope of ½). It is not as steep.
I: Awesome. Thanks. One more. (Triangle problem)
ANGELA: As the angle gets big, the slope will get larger.
I: What does that mean?
ANGELA: The steepness of the line will get larger. It will be more steep.
I: Thank you.

Brianna

I: How would you define slope?
Brianna: I would define slope as rise over run. Like the change in y over the change in x. It's the steepness of a line. So either how far it goes up (indicates an incline with her arm), or down, or if it does it at all. It does correlate with the angle of measurement for what's happening, but it's not the angle - it's just how steep the line is.
I: And what do you mean the angle of measurement?
Brianna: So if I said this was the ground (draws a horizontal line), and this was the line, or the hill that I'm talking about (draws an incline moving from lower left of paper to upper right), it correlates with this angle of measurement (indicates the angle between the vertices of the two lines drawn - lower left corner). So the larger the angle is here, the bigger, or greater the slope of this line (slanted). The smaller this angle is - so if this is still the ground and this angle is more like...hmm...let's say this angle (first one drawn) is 40 degrees, and this is more like 15 degrees (second angle drawn) then I would expect the slope to be less here (second angle) and greater here (first angle), because the angle helps to justify. It correlates but it doesn't determine the slope. It just helps determine how great it is.
I: So if I had a 30 degree angle, would that be enough information for me to determine the slope?
Brianna: No. It wouldn't. But if I had a 30 degree angle and a 50 degree angle, I should be able to tell you that the slope of the line of the 30 degree angle is smaller than the slope of the line of the 50 degree angle. I should be able to make some generalizations about it, but I cannot calculate the slope from the angle. It's just to help you see that this (the angle) is closer to the bottom, so it's (second line she drew) flatter and is going to be smaller. This one (first angle she drew) is higher and is going more vertical toward the y-axis so that slope will be greater.
I: Awesome. Very very cool. So, again another follow-up question. What images come to mind when you hear the word slope?
Brianna: Hills, valleys, mountains, peaks, roller coasters...things like that.
I: So, in order to generate the equation of a line, what do you need?
Brianna: Two ordered pairs.
I: Ok, so when I have those two ordered pairs, what do I...why do I need those?
Brianna: I need the two ordered pairs on the line so I can determine the slope of the line, so I can determine how steep it is. And then from there I can use that and the slope to determine where it actually goes through the y-intercept and actually what the entire equation of the line is at that point.
I: Could I calculate, or could I find the equation of a line with just one point?
Brianna: I can find it with one point if I had the slope. Or if I was given...say the y-intercept was two, I could determine a point from a y or x-intercept if I wasn't given the full ordered pair but I would have to have the slope or something that would help you get to an ordered pair to determine the slope and the equation of a line.
I: So I need the slope and a point or two points that would allow me to calculate the slope?
Brianna: Yes.
I: What are parallel lines?

Brianna: Parallel lines are lines that go on forever in the same direction and never touch, never intersect, never meet.
I: How do parallel lines relate to the concept of slope?
Brianna: So parallel lines have the same slope but they have a different y-intercept. So, if they both have a slope of two but they will be located either higher or lower on the graph then the other, and they won't touch because their slopes are the same and they're running in the same direction; they'll just have different starting points.
I: What about perpendicular lines?
Brianna: Perpendicular lines are two lines that intersect to create a 90 degree angle. Thinking in the world of vectors they call them orthogonal or something like that. But those lines have opposite reciprocal slopes so if one slope was ½ then the other slope would be a -2. The y-intercepts don't necessarily matter in that case just so long as you know that the slopes are opposite reciprocals of each other.
I: What about a tangent line?
Brianna: A tangent line, well in my mind, it's where it touches one point on a curve. So the tangent line changes depending on where you are at on the curve. So it has a slope; it has a y-intercept; there is an equation to the line but that equation changes depending on where I am at on the curve, or circle, or something that is not linear.
I: And it shares one common point? Could it share more than one point?
Brianna: Umm...like when I taught it when I was in Calculus, each point had a specific tangent line. It could have similarities to another point, but because it's on a curve and it's not the same change in y/change in x because of the curve - like tangents lines have different equations.
I: So, like I said, this is a follow-up and I analyzed all the first interviews and one of the teachers when I asked them to define slope they said, "It's m." So, how would you respond to a student or a teacher if they responded the same?
Brianna: I would tell them that, yes, m does represent slope and we do use the letter m in equations instead of writing the whole word slope, as like a shortcut, but that's not a symbol of it. It's not what it is. And I would take them through the process of examples, things that they've done and how hard or how easy something has been for them to do in a situation. So if I was riding a bike up a mountain, versus riding down the mountain, versus riding on the street - tell me how that felt. And then use that example to correlate it to the definition of slope, but m is just a symbol just like pi symbol represents 3.14… So for me I would have said, yes, you're kind of right but m is a symbol to represent what you're trying to calculate.
I: (introduces Tom problem) 9:25
Brianna: (draws and writes m=2)
I: How did you get m = 2?
Brianna: So I started at the origin which was (0, 0) and I used that as one of my ordered pairs. And I used the next point, the one that I could really tell what it was. I always tell my kids to make sure it goes through the corner of the graph, so you know what it is. That point was (20, 10). So instead I did rise over run so it goes up 20 and over 10. So 20 over 10 is 2. So because this was distance over time, he was walking at a speed of 2 m/s from 0 to 50.
I: Awesome. So you used rise over run?

Brianna: Mm hmm. (shakes head yes). Um, from 50 to 70, that's 100 - 70 which equals 30 over 20 which is 3 over 2, which is 1.5. So this is 101.5 and the slope here is 1.5 m/s. So he slows down at this point, but to me he just goes back, like this is the distance from home. So he went from home 50 meters, then he went back 20 meters. He either took a detour or he forgot something - so he went back towards his house. He went back toward his house 60 meters - because there was 100 here and this is now 40. So he went back towards his house 60 meters and did it in 20 seconds which is how I got a speed of 1.5 m/s. So he kind of slowed down. And then at the 70 second mark he was at the 40 meter spot, and now he's at 100 - so I'm going to do rise over run over here. So it's up 40 over 10. Ok, so this over here he has a slope of 4. So I don't know...here's my story in my mind. He realizes he's running behind time and he's going to miss his bus so he speeds up walking from 40 meters to 160 meters because he's doing about 4 m/s in his walk toward his bus stop. Then he gets here and he's like - whew - I'm almost there, and so this slope is a 0 slope. So it's like, to me, he gets here and then he's waiting for his bus to arrive at the 0 slope because he's not moving at all.
I: So what time does he get to the bus stop?
Brianna: In 100 seconds and he's waiting 20 seconds for the bus to arrive.

13:06
I: Follow-up question: Does that graph seem realistic?
Brianna: I don't see why not. Like you could always have a detour. They say the shortest distance between two points is a line; a straight line. But if you have a detour you have to do what you gotta do. 13:29
I: (hands over birthday present problem)
Brianna: Ok, so how much did she receive for her birthday? So that's the y-intercept. How much did she save each week is the slope. So she starts out with 5 weeks and she's saved $105 - so I'm going to use that as my first ordered pair. At 8 weeks she has $190 - I'll use that as my second ordered pair. And I use the slope formula actually to do this one.
I: Ok, why are you leaning towards that instead of rise over run?
Brianna: Because I don't have a graph. So when I have a graph I tend to use rise over run because it's easier to see. If I don't have a graph I tend to lean towards using the formula. (pauses) So she is saving about 5 dollars per week, and so that means she received $150 dollars for her birthday.
I: How did you figure out the 150?
Brianna: Because she saved $5 a week so I did 5x5 is 25 so that would tell me how much she saved for the 5 weeks all together. Then I subtracted it from what she had total. So 175 - 25 is 150 and that should give me my starting point.
I: (Hands over another problem) How would you describe the slope of that function? Can it have a slope and if so how would you describe it?
Brianna: It has slopes at points - or at sections of the graph. I would describe it as increases and decreases at phases and use that to kind of determine what the slope would kind of be or those tangent lines what kind of slopes they would have. I wouldn't necessarily be able to tell you the slope at every point because it has infinitely many points so it should have infinitely many slopes, some of them being the same. If I looked at, say, negative infinity to negative 3, that slope on that side of the graph would be

positive, because it's increasing as I go from left to right. Right here across the top - I don't know what that point is called - and it may not be exactly 3. It's kind of like a little bit more like negative 2 and two thirds or something like that. And then at that point where it changes from positive to negative, it has a zero slope. I tell kids it's like being on a roller coaster. So you climb up the mountain, and it's that point at the top where you pause right before you go over the edge. This point where it's zero, at like -2.75 or something like that, down to this point here which I would call about positive one half - that slope would be negative because it's going down as I move from left to right. Then again where it changes from positive to negative - at this flat line that I'm drawing - it would have a zero slope. And then from that point at negative one half all the way up to positive infinity, would have a positive slope because it starts to rise again and go up and increase to the right as I move over, and so that would have a positive slope.
I: (Hands problem to interviewee - explains similarity to water filling up a container problem from interview #1) So if you could graph the height of the water versus the time. So, the x axis would be time and the y axis would be height. Assuming it starts empty, how would the graph of the water look as we continually fill it up at a constant rate?
Brianna: (Asks clarification of axis labels.) (Draws a line.)
I: So as it fills up that container, how did you generate the function that you drew?
Brianna: So the water coming down at the same rate is not going to change, but as time moves on (because nothing is coming out of it per se) the height of the water should increase just like me filling up a bathtub. Even though the shape is not uniform, the height will still increase as the water fills up. Now, it may be a little less steep or a little more steeper, but to me it's still going to be a line because the rate's not changing.
I: (explains faucet leaking problem)
Brianna: Square root. Because it looks like it fills up and then starts to level off. *asks for clarification of the volume of water going out.*
I: Yes. So if it all accumulated into a container, this is what it would look like.
Brianna: So, it starts out coming out a little bit, then it comes out more as water is going in. So I'm guessing that there's more water that's going in then it started with. And then it looks like the rate of it coming in and going out equals out - it's kind of steady at a point because once the container gets at a certain water height or the faucet's at a certain water height, what's going in and coming out is like the same.
I: So, in the beginning does it begin to leak faster? Does it leak the same? Does it leak less? Does it not leak at all?
Brianna: At the beginning it looks like it's leaking less and then it's increasing to a point. The water's going in and then it's leaking. Because this slope is greater so it's leaking faster and then it kind of leveled off, and it's leaking.
I: So it's leaking the entire time and in the beginning it leaks faster.
Brianna: Yes, because there's less water in the faucet so what's going in is what's coming out. But as the water starts to build up in it - kind of like a tub - when a tub is kind of stopped up at first, it's leaking, and then after a while you have water that starts to stay in the tub, but you do know that some is going out while water is coming in. But it's stopped up, so that's kind of what's happening here.
I: (introduces three line problem) What's different about them?
Brianna: Their slopes. Their y-intercepts are all the same they're all zero.

I: So, their slopes are different? And how did you see that right away that their slopes are different?
Brianna: The position of the line. This line (far left line) is closer to the y-axis. This line (bottom line) is closer to the x, and this one is in the middle. So to me, a line that is closer to the y-axis has a greater slope. A line closer to the x-axis has a smaller slope, and the one in the middle has - it's kind of like numerical order. So if I would say that this one (line on right) is one half and this one (line on left) is a whole, then this one (in middle) would be somewhere between one half and a whole. Like two thirds. But their y-intercepts are the same.
I: Their y-intercepts are the same, but their slopes are different?
Brianna: Yes. So the slope for this one actually is a half (bottom line).
I: How did you figure that out?
Brianna: It's up one and over two. This one (middle line) is up one over one - so this is actually one. And then this one (far left line) is up two over one - the slope is two.
I: (explains that the last question has already been answered by interviewee during a previous question during this interview.)
End at 23:52

Carrie

Begin with small talk.

I: How do you define slope?
CARRIE: My definition of slope is the steepness of a line
I: What mental images come to mind when you hear the word slope?
CARRIE: Well, unfortunately, I think of a line on a graph. 2:24 Then I start to think of other images like a ski slope, a wheel chair ramp or any kind of elevated surface.
I: What do you need to generate the equation the line?
CARRIE: Basically, you need the start height and the ending height. The difference between the start distance and the end distance. So a vertical and a horizontal measurement.
I: What are parallel lines?
CARRIE: Two lines that will never intersect. 3:26
I: What makes them never intersect?
CARRIE: They have the same slope.
I: What about perpendicular lines?
CARRIE: Perpendicular lines intersect at a 90-degree angle.
I: How does that relate to their slope?
CARRIE: Its, I always say this wrong, but it is the opposite inverse. So negative two to one half.
I: So, if kid put his hand in the air and asked why is it the opposite inverse, could you provide an explanation?
CARRIE: It would be more of me having that student generate a few examples and see if they could come up with that on their own. Rather than me just telling them.
I: But could you prove it mathematically?
CARRIE: Yes.
I: Okay. What are tangent lines?
CARRIE: They are, oh wow, I don't know. (Laughs). Cause when you say tangent, I am thinking of a line touching a circle. I am thinking geometry not slope.
I: It is all right. During the first interview, one of the teacher's responses was m. How would you respond to a student or teacher that said m?
CARRIE: I would ask them what is m? How did you get it? What does it mean?
I: SO you would look for more?
CARRIE: Yes.
I: Would that be a sufficient answer?
CARRIE: No. 5:22
I: So here are a few tasks. (I give her the Tom problem)
CARRIE: Okay, so if he is on a straight road. It looks like he is going 100 meters in 50 seconds. So, your rise, or your vertical is 100 over your run or horizontal of 50. So he is going 2m/s just straight towards school. I don't know, for some reason he turns around and he goes backwards. SO he goes back 60 meters because he goes back to a distance of 40 from his house. He does that is 20 seconds. So, he goes back at a faster rate of 3 m/s. The slope is steeper there, so that also indicates a faster rate. And then he has to get back

to school, so he is 40 and has to get to 160, so that is a difference of 120 meters in 30 seconds. Is that right? Yes. 30 seconds. So he is going 4m/s to get to school?
I: When does he get to the bus stop?
CARRIE: In 100 seconds.
I: So you leaned towards rise over run?
CARRIE: Yes.
I: Why? 7:25
CARRIE: Because it is on a graph.
I: Okay, so part b?
CARRIE: Yes.
I: (Give her word problem Anjelica)
CARRIE: Okay, so she saves the same amount each week. So that is going to be a constant rate of change or a constant slope, if you are talking in those terms. She saves, five weeks (Can tell she is working it out in her head), she saves, 175 and after 8 weeks she weeks she saves 190. So, you are going to come up with your difference there. So you have a difference of 15 dollars and that happened in three weeks. So, that means that she saves 5 dollars a week. If she saves 5 dollars per week, that means in eight weeks she has saved 40 dollars. Which means that she got 150 dollars for her birthday. (Did not write down anything).
I: Awesome. That is perfect.
CARRIE: (Laughs)
I: So you kind of of did it in your head, did you rise over run? Did you use change in y over change in x? Di you use something different?
CARRIE: It is a little bit of both. I think when I am using rise over run, I think I am thinking about change in y over change in x. I don’t really separate them very much. I feel like I am using the same method when I am using either method.
I: So you wood not separate the two ideas?
CARRIE: To me, they are the same.
I: (Give her the cubic graph) Does that function have a slope?
CARRIE: Lots of them. (Laughs)
I: How so?
CARRIE: Well it gets depends on your period that you are using along your x-axis. Like where along the function are you trying to determine the slope?
I: Okay, so how would you describe the slope of that function? 10:21
CARRIE: There is not a constant slope. So there is a several, an unlimited number of slopes.
I: Is there anywhere that you can definitely say the slope is this based on that graph?
CARRIE: The easiest points are the ones that have a definite integer.
I: why would those be the easiest?
CARRIE: The subtraction is easier. (Laughs) It is easier to do the math mentally if you are using straight integers. X is negative one to x is zero is your easiest. (Talking about calculating a secant line not a tangent line). You have another relatively easy spot from negative one to negative two.
I: (I clarify that she is trying to find the slope of a secant line)
CARRIE: Yes.

I: (I give her the 3 lines coming out of origin) How would you describe the difference between those three?
CARRIE: Basically, they all have different slopes. The top line has a slope of two, then a one and a slope of one half.
I: How did you calculate those slopes?
CARRIE: I guess rise over run.
I: What else about those lines make you think that they are different?
CARRIE: Just the fact that they have different slopes. They have the same intersection point, so you don't have to worry about them being parallel. So visual…
I: That is what I mean. What about them being visual?
CARRIE: Just the steepness.
I: Is there any other characteristics or another way to say that they are different?
CARRIE: Maybe. I don't know. Just the fact that they are on the same coordinate plane. If they were on different coordinate planes, then I would have to check out intervals of the axis and stuff like that. The fact that they are graphed on the same coordinate plane makes it a little more obvious.
I: (Explain number 16)
CARRIE: It either increases or decreases (Laughs)
I: What is the relationship?
CARRIE: If theta increases then the length of your hypotenuse increases and your slope is increasing because your opposite side is getting longer so it is rising. If you make theta smaller, the length of your opposite side is decreasing so that is going to bring down the steepness of your hypotenuse.
I: Awesome. Thank you.

Deborah

I: Thank you for coming.
D: No problem
I: How would you define slope?
D: To me in the simplest form, slope represents steepness and direction. How steep is the rate of change for the given situation and is there a positive or negative trend of that rate?
I: What do you mean by steepness?
D: The steepness of a line.
I: Of a line?
D: Yes. When a middle school student analyzes slope they might think of rise over run when viewing a line on a graph. Higher-level middle school students can determine the rate of change within a word problem and conclude that this figure represents the slope of the linear equation used to represent the given scenario. **PHONE RINGS**

She answers phone and states that she has to leave. Apologizes.

Elizabeth

I: Good afternoon. Thank you for agreeing to participate in this second round interview. Lets begin with a review of small questions that I asked in the first interview. First, can you define slope.
ELIZABETH: Slope always reminds me of landscaping. It is the grade or gradient of how something decreases or going down.
I: So whenever you hear slope, you mentioned landscaping, what else comes to mind?
ELIZABETH: The truck signs on the interstate (From last interview). You know, not quite level. Watch that turn.
I: Anything else.
ELIZABETH: No, not really.
I: Okay, so if I asked you to generate the equation of a line, what type of information would you need?
ELIZABETH: I would have to have an idea of the direction of the line, was it on a number line or was it directional with coordinates.
I: Okay, how many coordinates would I need?
ELIZABETH: You would actually only need two.
I: And if I had those two coordinates, what would I be able to do?
ELIZABETH: Make a line. You could intersect them.
I: What if I only gave you one point?
ELIZABETH: You could make a line but it would go in any direction.
I: What additional information would you need if I only gave you one point?
ELIZABETH: Which direction it would go.
I: Would you need anything else?
ELIZABETH: No.
I: Okay. So in your own words, what are parallel lines?
ELIZABETH: Lines that go side by side. They never intersect. They are side by side.
I: What is the unique characteristic of parallel lines?
ELIZABETH: They are never ending.
I: What are perpendicular lines?
ELIZABETH: I am not sure. I have to think about this. Perpendicular lines are. They intersect. I can’t remember my math. (Laughs)
I: Okay. What about tangent lines? Do you know the definition of a tangent line?
ELIZABETH: Not at all.
I: Okay, so I interviewed ten middle school teachers and when I asked one the definition of slope, the teacher said m. How would you respond?
ELIZABETH: They are using the formula.
I: What formula?
ELIZABETH: I don’t know the formula. Like I said, we don’t teach slope in the sixth grade. I don’t know it. I have never used it.
I: You never used it.
ELIZABETH: Not since college.
I: (Three graphs problem) How would you describe the differences in these lines?

ELIZABETH: So the x is all zero and the y, evidently, where the number is going, it is going up higher for each (line). So it is just different points that you would be graphing. They all start with zero at the x.
I: Are the slopes the same?
ELIZABETH: I think, yes.
I: How did you figure out the answer was yes?
ELIZABETH: When I am looking at the boxes, if is the same distance between each of the lines (Pointing to the space that exists between each of the lines)
I: Okay, so how do you calculate the slope of a line?
ELIZABETH: I know there is a formula but I do not know how to use it. So if I was looking at it, I would think yes it is similar. The slope would be the same for each.
I: The slope is the same for all of them?
ELIZABETH: Yes.
I: So outside of different coordinates, is there anything else that would allow you notice is different about the lines?
ELIZABETH: No, they are all going off the chart. Continuing.
I: Thank you.
ELIZABETH: Being creative in my answer.
I: You are indeed. (Explain triangle problem) What happens to the slope as the angle gets bigger?
ELIZABETH: I am thinking that the line would go up so the slope would decrease. The gradient of the slope would decrease.
I: Okay. Excellent. What happens as the angle decreases?
ELIZABETH: The slope would be greater.
I: The slope would be greater?
ELIZABETH: Yes.
I: Excellent. Thank you so much. That is all that I have.

JACKSON

Begin with small talk about the upcoming school year

I: What is your definition of slope?
JACKSON: It is a rate of change. That is how I define it.
I: What images come to mind when you hear the word slope?
JACKSON: I think of a roof, I think of a a graph. I think of a hill. Gutters.
I: So when you think of that hill, do you see a curved mountain or a linear straight line?
JACKSON: I am thinking a linear, straight line.
I: Okay. What about ratio?
JACKSON: It compares two quantities, usually through division
I: What about rate?
JACKSON: A ratio that compares different units such as miles per hour or feet per second.
I: What do you need to find the equation of a line?
JACKSON: The slope and the y – intercept.
I: Could you do it another way?
JACKSON: Yes, I could do it with two points.
I: What are parallel lines?
JACKSON: Lines that never intersect.
I: What makes them never intersect?
JACKSON: They have the same slope.
I: Awesome. What are perpendicular lines?
JACKSON: The slopes are opposites?
I: Opposites?
JACKSON: We always say it is the opposite flipped. Like if I had ¾ if would be -4/3.
I: Do you know why that relationship holds true?
JACKSON: Yes. It has to do a lot of the points.
I: Okay, but if a student said "Hey Coach, why?", could you explain it?
JACKSON: Yes. (Does not want to elaborate)
I: What about tangent lines?
JACKSON: I have no clue.
I: Okay.
JACKSON: I have not done tangent since tenth grade (Laughs)
I: No worries. (I explain teacher responded m problem)
JACKSON: (Laughs awkwardly) That is a tough one. M just represents the slope. Like it the variable for slope. That is what I would say. It is not what the slope is.
I: Okay. (I explain the Tom problem)
JACKSON: Okay. (Pauses for 4 seconds) (Does not write anything) At first he is just walking to school and then he took a wrong direction somewhere and maybe slowed down a bit. And then he realizes that he was late and he went real quick. He had to hurry up. That is the way that I would look at it.
I: Could you figure out how fast he walked?
JACKSON: Yes you could be using the time and how fast he walked.
I: Okay. How fast did he walk?

JACKSON: Yes. He is walking ten meters every ten seconds. I am sorry, 20 meters every ten seconds.
I: How did you figure that out?
JACKSON: I used the formula. I just subtracted in my head.
I: And that middle part that went down, what did you say was happening?
JACKSON: I think he might have took a wrong direction, or was being lazy, maybe he went somewhere. I think he was taking his time. If you look at it, he went down from 100 meters, maybe he went the wrong direction and all of the sudden he realized it.
I: What time did he get to the bus stop?
JACKSON: If I am looking at this correctly, does it flatten out there? (Points to the horizontal portion of the graph)
I: Yes.
JACKSON: It took him 120 seconds, well…I would say around 120 seconds. Yeah.
I: Okay, good. Is that graph realistic? With respect to the shape and everything.
JACKSON: Yes, I would think so.
I: (I explain problem number 8) How would you solve that problem?
JACKSON: (He reads it) Well I would look at the rate that she has, after five weeks she saved 175 dollars and after eight weeks she saved 190 dollars. I am looking at it as if, if that the was the same that she saved from five weeks to eight weeks, it does not say (reads the problem again). She has the same amount each week. So after five weeks she has 175, so I would subtract that from 190 and find the rate that she saved.
I: And then what?
JACKSON: I would find the difference between five weeks and the eight weeks and then the difference in the money. So, 15 over 3.
I: So five?
JACKSON: Yes. (Hands me paper)
I: (Explain rate of change of cubic graph) You talked about the slope of a linear function, can that graph that is non-linear have a slope?
JACKSON: I would say yes.
I: Could you describe the slope?
JACKSON: Ahhh. I mean it goes…it goes up, of course, then it takes a steep down, then all of a sudden it goes extremely up. Like a very steep incline from the bottom point on the curve section all of the way up.
I: Okay. Are there any points on the curve that you could definitely say that the slope is this numerical value?
JACKSON: Ahhh. Definitive points?
I: Yes, like you can give a number as the slope.
JACKSON: Not that I know of.
I: Not that you know?
JACKSON: Nope.
I: (Number 15: Three Graphs). First question, Do those graphs have the same slope?
JACKSON: No.
I: Second Question, how would you calculate the slope?
Jackson: I would pick two coordinates and subtract the y values and subtract the x values.
I: How else could you proof that the slopes were different?

Jackson: Other than finding the slope, I would talk to them about the steepness of the lines. That is how I would visual look at it.
I: Any other way?
Jackson: Not that I know. No
I: Okay. Last one. (Question #16: triangle). What happens to the slope of the hypotenuse as theta increases?
Jackson: As theta gets increases, the slope would increase.
I: And as theta decreases?
Jackson: The slope would decrease.
I: Awesome. Thank you sir.

Liam

I: How do you define slope?
LIAM: Slope is the steepness of a line. Rise over run. The greater the slope, the steeper the line. The smaller the slope, the more horizontal the line.
I: When you hear the word slope, what do you think of?
LIAM: Ski slope. Going up or down a ski slope. The incline or decline of a certain line.
I: So when you visualize that ski slope, what do you see? A straight line or a curve?
LIAM: A straight line. That is how I try and teach it. A negative slope is going straight down a mountain. A positive slope is going straight up a mountain. Trying to get the positive and negative difference. Basically, a straight line. A linear equation.
I: So what do you need to write the equation of a linear function?
LIAM: The slope and the y – intercept. The y-intercept is where it crosses the line. Or you could be given two points and you could calculate the slope given the two points and find the equation that way.
I: What are parallel lines?
LIAM: Parallel lines have the same slope but not the same y-intercepts.
I: What about perpendicular lines?
LIAM: Are two lines that intersect and their slopes are the negative reciprocal of each other.
I: Do you know why they are negative reciprocals of each other?
LIAM: Because they have to meet at a ninety-degree angle and when flipped and taking the negative, would give you a ninety-degree angle.
I: Could you prove it mathematically?
LIAM: (Pauses) Well if you are given two or four different points (Trails off) Have to show they are 90 degrees. I don't think that I can right now.
I: Okay. No problem. What are tangent lines?
LIAM: Tangent lines. Are two lines that touch. (Thinking). Yeah, I don't know. I can't think what they are. In pre-calc, we learned about sine, cosine and tangent. (Thinking). I don't know.
I: (Ask *m* question)
LIAM: M is the variable of slope. Which if you are in slope intercept form, y = mx + b, which is the equation in slope intercept form, then m is the variable for slope. If a kid told me that, I would probably say, I would try to get more out of him. Okay, I get what you are saying, but what exactly is it. What does m mean? What does it stand for? How do you find it? More or less.
I: So you would look for more?
LIAM: Yes. By asking more questions. If is not wrong, but it does not tell you what it is. Like the steepness of a line.
I: (I explain the Tom task) **5:39**
LIAM: He walks for every 20 meters, for the first line, for every 20 meters, it takes him 10 seconds. So he walks at 2m/s.
I: How did you figure that out?

LIAM: Rise over run. I used the first two points that I could find. You could use any two points. So rise, I rose, one block which was 20 and ran one block which was 10. So I rose 20 and ran 10. So that is 20 over 10 which 2. So he is walking at a constant rate of 2m/s. He stops. He goes farther away. So maybe he stops by a friend's house. This is a negative slope. So for 50 to 70 seconds, 20 seconds, he went 60 meters farther from the school. He stops, maybe met a friend. Walked the wrong way for a second, took a little break, I guess he did not take a break or he would be sitting still. Once he was back at 40 meters from his house, he walked the last, 160 minus 40 is 120. He walked the last 120 meters at a rate that was twice the rate that we walked at first. He was walking about; I guess it was 4m/s.
I: Awesome. When does he get to the bus stop?
LIAM: He gets to the bus stop at 100 seconds.
I: How do you explain 100 to 120 seconds?
LIAM: I guess he got there and stopped. He did not walk any further. He got to his destination. He told for 20 seconds.
I: Is that a realistic graph?
LIAM: Yes, it could be. I don't see why not.
I: (I explain the Angelica problem)
LIAM: Can I write on this?
I: Absolutely.
LIAM: (He reads problem aloud) How much she received. Let's call that x. She has the same amount each week. (He solves this problem by setting up a system) (Walks through he step by step approach) **10:29** I'm going to solve for x and substitute it in. Combine like terms. (Does it correctly)
I: (Hypotenuse problem)
LIAM: The less theta, the less steep the line is. The greater theta is, the larger the slope is. The larger this number is, the steeper this line will be. If the angle was one degree, it would be very level. It the angle was 89 degrees; it would be very steep.
I: (Give cubic) Does this cubic function have a slope? **13:37**
LIAM: (Long pause) It is not a consistent slope but that is not necessary true. We only find slope of linear equations but,
I: Lets assume that it does have a slope, could you describe the slope?
LIAM: Yes, just find two points on the line itself, wherever you stop, wherever your vertices are, so you could do the top and bottom, and count rise over run. (Calculating a secant line) And she how much it increases or decreases. Still not sure if it does or not.
I: Thanks. (Three-line problem)
LIAM: How they are different? They are start at zero. The slope all three are different. They all go up at a different rate. If they were the same, there would only be one line. Since there are three lines, they all must go up at a different rate. The one closest to the x-axis being the least steep. The one closest to the y-intercept is the steepest line or the greatest slope. They all are going up. So they all are positive. So that is what is similar.
I: How would you calculate the slope?
LIAM: Well, if something crosses the origin, I would always use the origin as one point and find the next place where it crosses, intersects at a point. The first one would go up ½ over 1. The second has a slope of one. It crosses or intersects each box. This one

(Steepest) has a slope of two. I would go up two and over one. Yes, Up two boxes and over one. Or you could go to the point two one.
I: Awesome. That is it man. Thank you.

Luke

Begin with small talk
I: How would you define slope?
LUKE: Steepness of a line.
I: Any other definitions come to mind?
LUKE: Well, I play a lot of golf and slope coordinates to how difficult the course is. (Laughs) No, but really, steepness of a line.
I: What other images come to mind?
LUKE: I guess I look at whether it is going up or down. Positive slope or negative slope.
I: So, what do you need to generate the equation of a line?
LUKE: Well you need an x and a y. Two variables and a constant. Well not always a constant.
I: So, if you asked your students to find the equation of a line, what would you need to give them so that they could complete this task?
LUKE: Traditionally, when I taught it, we were always trying to put it in slope intercept form. So we would always solve for y.
I: How many points do you need?
LUKE: Two. I am guessing. I think it is two. Yes, two points. That way I can do y2 minus y1 over x2 minus x1. If I remember correctly.
I: You do. Could I do it with one point?
LUKE: I am going to say no. Cause you would not know if it is positive or negative with just one point. You have to have two points to create a line.
I: What if I gave you a point and the slope?
LUKE: Yes, that was on one of my quizzes. Yes, cause you can create your line once you have the slope.
I: What are parallel lines?
LUKE: Parallel lines are two lines that are on the same plane that will never intersect.
I: What makes them not intersect?
LUKE: They have the same slope.
I: Why do they need to have the same slope?
LUKE: Cause if they don't have the same slope, they will intersect.
I: Great. What about perpendicular lines?
LUKE: They intersect at ninety-degree angles.
I: How does that relate to slope?
LUKE: well, 90 degree angles, the straight up and down will be undefined and the horizontal is zero. But it does not have to be an undefined and a zero slope, perpendicular lines could be, if the slope is two the other would be, negative one half, would it be?
I: Yes.
LUKE: Does the product have to be one or is it negative one?
I: Negative one.
LUKE: See I remember
I: Do you know why it has to be negative one?
LUKE: Boy that is a good question. Off the top of my head, no I can't.
I: Explain m question. How would you respond?
LUKE: Well that would be the coefficient in front of x. When you are using y=mx+b.

I: Well you defined it as the steepness of the line, would you be okay with the response m?
LUKE: I would probably hand it back and say give me more detail, give me more information. Yes, you have it from the standpoint of the equation. But you did not really tell me what slope was.
I: (Explain Tom problem and the purpose of the interview)
LUKE: Well it looks like he was, at first he started off walking at 20 meters per 10 second. This reduces to 2 m/s. At a very constant rate for the first 50 seconds
I: What are you using?
LUKE: Rise over run. That is all a constant pace. Then he slows down substantially for something, if he is like any kid, he is probably messing around or talking to someone. He is going slower.
I: So he is going slower from 50 to 70?
LUKE: Yes. And then he picks up the pace again and it looks like he is even walking faster than he was previously.
I: How did you make the distinction that he is walking faster?
LUKE: Well he is definitely walking faster because instead of walking up one block and over one block, he is actually going up two blocks and over one block. So his slope has increased. Instead of being one to one, it is two to one.
I: Before you calculated the slope, you said it looked like he is going faster. How did you determine that?
LUKE: It is steeper.
I: When does he get to the bus stop?
LUKE: 100 seconds.
I: And then?
LUKE: He just stands there and waits for the bus.
I: Is that a realistic?
LUKE: I don't see why not. I could see me walking to school like that. (Elaborates more)
I: What about from 50 to 70 made you think he was slowing down? Still walking towards school but at a slower rate.
LUKE: Instead of increasing, it was decreasing.
I: So that was a graph, this was a word problem. (Explain problem with Anjelica)
LUKE: (Reads problem) You are starting to put pressure on me. It should be simple enough that I should be able to work out. I would take the difference in the weeks and the difference in the money. I think. Not really sure right now. Drawing a blank.
I: Okay. You were on the right track.
LUKE: Not sure.
I: How would describe the slope of that line?
LUKE: I would say it was a negative slope.
I: Why?
LUKE: It is like I told you in the previous interview. An easy way to remember is that right handed people are better than left handed people. (Raises right hand in the air) So when the right hand is above the left hand, that is a positive. When the left hand is above the right hand, it is a negative. So that how, well my left handed kids get upset with that
I: I would image.

LUKE: But they remember it. Especially, at the sixth grade level. We are not trying to calculate slope. We are looking at positive and negative.
I: Could you draw a line with a larger negative slope?
LUKE: It would be steeper.
I: Smaller
LUKE: It would be flatter or not as steep.
I: (Cubic) Does that graph have a slope?
LUKE: Messing with now with parabolas and stuff like that. It has been a long time. Because. It would have to be; would the x factor have to be a square?
I: For that one, it is cubed. So it would be a three.
LUKE: Cubic. Okay. That is right it is going positive, then negative then positive.
I: What are you referencing when you said positive, negative positive?
LUKE: It is going up, when it is going down, it is going negative and when it is going back up, it is going positive. Now I can see the cubic.
I: Based on that graph, could you give me the slope at any one point?
LUKE: No, that is beyond my reach.
I: No worries. Thanks.
LUKE: Too many years.
I: (Three lines) Do they have the same slope?
LUKE: No.
I: What is different about those lines?
LUKE: You could count the rise over the run. First one would be ½, the second would be one and the last would be two.
I: Okay. Any other argument that you could give?
LUKE: You could create an equation. Pick two points. Get your slope. Pick a point. Generate the equation.
I: Any other ways? These lines have different slopes because
LUKE: You could do y2 minus y1 …. Those are the three things that I fall back on.
I: Awesome (I provide him with the Triangle problem) What happens to the slope as the angle increases?
LUKE: Slope will get greater.
I: Angle decreases?
LUKE: Slope will get smaller.
I: Brilliant. Thanks.

Rachel

I: Define slope.
RACHEL: Rise over run. Change in the y axis over change in the x axis. For me, it is the steepness of the line or a hill. So it is the steepness of something.
I: Anything else come to mind?
RACHEL: The formula. Change in y over change in x. The grade of a road. The slope of a mountain. How much it rises over how horizontal it goes.
I: What images come to mind?
RACHEL: A ski slope.
I: What do you need to find the equation of a line?
RACHEL: Your slope and y-intercept. Or two ordered pairs.
I: Awesome. What are parallel lines?
RACHEL: Lines that never intersect.
I: How do they never intersect?
RACHEL: The have the same slope but cannot have the same y-intercept.
I: What about perpendicular lines?
RACHEL: They are lines that intersect at a right angle. So they have slopes that are negative reciprocals. Or opposite reciprocals. I don't know what you call it at the high school. You flip it and change the sign.
I: What about a tangent line?
RACHEL: That is coming back from college. That is a lines that touches. I always think of a tangent line with a circle. A tangent line is a line that touches a circle at one point.
I: What do you mean touches?
RACHEL: Like it does not go through the circle, it only touches at one spot.
I: Can other things have tangent lines? Other than circles.
RACHEL: Yes, I am sure that they can. But that is what I remembered about tangent lines. I can see that picture.
I: Does that have any relationship to slope?
RACHEL: I am sure it does but I cannot remember.
I: Did you learn it before and cant remember?
RACHEL: I have no idea actually.
I: During the first round of interviews, I asked a teacher to define slope and the teacher replied m. How would you respond to a teacher or student that said m?
RACHEL: Wow. That is like the last thing that I would think. Just because that is from the equation of the line. I would hope, but this has nothing against anyone that would say m, but I would hope that when you think of slope that you are not thinking of a formula. I think for you to get a conceptual understanding. I do not think that kids need to memorize formulas because they are never going to remember what m stands for. So I think for me, I am hoping that when my students are asked what is slope, they would not say m because that is not what they (Trails off). They should know that m stands for slope in the equation but I am hoping when you ask them what slope is they will say something like, almost a picture in their head because that is conceptual, that is understanding. Okay, so if they say, if I said slope and they said m, I would say what does that mean. That would be my response. That should be one of their responses when they are asked what is slope?

Just like I do not tell them to memorize y2 minus y1 unless they understand what that is. Why are you subtracting y values? They should understand if I subtract the y values that is actually the vertical. So you know, that is just understanding why you do certain things because they will hold onto it longer. So m is just memorizing. So for me, if someone said m it was just memorizing the y = mx+b. That is not the purpose of learning how to do lines, equations. That is just memorizing the pieces of an equation without understanding what they really stand for.
I: Great response.
RACHEL: That is just bizarre.
I: (I explain the Tom problem)
RACHEL: Can I make up anything?
I: Absolutely. But please state how fast he is walking.
RACHEL: So he is walking to school like any normal day and he is texting his friends on his phone. His friend says “Come on over and grab me”. So he has to backtrack (Tracing negative line segment). So he walks a little bit back towards his home. Just a little bit to go pick up his friend at his house. Then they realize they are late and have to walk a little bit faster than he was at the beginning. Then they get to school and that is why it levels off.
I: How did you know it was a faster pace?
RACHEL: The slope is steeper. This is like a consistent slope (First positive line segment). I don’t know how fast or slow, well I guess I could figure it out.
I: How? How fast is he walking during that first portion?
RACHEL: Two. Two meters per second.
I: How?
RACHEL: One of the pairs is (0,0) and the other is… So rise over run. Up 20 and over 10. 20 over 10 is just two.
I: SO you used rise over run with the (0,0). Would you use rise over run with the negative and other positive portion of the graph?
RACHEL: Sure. You could say he went down twenty, down forty, down sixty and over 10, 20. So negative 30 (Does calculation in her head)
I: So, negative 30?
RACHEL: Yes. (Shakes head)
I: So that (Rachel interrupts me)
RACHEL: Negative three. Sorry. (Laughs) I was like that cant be right if this was positive two.
I: So based on that you would use rise over run?
RACHEL: I would. But I think if I was doing it with lower kids, I would not teach it that way. I would so them rise over run but I would tell them not to use it because a lot of times they will get stuck on the standardized test because this is going up (The vertical) in different increments than this (the horizontal) is and it catches them. So a lot of times they would say that this is up one and over one, so the slope is one. And I am like no, no, no, they went up 20 and over 10. So the slope is two. So a lot of times I would tell them just to double check, pick two ordered pairs and calculate the slope using the formula and get the same as your got with rise over run.
I: So is that graph realistic.

RACHEL: I would say yes but I have no idea how fast I can walk meters per second. But in general, sure! (Talks about meters and second)
I: Excellent. (Explain Angelica problem)
RACHEL: So this why I have the paper? (Reads problem and solves) So she saves five dollars each week and she starts with 150 for her birthday.
I: How did you approach this problem? Did you use rise over run again?
RACHEL: I just graphed the two points. I did not use rise over run. I subtracted the x and y values. I used the formula. So then I got 15/3 which is 5. Then I plugged into the equation of y=5x+b and solved for b and got 150.
I: (Explain cubic function) Does that function have a slope?
RACHEL: The slope is not the same, not consistent. That is why it is not linear. It is increasing, then decreasing, the increasing. You could find the slope between two ordered pairs but it would be different at each point on the graph.
I: Could you calculate the slope at any one value.
RACHEL: I am sure that you can but I don't know how.
I: Okay, thank you. (She checks her watch) (I explain the three lines)
RACHEL: They all have different slopes. They have the same y-intercept. They all have positive slopes but they are not the same slopes.
I: How did you know they were different slopes?
RACHEL: If they were the same slope, they would be laying on top of each other or they would be parallel.
I: Awesome. Last one. (Triangle Problem)
RACHEL: (Thinks about it) As this angle gets bigger, the slope… I don't know. I honestly don't know.
I: Could you find the slope of that line if I gave you the angle measurement and a point?
RACHEL: I probably could. I know I should be able to but I don't remember. It has been too many years.
I: No problem. Thank you so much.