# STUDENTS' UNDERSTANDING OF QUADRATIC FUNCTIONS: A MULTIPLE CASE STUDY

by

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A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Curriculum and Instruction

Charlotte

2011

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# ABSTRACT


VOLKAN SEVIM. Students' understanding of quadratic functions: A multiple case study. (Under the direction of DR. VICTOR V. CIFARELLI)

The purpose of this study was to explore how individual students understand various aspects of quadratic functions such as quadratic growth, quadratic correspondence, quadratic graphs, vertex points, x-intercepts, y-intercept, line of symmetry, parameters of general quadratic functions, and quadratic equations, in order to provide detailed characterizations of the scope and depth of students' understandings of these concepts. To this end, a qualitative multiple case study methodology was used. Semi-structured, video recorded, in-depth interviews with three university students and one high school student, who either recently completed a formal pre-calculus course or were currently enrolled in a pre-calculus course, constituted the study's primary data source. Students were given a twelve problem task instrument and their problem solving activities were analyzed using cognitive constructivist theories in which the participants' acts of understanding, bases of understanding, and cognitive structures were explicated and modeled.

The first case, pseudo named Ken, yielded an understanding of *quadratic function as a unique type of equation where one "solves for y*." The analysis of the second case, of Sarah, led to the emergence of an understanding of *quadratic function as a unique type of graph where every value of x has only one y value on the parabola shaped graph*. And, three of all four cases suggested a way of understanding *quadratic functions as a collection of things that are compartmentalized in multiple ways*. In addition, all four cases confirmed some of the major findings in the literature on students' understandings

of functions. All four cases were compatible with both the action view of functions and the compartmentalization of function knowledge. They thus added to the existing findings in the literature by providing holistic fabrics of common ways of understanding *quadratic* functions.

These findings emerged through several cross analyses between and among the multiple cases of the study. The design of the study allowed this multiple layers of analyses, while yielding rich descriptions and explanations throughout.

# ACKNOWLEDGEMENTS

I would like to first thank my advisor Dr. Victor V. Cifarelli for his help, support and advice throughout my studies at the University of North Carolina at Charlotte. I am indebted to him for spending countless hours with me since I began my studies four years ago. I can never forget his contribution to my academic and professional growth.

I would also like to thank all my professors at the University of North Carolina at Charlotte, including Dr. Greg Wiggan, Dr. David Pugalee and Dr. Jae Hoon Lim for their significant intellectual input. I would like to thank my advisor Dr. Glendon Blume at Penn State University, who had notable impact on my growth as a student, a teacher and a scholar. I would like to thank my colleague Dr. Charlene Sheets, with whom I shared the same office in the last academic year as a Visiting Lecturer and Teacher-in-Residence. I am indebted to her for the continuous help, support, and acts of kindness she showed me throughout the year. I also benefited greatly from my many conversations and interactions with Dr. Adalira Saenz-Ludlow. I also would like to thank my close friends, Kelli Allen, and Larry Drucker, for their continuous encouragement and support during my six years in Charlotte, North Carolina.

Finally, I would like to extend my gratitude to my brother Hakan Sevim, my father Mehmet Sevim, my mother Emine Sevim, and my grandmother Munevver Aygokce for their endless love and care.

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# CHAPTER 1: INTRODUCTION AND PURPOSE OF THE STUDY

As the current National Council of Teachers of Mathematics standards document recommends, (NCTM, 2000) helping students make connections among mathematical concepts in K-12 mathematics classes should be a critical component of the mathematics curriculum and instruction. To this end, it is important for researchers and educators to have a substantial knowledge and understanding of how students think and make connections among mathematical concepts. In order to guide instructional practice on effective methods that bring about the desired connections and deep understanding of mathematical ideas, mathematics education researchers are investigating students' understanding of various mathematics concepts. Although a substantial amount of research has been conducted concerning the understanding of functions, research on quadratic functions has been scarce. There have been only a few studies conducted in this field. While these studies provided much needed knowledge and understanding of how students think and make connections among certain aspects of quadratic functions, they have not provided a complete explanation of how individual students think, understand and make connections among multiple properties of quadratic functions.

According to Ellis and Grinstead (2008), the studies that have investigated students' understanding of quadratic functions provided insights into: (1) the ways students reason about the influence of the parameters $a$, $b$, and $c$ in $y = f(x) = ax^2 + bx + c$ on the graph of a quadratic function; (2) students' overgeneralization of the properties of linear functions to quadratic function tasks; and (3) students' misconceptions and difficulties in making connections between algebraic and graphical representations of quadratic functions, and between quadratic equations and quadratic functions.

In order to further contribute to our developing understanding of how individual students think, reason and understand various aspects of quadratic functions, additional research is needed that provides both a more holistic as well as a more complete description of students' understanding of quadratic functions. Thus, this study was designed to address these by providing detailed characterizations of the scope and the depth of the participants' fabric of understandings regarding quadratic functions, including their informal reasoning such as the intuitive quadratic function models they use and meanings that they associate to the models.

The purpose of this study is to explore how individual students understand various aspects of quadratic functions such as quadratic growth, quadratic correspondence, quadratic graphs, vertex points, x-intercepts, y-intercept, line of symmetry, parameters of general quadratic functions and quadratic equations. The study is intended to contribute to our in-depth understanding of individual students' conceptions of quadratic functions through analysis and interpretation of their mathematical behaviors in an open-ended problem solving environment. Four students were interviewed in order to obtain rich descriptions, characterizations and explanations of the scope and the depth of individual students' unique fabric of understandings with respect to multiple aspects of quadratic functions.

Many research questions are yet to be answered in this area of mathematics education research. Because quadratic functions are one of the most frequently used families of functions in the 6-12 grade curriculum (perhaps second only to linear functions), and because their real world applications make them an important part of school algebra and calculus, it is important that researchers study students' understanding of quadratic functions more in-depth.

Prior research has identified some misconceptions and student errors in dealing with quadratic functions. This study aims to provide a more systematic characterization of individual students' rich fabric of conceptions or understandings, reasoning and meanings about quadratic function concepts using in-depth qualitative data (Oehrtman, 2009). It employs "extensive open-ended tasks to reveal the conceptual structures [or fabric of understandings or *basis of understandings* (Sierpinska, 1994)] that students spontaneously apply to resolve difficult quadratic function problems" (Oehrtman, 2009, p. 398). Indeed if such knowledge of individual students' scope and depth of their prior knowledge and understanding of various classes of functions were made available, educators could then make more informed decisions in teaching and curricular practices in school mathematics. This study attempts to analyze students' understandings of only one of those classes of functions: quadratics.

This study focuses on explicating students' understanding of quadratic functions through description, analysis and explanation. It investigates the ways that individual students (1) operate with various aspects and properties of quadratic functions in problem situations, (2) understand various aspects and properties of quadratic functions, and (3) make connections between various aspects and properties of quadratic functions. The study addresses the following research questions: What are students' understandings of quadratic functions? How do individual students understand and organize various aspects and properties of quadratic functions? How are these understandings constituted within situations involving quadratic functions and their properties?

These questions are posed from a cognitive constructivist theoretical perspective in the field of mathematics education. In other words, as these research questions indicate, the focus of the study is students' existing conceptions and mental processes; not social processes involved in their mathematical learning experiences in classroom settings. According to this cognitive perspective, the a priori instructional representations of mathematical ideas are not the primary source of students' mathematical knowledge (Cobb et al., 1992). Instead, students' own constructions constitute their primary knowledge source. To place this broad theoretical framework into perspective, Cobb's (2007) overview of four major theoretical perspectives in mathematics education research and practice will be provided in the next section.

Because of the qualitative nature of the research questions and the study's cognitive constructivist theoretical perspective, a multi case study, with a primary data source of two sets of 75 minute-long semi-structured clinical interviews with four participants was conducted. Students' mathematical problem solving activities within quadratic function situations were audio taped and video recorded. The students solved problems that involved both familiar and unfamiliar quadratic function situations. Their written work and self-evaluations of their mathematics background were also collected as supplementary data sources. The two semi-structured clinical interviews consisted of participants responding to a task instrument (Appendices A and B) with several non-standard problems on translations between graphical and algebraic representations of quadratic functions that require minimum quantitative calculations (Appendix B). The tasks were designed as free-response questions, and they required participants to provide written explanations to either justify their answers or to refute other choices. The participants of the study consisted of three university freshmen students who recently completed a pre-calculus course and one newly graduated high school student who recently completed a sequence of pre-calculus, AP calculus and AP statistics courses. The following section provides the relevant theory as well as the particular theoretical framework and constructs used in the study, and a review of existing literature on students' understanding of quadratic functions.

# CHAPTER 2: RELATED RESEARCH AND THEORY

Cobb's (2007) account of the current major theoretical perspectives in the field of mathematics education, which was published in the National Council of Teachers of Mathematics (NCTM)'s second handbook of research in mathematics education, serves as a useful overview that provides a theoretical context for the current study.

According to Cobb (2007), four major theoretical perspectives underlie current research and practice in mathematics education: Experimental psychology, cognitive psychology, socio-cultural theory, and distributed cognition theory. Below, each perspective is summarized in order to provide a theoretical context for the study. In-depth historical origins and more detailed accounts of each perspective can be found in Cobb (2007) and elsewhere.

Experimental psychology can be best described in terms of its strong emphasis on using quantitative methods for understanding the effects of manipulable independent variables, found in classrooms and school settings, on isolatable psychological characteristics such as student performance. According to Cobb (2007), this perspective is commonly used by educational policy makers who are concerned mostly with administrative aspects of schooling. Determining which instructional or school conditions (as independent variables) contribute to student performance is the primary focus of the educators and researchers who work within this perspective. In seeking to better manage the realities of schooling (often from a certain distance), conducting statistical studies that offer compact group data is the most viable approach for educational administrators. Within this perspective, an individual student is treated as an abstract, somewhat statistically constructed, collective entity whose performance is discerned by its deviance from the group norm that measures collective performance. All educational issues can be studied quantitatively within this approach.

Cognitive psychology, on the other hand, mainly concerns with cognitive analyses of specific students' mathematical reasoning. Within this perspective, internal cognitive structures and processes are central to research. Characterization, specification and explication of these structures and the study of specific individuals' active construction of increasingly more sophisticated understandings of mathematical concepts are the major focus of most of such research programs. Under this umbrella of psychology, the constructivist perspective was originated in the early works of constructivist philosophers such as Kant, and more recently the genetic epistemology of Piaget and his colleagues in the Geneva School—not as invariant stages of development but as ongoing processes of learning. Within this theoretical perspective, learning is viewed as an internal construction and reorganization of sensory-motor as well as cognitive activity, and an individual student is considered as an epistemic individual. As Cobb (2007) points, educational factors regarding the social context of teaching and learning are often seen as unsuitable for explanation from this line of research. Mathematics educators, who conduct research with this theoretical perspective, explicate domain specific cognitive frameworks, or develop general learning theories of mathematics. The results of such studies are especially valuable to practitioners, who need to know their students' current knowledge and understanding of various mathematical topics, as well as researchers and curriculum developers, who need to specify experiential and cognitive developmental steps toward important mathematical ideas that all students should learn and to characterize significant shifts that occur in students' mathematical reasoning as they study mathematics.

Socio-cultural theory, originating from the works of Russian psychologists Vygotsky and Leont'ev, focuses on the notion of participation in established cultural practices. The main area of study is the nature of the supported progressive participation of members in established (and evolving) cultural practices of a society (Cobb, 2007). In this perspective, cognition is viewed as inherently social, and learning is characterized as an internalization process from the 'inter-mental' social interactions to 'intra-mental' individual thought. Such internalization processes are believed to be accomplished through full participation in the existing cultural practices and appropriation of signs and other artifacts of the society. Mathematics educators, who conduct research within this theoretical perspective, study forms of reasoning inherent in various cultural practices. The study of mathematical reasoning and cognitive development in out-of-school settings, in daily life, is very relevant to this perspective. According to Cobb (2007), throughout the years since its inception in the works of Vygotsky, there has been a shift of focus within this perspective from studying the relationships between social interaction and cognitive development to the relationships between cognitive development and cultural practices (as seen in Leont'ev's work). This perspective considers individual students as individuals-in-culture.

Lastly, distributed cognition, as a theoretical perspective, is concerned with the immediate physical, social and symbolic environments to which cognition extends out.

The activities of individual or small groups of students in such environments are studied in terms of the ways their cognition is distributed over an immediate system of people and artifacts. Cobb (2007) argues that distributed cognition theorists in mathematics education see an individual student as an element of a larger reasoning system. In this tradition, the roles played by classroom norms, discourse and tool use are valued as significant cognitive resources that support learning—which is viewed as the increased relations between the material, social and symbolic resources of the immediate environment. Such classroom processes are viewed as emergent in the sense that they are not already established. Activities in real life settings are also investigated in an attempt to understand how cognition is distributed across elements of this immediate environment. Cobb (2007) writes: "In addition to questioning whether people's reasoning on school-like tasks constitutes a viable set of cases from which to develop accounts of cognition, several distributed cognition theorists have also critiqued current school instruction. In doing so, they have broadened their focus beyond mainstream cognitive science's traditional emphasis on the structure of particular tasks by drawing attention to the nature of the classroom activities within which the tasks take on meaning and significance for students" (p. 26).

As this overview suggests, for the current study, which focuses on explicating students' fabric of understandings of quadratic functions through description, analysis and explanation, and which investigates the ways that individual students operate, understand, and make connections among various aspects and properties of quadratic functions in problem situations, theoretical constructs within cognitive psychology were considered more suitable.

To address the research questions: What are students' understandings of quadratic functions? How do individual students understand and organize various aspects and properties of quadratic functions? How are these understandings constituted within situations involving quadratic functions and their properties? it was considered to be necessary to explicate domain specific cognitive frameworks using constructs from this particular theoretical perspective.

Furthermore, to obtain the desired broad description and systematic characterization of students' quadratic function conceptions, qualitative data were analyzed within the large framework of constructivist thought (and the narrower framework of cognitive constructivist thought) in mathematics education. As Noddings (1990) suggests, from this perspective we need to ask: What conceptions do students hold? What can they do with it? What are the characteristics of their reasoning? "In research this means that we have to investigate our subjects' perceptions, purposes, premises, and ways of working things out if we are to understand their behavior" (Noddings, 1990, p. 14). "In order to teach well, we need to know what our students are thinking, how they produce the chain of little marks we see on their papers, and what they can do (or want to do) with the material we present to them" (Noddings, 1990, p. 15).

Cognitive constructivism, the guiding overall theoretical framework of this study, focuses on the learners' internal mental actions and conceptions (Steffe and Kieren, 1994). Characterizing it as a cognitive position as well as a methodological viewpoint, Noddings (1990) argues that constructivism can be best offered as a post-epistemological perspective. According to Noddings (1990), constructivism is a form of cognitivism, a particular perspective in cognitive psychology. She points out that in 1960s and 1970s there has been a philosophical shift from behaviorism to structuralism and cognitivism. Building on Piaget's work on genetic epistemology, constructivist thought in mathematics education evolved throughout the second half of the past century to include many directions (von Glasersfeld, 1995). The central question however remains unchanged: how do learners, using their existing conceptions, make sense of their experiences while they are engaged in a mathematical activity? Piaget (1970) offered a convincing explanation to this, in which individuals construct their knowledge by using two cognitive processes, assimilation and accommodation, that are triggered by internal cognitive conflict. According to Piaget (1970), these two processes result in the resolution of a cognitive conflict or a mental disequilibrium, and thus, in the construction of higher level equilibriums called schemes. Piaget viewed cognitive structures as products of development. These structures were seen as not innate but as developed through the coordination of actions and reflections (Noddings, 1990). "Constructivism is rooted in the idea of an epistemological subject, an active knowing mechanism that knows through continued construction" (Noddings, 1990, p. 9). And, according to Piaget, all construction or all constructive activity begin with base structures called assimilatory structures (which are themselves constructed).

Noddings writes: "Finally, Piaget's cognitive constructivism leads logically to methodological constructivism. The need to identify and describe various cognitive structures in all phases of construction suggests methods such as the clinical interview and prolonged observation that permits us to make inferences about the structures that underlie behavior" (p. 9). As Noddings (1990) further notes, students inevitably have "multiple selves behaving in consonance with the rules of various subcultures" (p. 12).

Although not the focus of this study, the participating students indeed came from different set of experiences and were exposed to pedagogy and content about quadratic functions in varying degrees. The variety of these students' experiences and understandings of quadratic functions strengthen the purposes of providing rich descriptions and explications of large spectrums of fabrics of understandings. Through a qualitative multiple case study using purposeful selection (Patton, 1990) this research aims at "achieving representativeness or typicality of the [...] individuals," and "adequately capturing the heterogeneity in the population" (Maxwell, 1996, p. 71) through what Guba and Lincoln (1989) refer to as *maximum variation sampling*.

Perhaps the first constructivist case study in mathematics education literature that investigated the mathematical thinking of individual students is that of Erlwanger (1973). Steffe and Kieren (1994) succinctly summarized Erlwanger's study of a young child pseudo-named Benny:

> Erlwanger demonstrated the power of interpretive research as well as the need for alternative methodologies [...]. This work in fact, was also one of the first to focus on [...] the structural dynamics of an individual, as interpreted from the actions and words of Benny. (Steffe and Kieren, 1994, p. 718)

Case studies of students with different cognitive structures were also part of Piaget's genetic-developmental epistemology research, which included a particular (and useful) cognitive learning theory. The learning theory of Piaget can be summarized by its six principles. Gallagher and Reid (1981) provide a clear and detailed description of this theory. First, according to Piaget, the learning process involves more than observation and experience. As an *internal process of construction*, it requires an internal coordination of one's actions and their results. Thus, learning involves reflection on activity. In addition to performing such activities as looking, touching, moving things around, and counting, a learner must also reflect on the results of these activities in order to learn.

Second, learning is a process of continuously *reorganizing on a higher mental level* that which was initially constructed at some lower mental level. While a learner may abstract physical properties of objects by observation or experience alone, coordination and reorganization at successive hierarchical levels of cognitive structures are necessary in order to abstract rules and principles that govern objects. All of these constructive mental activities are internal and involve continuous reorganization of action (Gallagher and Reid, 1981).

Third, in Piaget's learning theory, *learning is subordinated to development*. In order for internal construction, coordination and reorganization on a higher level to occur, learners must first have the capacity to respond to new experiences. In other words, the necessary development must have already taken place before one can learn. According to Gallagher and Reid (1981), Piaget and his Geneva School define development as a spontaneous biological and psychological process, which is an integral part of a person's total growth. Once the necessary functions of the body, such as the nervous system and the intellect, develop, then the relevant constructive mental activities become possible.

Fourth, a *feedback process that involves questioning, contradictions and consequent mental reorganizations* is an essential element of learning. Often this process triggers growth in knowledge. According to this theory, learners are able to self-correct themselves through successive approximations (Piaget, 1970). When they observe contradictions between their initial expectations and the actual results of their actions, they revise their expectations accordingly. Through this process of self-correction, they eventually become "able to anticipate a correct solution from the outset" (Gallagher and Reid, 1981, p. 9).

Fifth, *questioning, contradictions and consequent mental reorganizations are often stimulated by social interaction.* Note that although this environmental stimulation is important, and often leads to acquisition of knowledge, mostly the learner's own activity and the internal construction of hierarchical levels of cognitive structures are responsible for learning. Sixth, u*nderstanding lags behind action.* According to Piaget, understanding and conscious realization are not sudden insights. Rather, they are products of a continual process of reorganization (Piaget, 1970). One of the pedagogical implications of this last principle is that, as Gallagher and Reid (1981) write, if "children are able to perform tasks successfully without understanding why what they are doing works" (p. 10), then the goals of instruction must be carefully examined and revised.

These six principles indicate that in addition to experience, reflection and internal construction are necessary for acquisition of knowledge. They point to the critical role placed on the learner's own activity. One of the implications of Piaget's theory of learning in designing the current study is that, as we shall see in later sections, the interview tasks were developed around what students were expected to currently know and were capable of doing. They had access to the tasks in terms of being able to act on them, and they were able to meaningfully reflect on their problem solving actions and processes. In addition, the students were purposefully selected so that they already had the necessary basic prior knowledge.

As the above theory of Piaget suggests, children are able to observe and successfully manipulate physical objects through experience at the practical (action) level. However, they may have not yet constructed or abstracted the relationships, principles or rules that govern those objects. If this is the case, which is believed to be so, we educators should survey their current levels of understanding (or understandings) and design learning and teaching environments accordingly. Cobb's (2007) suggestion regarding the importance of the development, testing and revising of designs that support students' mathematical learning attests to the need for knowing what students understand about particular mathematics concepts in the design of effective learning environments.

This theory, especially its sixth principle, can be applied to the teaching of quadratic functions in a pre-calculus class. For example, we can assert that most pre-calculus students can successfully use the so-called "completing the square" technique in order to convert a quadratic expression from its general form $(ax^2 + bx + c)$ to its standard vertex form $(a(x - h)^2 + k)$. Students may first observe their teacher completing the square for them, talking out loud: "I make sure that the leading coefficient of the quadratic polynomial is 1; then I take the middle coefficient of the three-term expression and divide this middle coefficient by two; then I take the square of this half, and add and subtract the resulting square in between the second and third terms; and finally turn the first three terms (of the now five-term expression) into a complete square and take care of the remaining numbers at the end." "What happens if the leading coefficient is not 1?" "How do you know what to write inside the completed square at the end?" "What do you do with those fourth and fifth terms at the end?" "Or how do you take care of them?" "Why do you add and subtract the squared term in between the second and third terms?" or "Do you add first or subtract first?" might be some of the questions that students could ask.

This instructional method of telling, demonstrating, modeling, or showing a sample solution might be considered as very traditional. Now, let's incorporate a more authentic and difficult activity involving the completing the square technique. As Piaget's learning theory suggests, students' understanding lags behind their action and they can successfully carry out tasks without understanding the underlying principles and rules that govern mathematical symbols and objects (i.e., without knowing why those rules work). As an example, consider the following task that requires the same solution technique. "Sketch the graph of $y = x^2 + 4x + 9$ without using a graphing calculator, and by transforming the parent graph $y = x^2$." To start, some students may erroneously try to shift the parent graph upward vertically by 9 units and then apply a dilation of 4 to the graph.

In order to establish the falseness of the vertical shift by 9 units and dilation by 4, the teacher could quickly flash the correct graph on an overhead calculator screen (with a marked y-axis with point $(0, 9)$ in bold letters somewhere far above the correct graph) that can point at the contradiction between their expectations and the actual graph of the function. After students are convinced of the falseness of the proposed solution, a feedback process that involves further questioning, more contradictions and consequent mental reorganizations through interaction and discourse can be encouraged. Constructions are made by personal puzzlement, goal-setting, and testing-hypotheses (Noddings, 1990).

This instructional activity is more compatible with both Piaget's six principles of learning and the recommendations of the current reform-based documents such as the *Principles and Standards for School Mathematics* by the National Council of Teachers of

Mathematics (NCTM, 2000). According to NCTM (2000), students should "understand relations and functions and select, convert flexibly among, and use various representations for them," and they should "understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions" (p. 296). After students are convinced that the correct graph is not a result of vertical translation by 9 units, a whole class or small group discussions could ensue. The classroom teacher could also suggest that in $y = x^2 + 4x + 9$ students break 9 into $4 + 5$ (using their 'number sense') or try an easier related problem such as graphing $(x + 1)^2$, $(x - 1)^2$, or $(x + 2)^2$ by hand and by transforming the parent function $y = x^2$. He or she could also suggest that students compare the problem to the binomial expansion $(a + b)^2 = a^2 + 2ab + b^2$.

Although these activities are more compatible with constructivist theories of learning and they may provide students with opportunities to reflect on the results of their activities and continuously reorganize their conceptions of binomial expressions and their squares on higher mental levels, the outcomes of students' work will depend on what they currently understand about the constituent parts of the activities and the mathematical concepts and relationships involved in the task. For instance, a more mature learner of mathematics may understand $x^2 + 4x + 4$ as a single entity (as one expression) and operate on it accordingly, whereas other learners may understand $x^2 + 4x + 4$ as a 'problem' or as an 'equation.' It is the purpose of this research study to elicit student conceptions such as these that help or hinder them in various quadratic function situations.

In sum, in designing instructional activities that are consistent with Piaget's learning principles, it is hypothesized that students' own meaningful mathematical activity—stimulated by social interaction and led by a feedback process involving self-corrections and successive approximations—would play a central role toward the goal of internal construction of the underlying relationships between symbols. Although teaching is not a focus of the proposed study, the significance of ascertaining students' conceptions, that are meaningful to them, becomes more transparent when we keep this practical goal in mind.

In providing a rationale for their choice of students' understanding of quadratic functions as a research domain, Ellis and Grinstead (2008) succinctly summarize the existing research literature on the topic:

> Quadratic functions served as an appropriate domain both for its extension beyond linear functions, thereby providing a site for studying students' generalizations across families of functions, and for its relative lack of attention in the literature compared to work on linear functions. Studies focusing on quadratic functions have mainly detailed students' difficulties in a few key areas, including (a) connections between algebraic, tabular, and graphical representations, (b) a view of graphs as whole objects, (c) struggles to correctly interpret the role of parameters, and (d) a tendency to incorrectly generalize from linear functions. (Ellis and Grinstead, 2008, pp. 277-278)

These studies and other broader studies on students' understanding of quadratic functions will be discussed next.

Mathematics educators and researchers have offered useful insight into teaching quadratic functions (Bosse and Nandakumar, 2000; Buck, 1995; Craine, 1996; Edwards and Ozgun-Koca, 2010; Edwards, 1996; Gibbs, 2006; Moore-Russo and Golzy, 2005; Movshovitz-Hadar, 1993; Olmstead, 1995; Winicki-Landman, 2001). These practical suggestions included the exploration of the parameters of a quadratic function, moving from geometric to algebraic representation of a quadratic function, exploring the graph of a quadratic function through graphs of lines, and using historical development of the concept as a motivating factor.

However, research on this important mathematics concept has been scarce. A few studies documented students' difficulties with translations (i.e., horizontal shifts) when manipulating the graphs of quadratic functions (Zazkis, Liljedahl, & Gadowsky, 2003; Eraslan, 2008). Ellis and Grinstead (2008) provided some explanation of students' understanding of the roles played by the parameters in a quadratic function. For example, they found that two thirds of their eight participants from an urban high school in a Midwestern city in the United States identified the parameter $a$, in $y = ax^2 + bx + c$, as the "slope" of the parabola. These students' grades in their advanced algebra or trigonometry classes were in the A-C range, and they were identified as more articulate than their peers.

Schwarz and Hershkowitz (1999) and Buck (1995) documented students' tendencies to generalize from linearity. They found that students often use linear interpolation and extrapolation in quadratic situations that require non-linear interpolation and extrapolation. Inappropriate generalization from linearity was also documented in Zaslavsky's (1997) seminal study on students' understanding of quadratic functions. These studies have mostly explicated student misconceptions regarding quadratic functions. Leinhardt et al. (1990) offered a definition of misconception as "incorrect features of student knowledge that are repeatable and explicit" (p. 30).

Thus, prior research has identified some misconceptions and student errors. Sierpinska (1994) writes: "… how would one practically check that a person's understanding is not contradictory with any statement of the theory? There may be an infinity of them. It is much simpler to prove that a student's understanding is not perfect: one contradiction would suffice. This is why the mathematics education literature is full of stories of students' 'errors,' 'lack of understanding,' 'misconceptions,' ' misunderstandings,' etc. Accounts of good understanding are rare, and those that exist are often poorly justified" (p. 113). To offer a richer and broader understanding of the ways students think and understand about quadratic functions, the current study provides a systematic characterization of students' conceptions, reasoning and meanings about quadratic function concepts.

In a study involving over eight hundred tenth and eleventh grade students, Zaslavsky (1997) investigated conceptual obstacles that may impede students' understanding of quadratic functions. All participating students had studied quadratic functions in their mathematics classes at most six months prior to the study. The students were given a questionnaire with several non-standard problems on translations between graphical and algebraic representations of quadratic functions that required minimum quantitative calculations. The tasks were designed as multiple-choice items, and they explicitly required students to provide written explanations to justify their answers and to refute other choices.

Zaslavsky's (1997) study can be considered as the first systematic study on this topic; and it is used as a backdrop for the subsequent (few) studies on quadratic functions. It is important for the current study to build on Zaslavsky's findings on students' conceptual obstacles, or the stumbling blocks in their thinking and problem solving, because they provide some of the central components of most students' understanding of quadratic functions.

The results of Zaslavsky's (1997) study revealed 5 conceptual obstacles surrounding students' experiences with quadratic functions: *1) The interpretation of graphical information (pictorial entailments)* involves the consideration of only the visible part of the graph of a quadratic function and not taking into account its infinite domain. For example, treating a graph as a picture rather than a symbolic representation of a function with analytical properties led students to infer that there was no y-intercept when the y-intercept did not show on the graph. *2) The relation between a quadratic function and a quadratic equation* also appeared to impede students' understanding of quadratic functions. The underlying obstacle relates to equivalent quadratic equations, which have the same truth-value for any value of the variable in the two equations. Zaslavsky (1997) noted that students are usually taught that "if two equations are equivalent they can be treated as if they are the same" (p. 31). In the case of quadratic functions however, two quadratic functions with the same x-intercepts differ in their values for all the other $x$ values. In her study, many students constructed a quadratic function based on only the given x-intercepts and failed to check whether a third point was on the function. Therefore, they treated quadratic functions as quadratic equations.

*3) The analogy between a quadratic function and a linear function* refers to students' "excessive adherence to linearity" (p. 33). Despite the fact that no three points on the graph of a quadratic function are collinear, many students in the study argued that the midpoint of a line segment connecting two points of a parabola is also on the parabola. Zaslavsky's (1997) account for this explanation combines students' excessive adherence to linearity with the pictorial entailments of graphs (e.g., if the relevant part of the parabola looks like a line, it can be treated as a line). Another obstacle related to linearity is students' over-generalization of the characteristics of lines (e.g., constant slope) to quadratic functions. Zaslavsky (1997) posited that the common use of the letters $a$, $b$ and $c$ as coefficients of the standard form of quadratic functions and the letters $a$ and $b$ as coefficients of linear functions might have caused students to find the difference quotient between two points on a parabola in order to find the value of the leading coefficient $a$.

*4) The seeming change in form of a quadratic function whose parameter is zero* is shown to be an obstacle when students dealt with functions with an equation of the form $y = ax^2 + bx$. Students argued that this quadratic function does not have a y-intercept because it does not have a "$c$" value. And the last obstacle, *5) The over-emphasis on only one coordinate of special points*, is related to students' familiarity with finding only the missing values of the $x$ and y-intercepts where the other coordinate is fixed-zero. When asked to determine whether two quadratic functions with equations $y = ax^2 + bx + 1$ and $y = ax^2 + bx + 5$ have the same vertex, many students responded positively, ignoring the y-coordinate of the vertex.

Zaslavsky contends that most of the above conceptual obstacles originate from the prior formal instruction that students have received. According to Zaslavsky (1997), the students seemed to lack experiences with visualizations of quadratic functions and in making connections between the symbolic and graphical representations of functions.

Zaslavsky provides two additional observations. In the study, students preferred translating from equations to graphs over translating from graphs to equations. For example, in most cases students used the parameters of the given equations to check the graphs. And although they were able to find the line of symmetry of a parabola by reading off of the graph, when determining whether certain points were on the graph or not, students were not able to utilize the line of symmetry. Zaslavsky concurs with Vinner and Dreyfus' (1989) notion of *compartmentalization of knowledge*, which seemed to account for students' lack of ability to use the line of symmetry to solve other problems.

Although not particularly focusing on quadratic functions, another study that investigated students' conceptual understanding of functions is Schwarz and Hershkowitz's (1999) large research and development project, which spanned over twenty years. Schwarz and Hershkowitz's (1999) study was developed in three consecutive cycles of curricula, the last of which incorporated an interactive learning environment based on multi-representational software and high-level discursive activity. The focus of the study was to characterize ninth graders' 'concept images' of function (Vinner and Dreyfus, 1989). As part of this research and development project, authors report their findings on their comparisons between the nature of students' function concept images in the last curricular cycle and of those developed in the previous, more traditional cycles. Each cycle was based on the same syllabus for a yearlong course that met 2 or 3 hours per week.

The first curricular cycle of the large project was completed in 1991, and it was traditional in the sense that the set-theoretic definition of function was presented at the outset and students worked on examples in their textbooks. The textbook examples were short and rarely emerged from problem contexts. The second cycle was similar, with the exceptions that it included more examples of functions besides linear and quadratic functions (such as polynomial, absolute value and greatest integer functions) and that its development was informed by research findings from the studies of the first cycle. It also included different representations of functions and encouraged some exploration of problem situations. Finally, the third cycle, which was named the Functions Project, "was based on the cumulative experience drawn from the previous two cycles and on research and theory concerning the role of multi-representational software in learning the function concept" (p. 368). In this cycle, students were first encouraged to explore problem situations involving functions and then they were given a short informal definition of function. Students had access to either multi-representational software or a graphing calculator. When exploring problem situations, students were free to choose a function representation that they wanted to use or to link different representations as they wished. They worked in small groups and shared their findings in whole-class discussions.

The Schwarz and Hershkowitz (1999) study is important in that it explains some of the 'frames of reference' that students use in coping with unfamiliar function tasks. Specifically, in characterizing students' understanding of functions, the researchers explored three aspects of function 'concept images' (Vinner and Dreyfus, 1989), which they named as: *Prototypicality, part-whole reasoning*, and *attribute understanding. Prototypicality* refers to the prototype functions that students use in handling various function tasks. According to the authors, prototypes are specific examples of a concept that serve as frames of reference in coping with unfamiliar tasks. The two most common prototypical examples of the function concept are linear and quadratic functions. These prototypes have all the critical attributes of the function concept as well as some additional self-attributes that are unique to them. Schwarz and Hershkowitz (1999) investigated whether students used prototypes exclusively or whether their use of prototypes was beneficial in handling new examples. The second aspect of students' function concept images, *part-whole reasoning*, refers to the ability to recognize a *representative* as belonging to a function representation. A representative is defined as a partial embodiment or display of a function representation. And the third aspect, *attribute understanding*, is the ability to recognize the invariants among representatives across representations. "It is important to note, though, that because of the difficulty of drawing graphs of functions and computing their values without appropriate tools, linear and quadratic functions have commonly been the only families of functions taught systematically" (p. 367). "[It is not clear] however, whether reference to linear or quadratic functions is a result of their central role in the teaching process or results from their intrinsic prototypical natures" (p. 367).

While the Schwarz and Hershkowitz (1999) study provided a broad understanding of how students see functions and their representations, Zazkis et al. (2003) offered an analysis of a particular property of functions, horizontal translation, in the context of quadratic function problems. Zazkis et al. (2003) examined how teachers and students think about horizontal translations in the context of quadratic functions. Ten eleventh and twelfth grade students, fifteen pre-service secondary teachers and sixteen in-service secondary teachers were asked to first sketch the graphs of $y = x^2$ and $y = (x - 3)^2$ and compare their answers with the graphs of these parabolas displayed on a graphing calculator screen, and then explain the relationship between the two graphs. Building on the prior research finding that for many students horizontal translation of functions is more problematic than other types of transformations (Baker et al., 2000; Eisenberg and Dreyfus, 1994) the authors searched for possible sources of students' difficulties with the direction of the horizontal translation of functions and looked for ways to remedy the situation.

Zazkis et al. (2003) reported that half of the students participated in their study sketched the graph of $y = (x-3)^2$ incorrectly by translating the graph of $y = x^2$ three units to the left along the x-axis. These students attributed the inconsistency between their expectations (that the graph of $y = x^2$ should move three units to the left) and the correct graph of $y = (x-3)^2$ on the calculator screen to the fact that they forgot to "do the opposite." Those students who graphed the parabolas correctly offered no explanation for how the two graphs are related. Most of the students simply accepted the rule that if a number is subtracted from $x$ "inside the parenthesis," then the graph shifts to the right; and if a number is added to $x$ "inside the parenthesis," then the graph shifts to the left. All of the students thought the location of the graph of $y = (x-3)^2$ is counterintuitive and is the opposite of what they would expect.

The teachers on the other hand, sketched the two graphs correctly. As Zazkis et al. (2003) state: "Unlike the uniformity in students' tendency to rely on memorized rules, there was more variety in teachers' responses to the interviewer's request to explain the movement of the parabola" (p. 441). Nevertheless, the majority of the teachers referred to the same rule of "opposites" that the students did. Whereas in vertical translation, adding a number moves the graph up and subtracting a number moves it down, in horizontal translation, adding a number moves the graph to the left and subtracting a number moves it to the right. Some of the teachers plotted various points, and preserving the shape and symmetry of $y = x^2$, connected the points to sketch the graph of $y = (x-3)^2$. And half of the teachers argued that the zero of $y = (x-3)^2$ is $x = 3$, which is also the vertex, and therefore the rest of the graph could be sketched by using the notion of symmetry and preserving the overall shape of the graph. The pre-service and in-service teachers' explanations did not differ significantly from each other and most of them were not completely satisfied with the reasons they gave for the counterintuitive direction of the horizontal translation of $y = x^2$.

These findings in Zazkis et al. (2003) suggest that the ability to translate $y = x^2$ on the horizontal axis correctly does not imply that the participants understand the underlying reasons for the direction of the horizontal translation. They viewed the direction as counterintuitive and their reasons involved memorization, point-wise graphing and attending to the zeros of functions, and none of the participants (students or teachers) claimed a satisfactory justification.

Zazkis et al. argue that one of the sources of the participants' intuitions might be a conceptual obstacle (Zaslavsky, 1997) that they possess. That is, the over-generalization of the fact that adding 3 to a number results in the positive translation to the right on the number line and subtracting 3 from a number results in the negative translation to the left on the number line (e.g., $x \to (x-3)$). To substantiate this claim for a potential obstacle in students' (counter) intuitions of horizontal translation of functions, Zazkis et al. provide the following detailed explanation.

"We suggest that the main source of difficulty here is in seeing this algebraic replacement [substituting $(x-3)$ in place of $x$] as a transformation ($x$ moves to $(x-3)$) and trying to infer the geometric transformation, the movement of the graph, from the algebraic substitution. That is to say that the transformation $f(x) \to f(x-3)$ is simplified to be viewed as $x \to (x-3)$. Such a view is in accord with what Hazzan

(1999) described as *reducing abstraction*, which is a strategy used by learners to cope with complexity" (p. 445).

Therefore, the over-generalization of the learners' experiences with number facts to reduce $f(x) \to f(x-3)$ into $x \to (x-3)$ with an exclusive focus on the algebraic representation of functions is what Zazkis et al. offer as the source of learners' difficulty, which differs from what Eisenberg and Dreyfus (1994) view the source of difficulty as visual processing of information or what Baker et al. (2000) view the source as the complexity of mental construction needed to process a horizontal transformation.

To overcome the above obstacle, Zazkis et al. propose a pedagogical approach that introduces translations in the context of geometric transformations before students study functions. It is argued that this pedagogical route can be much more beneficial for the conceptual understanding of function transformations. This curricular sequence is also supported by the NCTM's (2000) geometry standards that introduce informal transformations such as flips, turns and slides in grades PreK-2. The authors argue that the traditional instruction of translations of functions in the context of algebraic representations of functions may be the root of the above conceptual obstacle of reducing the translation $T((x,y)) = (x+3,y)$ or $f(x) \to f(x-3)$ into $x \to (x-3)$. However, studying translations of functions via geometric transformations can help students to develop a sense of the translation $T((x,y)) = (x+3,y)$ of $y = x^2$, whose image can be described by the equation $y = (x-3)^2$.

While Zazkis et al. (2003) offer these useful insights into how students think about horizontal translations of quadratic functions, another study that investigated students' understanding of quadratic relations is Vaiyavutjamai and Clements (2006).

Vaiyavutjamai and Clements studied 231 ninth graders' understanding of quadratic equations in Thailand. The authors investigated: (1) students' ability to solve elementary quadratic equations and (2) the effects of "traditional teaching approaches" on students' ability to solve quadratic equations and on their 'relational understanding' (Skemp, 1976) of quadratic equations. Vaiyavutjamai and Clements administered a pre-test (before the students received instruction on quadratic equations) and a post-test (after the students received instruction on quadratic equations) that consisted of 18 equations with varying forms and difficulty levels. They observed all of the participating students in six classrooms, and interviewed 18 of them before and after instruction. The two interviews with the students included the following four tasks: Solve the quadratic equations: (1) $(x-3)(x-5)=0$; (2) $x^2-x=12$; (3) $x^2=9$; and (4) $2x^2=10x$.

The instruction that students received consisted the common instructional sequence of review, introduction, model example, seatwork, and summary. The rationale for studying the effects of such traditional instruction was that if traditional instruction is the most widely used approach around the world (Hiebert at al., 2003), then the effects of such approach on students' understanding of quadratic equations need to be known. The instruction included solving quadratic equations by factorization (and the application of the "null factor law"), completing the square and using the quadratic formula.

To assess students' performances on the pre- and post-tests, the authors gave a score of 1 to a correct response and 0 to a wrong or no response. Overall, the 231 students' mean score (out of a maximum score of 18) on the pre-test was 1.75 and 6.17 on the post-test. To assess students' 'conceptual understanding' (Hiebert and Lefevre, 1986) of solving quadratic equations, Vaiyavutjamai and Clements developed a rubric that draws distinctions between different levels and aspects of students' understanding. The scores for each of the four interview tasks ranged between 0 and 4; 0 indicating a lack of comprehension of a task and 4 indicating correct solution of an equation with the use of appropriate methods, relating the answers to the equation and checking the answers if they are indeed the solutions of the equation. The 18 interviewees' pre-interview mean score was less than 1 and post-interview mean score was less than 2. Overall, the interviewees were able to provide a correct answer to most of the four interview tasks after the instruction (as reflected in the post-interview mean score between 1 and 2); however they were far from having a conceptual understanding of their solutions, which would be reflected by a score of 3 or 4.

In addition to the above findings, the authors also observed a misconception about students' conceptions of variables in quadratic equations. Most of the interviewees thought that in $(x-3)(x-5)=0$ the two $x$'s stand for different variables. They substituted $x=3$ in $(x-3)$ and $x=5$ in $(x-5)$, and argued that $(3-3)(5-5)=0$. Vaiyavutjamai and Clements argue that "at the post-teaching stage a minority of students in the six classes grasped the concept of variable in the context of quadratic equations" (p. 72).

In sum, the literature on students' understanding of quadratic functions provide useful information about how students think about quadratic equations, x-intercepts, prototypical graphs, quadratic function translations, domains and representations of functions. Although these studies have provided these valuable findings that helped deepen our understanding of students' (mis)conceptions of quadratic functions, more research is needed in order to describe fully what and how individual students understand various aspects of quadratic functions.

To this end, in the current study, the notion of *act of understanding* (Ajdukiewicz, 1974; Sierpinska, 1994) is viewed to be more compatible with the purpose of explicating students' fabric of actions in quadratic function situations as well as the fabric of their understandings of quadratic function concepts. The construct of *act of understanding* helps interpret how and what the four students were thinking about various aspects of quadratic functions while solving quadratic function tasks. Briefly, an act of understanding is defined by Ajdukiewicz (1974) as "an act of mentally relating the object of understanding to another object" (cf., Sierpinska, 1994, p. 28). Sierpinska (1994) further develops Ajdukiewicz's (1974) theory, and defines the 'another object' in the above definition as the *basis of understanding*. And, lattices of acts of understanding form the *process of understanding*. These theoretical constructs and other related theories will be further discussed in detail in Chapter 3.

In the following section, the details of the design and research methods of the study that helped explicate these bases of understandings of quadratic functions are discussed.

# CHAPTER 3: RESEARCH METHODS

## Research Design

To inform practice about effective teaching strategies and curricular resources that are built on students' own mathematical conceptions (or understandings) and the connections they make among them, mathematics education researchers have studied students' understanding of various mathematical concepts in-depth. These valuable efforts however, have not shed light on all concepts of school mathematics that are considered to be important. For example, research on students' understanding of quadratic functions, an important mathematical concept in school curriculum, have been scarce and have investigated the topic rather superficially. Providing new and more extensive insights on issues of importance is a significant contribution to the field of research in mathematics education (Heid, 2010). Thus, unlike earlier work documenting students' misconceptions, the purpose of this study is to describe and explain how students think and reason about quadratic functions. Studying students' thinking also enables us "to see how the socially defined, mathematical conventions expressed in [textbooks] resonate with or deviate from student thinking" (Confrey, 1991, p. 124). Such insight is especially useful for mathematics teachers and curriculum developers in designing effective resources and learning environments for students.

In this chapter, first, the research design is discussed in-depth. Second, a description of the characteristics of participating students is given, and the rationale for their selection is clarified. Third, data collection and data analysis methods are explained in detail. In addition, the results of a pilot study that design of the current study are discussed. Finally, the chapter closes with a discussion of issues regarding validity, generalizability and ethics.

This study investigates how students understand quadratic functions. To elicit data that could provide answers to the kind of research questions proposed in this study, a qualitative case study research design was used. Case study design was chosen as an appropriate method because the study demanded an in-depth analysis of individual students' thinking, reasoning and understanding of quadratic functions in a holistic manner. For the purposes of obtaining sufficiently rich and varied data, four participants (cases) were studied.

Semi-structured, video recorded, in-depth interviews with three university students and one high school student, who either recently completed a formal pre-calculus course or were currently enrolled in a pre-calculus course, constituted the study's primary data source. Each student was considered to have a unique set of understandings about quadratic functions, and therefore were treated as an individual 'bounded system,' or a case (Creswell, 1998; Maxwell, 1996; Yin, 1989; Stake, 1995), and thematic analyses within and across individual cases were conducted. As Creswell (1998) succinctly put: "When multiple cases are chosen, a typical format is to first provide a detailed description of each case and themes within the case, called a 'within-case analysis,' followed by a thematic analysis across the cases, called a 'cross-case analysis,' as well as assertions or an interpretation of the meaning of the case" (Creswell, 1998, p. 63).

As Noddings (1990) emphasized, students need building materials, patterns, tools, and sound work habits in order to construct abstract mathematical relationships. What are their current building materials, patterns, tools or work habits with regard to quadratic functions? The answers to this central question of constructivist perspective and other general research questions of this kind were believed to be obtained only from data rich in context and embedded within the students' own individual problem solving activity (Maxwell, 1996).

Creswell (1998) places qualitative data sources into four categories: Observations, interviews, documents, and audio-visual materials. Observations provide opportunity to record verbal and visual information in real time at original settings; interviews allow the researchers to elicit specific types of information from individual participants; documents offer the original words of the participants; and audio-visual materials supply rich images that help easily visualize and share reality (Creswell, 1998). In the current study, two of the above categories, interviews and documents, were used to obtain the necessary rich data. In addition, video recording of all the interviews, where students engaged in problem solving activity, allowed another category of data source, i.e., the audio-visual materials. According to Creswell, there are also various types of data under each of the four categories. The current study included two task-based semi-structured interviews with each participant, a survey of individual mathematics background, students' work on the task sheets, and transcriptions of audio and video recordings of problem solving events. Thus, multiple data sources were used in order to obtain rich data on each case.

Participant Selection and Data Collection Methods

Four students, who recently completed a formal pre-calculus course, constituted the cases of this study. After gaining some experience with linear and quadratic functions in middle school or high school algebra and geometry classes, most students in the United States formally study quadratic functions in a pre-calculus course. It was desired that all participating students have had experiences with quadratic functions within a variety of contexts—in their early algebra and geometry classes as well as pre-calculus classes. Moreover, it was believed that interviewing students who had some level of success in the formal study of mathematical functions would provide richer data for the purposes of this study. Therefore, students who had successfully completed a pre-calculus course at most one year prior to this study were selected.

Due to the teaching background of the researcher, it was possible for this study to include as participants students having a variety of experiences with the formal study of quadratic functions. It was believed that examining a wide spectrum of experiences would provide rich data in the investigation of students' understanding of this important concept. To this end, three university students and one student from a local public high school participated in the study. The researcher's prior work place, an urban high school in a large Southeastern city in the United States, was used in the selection of one of the participants of the study, i.e., the high school student. The researcher's current work place, an urban university campus in the same Southeastern city, provided the other three participants. The researcher's existing rapport with three of the students (two of the university students and the high school student) enabled easier access to participants in both schools. The researcher taught the pre-calculus courses that these three students completed.

At one end of the spectrum of success in pre-calculus, students that have had somewhat stronger mathematics background and that are substantially more articulate was preferred. In the study, participants were not only be required to have studied quadratic functions prior to data collection, but also they were asked to provide detailed explanations and reasons for their solutions to various non-traditional quadratic function tasks. At the other end of the spectrum, students who have demonstrated some level of mastery in pre-calculus (without a strong background in mathematics) were chosen. The students were also required to have passed their pre-calculus course with a grade of "C" or better; they all were asked to articulate their thoughts and approaches to all parts of the interview tasks during their problem solving activities. This spectrum of student experiences enabled the researcher provide a broader description of students' understanding of quadratic functions. Such purposeful selection (Patton, 1990) was desired in a qualitative study that aimed both at "achieving representativeness or typicality of the [...] individuals," and "adequately capturing the heterogeneity in the population" (Maxwell, 1996, p. 71) through what Guba and Lincoln (1989) referred to as *maximum variation sampling.*

Two semi-structured one-on-one clinical interviews with each of the four participants, in which their mathematical problem solving activity was audio-taped and video recorded, comprise the primary data collection method for this study. The two interviews (i.e., the initial and subsequent interviews) were semi-structured in that they included a structured interview protocol (Appendix A) with a fixed set of quadratic function tasks (Appendix B), and at the same time they allowed participants to lead the discussion, based on their own developing purposes, goals and thoughts, with minimum input provided by the interviewer (i.e., hints, suggestions, clues or directions). The tasks were also designed to be open-ended in that they could be solved in multiple ways. Each student was given a task instrument (Appendix B) with 8 to 12 quadratic function tasks depending on the pace of their problem solving activities during the two interviews. The partitioning of the 12 tasks between the two interviews was thus determined based on the pace and flow of each individual case. Each interview lasted approximately 75 minutes.

The tasks were designed as free-response questions, and they required students to provide written explanations to either justify their answers or to refute other possible choices. They were designed to be a mixture of traditional and non-traditional tasks that require students to demonstrate their knowledge of various aspects of quadratic functions in a variety of problem solving situations. Some of the tasks were adapted from standard pre-calculus textbooks, some from existing literature, and some were created by the researcher. Traditional or standard tasks included those that can be found in common pre-calculus textbook exercises located at the end of each section or unit (Larson et al., 2001; Sullivan, 2008; or Dugopolski, 2008). Non-standard tasks were those for which students had no familiar ways of approaching. They were intended to engage students in an inquiry into a problematic situation; therefore they were considered as rather unfamiliar, conceptual and problematic.

For instance, while a traditional textbook exercise may require students to recall a particular computational skill such as finding the $x$ component of the vertex of quadratic function graph using a formula, a non-standard task may ask students to compare two or more function graphs and reason through their similarities and differences. Most of these (latter) tasks include translations between graphical and analytic representations of quadratic functions that require minimum numerical calculations. As seen in Appendix B, some of these non-standard tasks ask students to make assessments on what is important to them, some ask them to self-generate definitions, some, without any symbolic or visual cue, ask them to coordinate certain aspects of a given quadratic function, and some ask for the bases on which they interpret certain forms of a quadratic function.

All interviews were conducted at the researcher's current work place, on a university campus, in a mathematics education conference room in the Department of Mathematics and Statistics. In addition to the interview transcripts, the video recordings and the written work of the students were used as other data sources.

The semi-structured interview enabled the researcher to elicit the ways in which students' spontaneously apply their conceptions or understandings of quadratic function concepts to resolve problematic situations (as presented in the form of interview tasks). One of the characteristics of clinical interview methodology is that when the participants are engaging in a problem solving activity, the researcher does not guide them in any way. The participants are not detached from their own purposes. The researcher tries to elicit all aspects of the problem solving activity that underlie participants' mathematical behavior (including their purposes, perceptions, premises, and ways of working things out (Noddings, 1990). Therefore this methodology was appropriate for this study that uses cognitive constructivism as its main theoretical perspective.

During the interviews the researcher made an effort to be conscious of what data would be useful for answering the research questions and therefore serve the research purposes (Creswell, 1998). Interviews provided useful data for investigating how the individual learners thought about mathematical situations. Researcher also strived to cope with spontaneous participant inputs during the interviews by frequently asking probing questions that required further elaboration of meanings and underlying understandings.

Besides the interviews, other data sources included: students' written solutions to the tasks, their self-generated diagrams and mathematical expressions, their responses to the individual mathematics background survey (Appendix C), and the researcher's observations throughout the students' problem solving activity. The interview protocol included a set of probing questions in order to facilitate students' problem solving activity and to ensure the think-aloud description of their attempts to make sense of their activity.

Data Analysis Methods

After stating the research problem, formulating the research questions, collecting appropriate data, and providing detailed descriptions of two of the four cases (Case 1: Ken and Case 2: Sarah) as illustrative examples, thematic qualitative data analysis techniques were employed in an attempt to best interpret all four students' quadratic function conceptions. In order to address the issue of lack of depth inherent in a multiple case study (Creswell, 1998), as compared to a single case study, the cases of Ken and Sarah were treated as illustrative cases that were described and analyzed more in-depth. The resulting descriptions, themes and assertions guided the analysis of the subsequent cases. In-depth analyses of these two cases also afforded the generation of tentative models and hypotheses against which the subsequent cases (Case 3: Seth and Case 4: Joseph) were interpreted. The interviews were transcribed verbatim. The structure of the analysis of each case study consisted of: An opening vignette that includes general information about the participant, the participant's past mathematical experiences, their overall approach to and performance on the quadratic function tasks presented in the interview instrument; selected quotes from the participant's responses to multiple data sources that characterize their reasoning and thinking about quadratic functions; and data analysis and discussion of the findings about the case. For each case, data analysis included: description, direct interpretation, generation of categories or themes, category aggregation, and generation of patterns among categories (Creswell, 1998; Stake, 1995). Video recordings and written documents were examined to discern problem solving episodes that reveal salient aspects of students' understanding of quadratic function concepts.

Themes or categories of how students understand quadratic functions, or the ways they reason about and act on problem situations involving quadratic functions, evolved through the phases of observation, transcription, in vivo coding, and identifying portions of interview data that seemed to represent participants' own meanings.

Video recordings enabled the researcher to not only infer students' inert actions but also observe their overt reactions, expressions and gestures. These reactions, expressions and gestures were included in all interview transcriptions as annotations. Students' localized goals and purposes within clusters of problem solving activity (Schoenfeld, 1985) were also analyzed. Other theoretical constructs of constructivism, such as cognitive structure, concept images and acts and bases of understanding were used as the interactive research process further evolved.

Results of the Pilot Study

In support of the proposed dissertation research, a pilot case study investigating students' understanding of quadratic functions was conducted of a high school student, pseudo-named Tim, who had studied quadratic functions in his pre-calculus honors class three months prior to the data collection. Tim described mathematics as his favorite and strongest subject in school. The pre-calculus honors class in which Tim was enrolled was one of the classes that the researcher taught at the same school. Before taking this pre-calculus honors class in tenth-grade, Tim took pre-algebra in sixth-grade, algebra (or Algebra 1) in seventh-grade, geometry in eight-grade, and advanced algebra (or Algebra 2) in ninth-grade. He received A's in all of these courses. Data for this pilot study included a 120-minute videotaped semi-structured clinical interview and Tim's written work. The interview protocol included a task instrument designed to elicit the student's approach to various aspects of functions and equations, as well as his understanding of quadratic functions. The task instrument seen in Appendix B evolved from this pilot instrument. The initial analysis confirmed the previous research findings that students can more easily coordinate representations of functions by generating and plotting points from a given function rule and that they cannot easily coordinate representations by looking at a series of coordinate points and generating the function rule that applies to those points. The table of $x$ and $y$ values that Tim generated in response to a task (Appendix B, Task 11) includes the set of points $\{(-2,4),(-1,2),(0,1),(1,2),(2,4)\}$. It seems that once he chooses a few arbitrary points and makes mirror images of them about the axis of symmetry, he feels confident that connecting all the points would result in the graph of a quadratic function. Tim seemed to believe that the vertex of a quadratic function is where the turn within a symmetric pattern of points occurs. If the points "go up" to the vertex, then they must "come down" past the vertex. If the points "come down" to the vertex, then they must "go up" past the vertex. Tim's investigation of "the points that could work," suggested that he coordinated the different coordinate pairs in his table in a rather unique way, and that he did not make sure that there is a quadratic function relation between the $x$ and $y$ coordinates of each pair. On the other hand, if he was given a quadratic function rule he made sure that he applied the rule to every $(x, y)$ pair.

Issues of Validity, Generalizability, and Ethics

The theoretical framework of the study, constructivism, is a position that is problem based and methodological. The clinical interview as the primary research method was dynamic enough to provide ample opportunities to see accurate expressions of the students' understandings as he or she negotiated the problematic situations. The interviews were therefore interactive and included well thought out probing questions. The variation in the interview tasks was sufficient enough for the tasks and data gathering to be able to probe into students' conceptions. In addition, maximum variation sampling allowed the conclusions to represent the entire range of variation (Maxwell, 1996).

No physical or psychological risk was involved in participating in the study. All participants were given a pseudo-name, and their identities were kept secret. All data including the student documents, interview transcripts and video recordings were secured in a secure location and kept anonymous. An IRB approval was obtained prior to data collection.

While these methodological practices are valid, generalizability is a complex issue. While the results are intended to explain the understandings of entire group of participants, it would be premature to say that the results further predict how other students view quadratic functions. For example, even though the students demonstrated understanding that suggests that they would perform well in an academic course that included quadratic functions, none of the students demonstrated what might be called a high abstract understanding of quadratic functions in the discipline of mathematics.

## Summary

The current study is a qualitative multiple-case study using a constructivist perspective to examine how students understand quadratic functions. Videotaped interviews with four participants from a local high school and a university constituted its primary data source. All participants were required to have been formally introduced to quadratic functions in a pre-calculus course. Thematic qualitative analyses were conducted for each case as well as across multiple-cases.

The purpose of this study was to explore how individual students understand various aspects of quadratic functions such as quadratic growth, quadratic correspondence, quadratic graphs, vertex points, x-intercepts, y-intercept, line of symmetry, parameters of general quadratic functions and quadratic equations. The study is intended to contribute to our in-depth understanding of individual students' conceptions of quadratic functions through analysis and interpretation of their mathematical behaviors in an open-ended problem solving environment. Four students were interviewed in order to obtain rich descriptions, characterizations and explanations of the scope and the depth of individual students' unique fabric of understandings with respect to multiple aspects of quadratic functions.

Many research questions are yet to be answered in this area of mathematics education research. Because quadratic functions are one of the most frequently used families of functions in the 6-12 grade curriculum (perhaps second only to linear functions), and because their real world applications make them an important part of school algebra and calculus, it is important that researchers study students' understanding of quadratic functions more in-depth.

Prior research has identified some misconceptions and student errors in dealing with quadratic functions. This study aimed to provide a more systematic characterization of individual students' rich fabric of conceptions or understandings, reasoning and meanings about quadratic function concepts using in-depth qualitative data (Oehrtman, 2009). It employed "extensive open-ended tasks to reveal the conceptual structures [or fabric of understandings or *basis of understandings* (Sierpinska, 1994)] that students spontaneously apply to resolve difficult" quadratic function problems (Oehrtman, 2009, p. 398). Indeed if such knowledge of broad characterizations of individual students' scope and depth of their prior knowledge and understanding of various classes of functions, as recommended in NCTM (2000), were made available, educators can make more informed decisions in teaching and curricular practices in school mathematics.

This study attempted to investigate students' understandings of only one of those classes of functions: quadratics. It focused on explicating students' fabric of understandings of quadratic functions through description, analysis and explanation. It investigates the ways that individual students (1) operate with various aspects and properties of quadratic functions in problem situations, (2) understand various aspects and properties of quadratic functions, and (3) make connections between various aspects and properties of quadratic functions. The study addressed the following research questions: What are students' understandings of quadratic functions? How do individual students understand and organize various aspects and properties of quadratic functions? How are these understandings constituted within situations involving quadratic functions and their properties?

These questions were posed from a cognitive constructivist theoretical perspective in the field of mathematics education. In other words, as these research questions indicate, the focus of the study was students' existing conceptions and mental processes; not social processes involved in their mathematical learning experiences in classroom settings. According to this cognitive perspective, the a priori instructional representations of mathematical ideas are not the primary source of students' mathematical knowledge (Cobb et al., 1992). Instead, students' own constructions constitute their primary knowledge source. To place this broad theoretical framework into perspective, Cobb's (2007) overview of four major theoretical perspectives in mathematics education research and practice will be provided in the subsequent section.

Because of the qualitative nature of the research questions and the study's cognitive constructivist theoretical perspective, a multi case study, with a primary data source of two sets of approximately 75 minute-long semi-structured clinical interviews with four participants were conducted. Students' mathematical problem solving activities were audio taped and video recorded within quadratic function situations that were familiar as well as unfamiliar to them. Their written work and self-evaluations of their mathematics background were also collected as supplementary data sources. The two semi-structured clinical interviews consisted of participants responding to a task instrument (Appendix B) with several non-standard problems on translations between graphical and algebraic representations of quadratic functions that require minimum quantitative calculations. The tasks were designed as free-response questions, and they required participants to provide written explanations to either justify their answers or to refute other choices. Three university freshmen students who recently completed a pre-calculus course and one newly graduated high school student that completed a sequence of pre-calculus and AP calculus courses within the last year were chosen as the participants of the study.

# CHAPTER 4: RESULTS

## Descriptions of Two Illustrative Cases: Case 1 (Ken) and Case 2 (Sarah)

In order to provide an in-depth analysis of how students understand quadratic functions, two participants were selected as initial cases, and their mathematical problem solving activities were described in full. In addition to the verbatim transcripts, these detailed descriptions enabled the researcher to generate codes, categories and themes against which the other two participants' responses were analyzed. These two initial students were considered as sufficiently articulate, and their responses were observed to be ample sources for generating working models that guided the subsequent analyses. The two initial cases were pseudo named Ken and Sarah. They were not selected based on the strength of their mathematical knowledge. As it will be further clarified, the analyses aimed at addressing the research questions only, which involved the explanation of the students' ways of understanding without focusing on their mathematical validity or correctness. The two subsequent cases, pseudo named Seth and Joseph, are analyzed in the cross analysis section.

### Case 1 Results: Ken's Solution Activity.

The first case of this study is a college freshman with a pseudonym Ken. In high school, after taking Algebra I and Geometry, Ken took Algebra II, Pre-calculus, Discrete Mathematics and AP Statistics courses. He characterizes his experiences in school mathematics as "good experiences; most errors come from lack of focus." In his individual mathematics background survey, and in his informal interactions with the researcher, Ken portrayed himself as confident and reasonably strong in mathematics. He claimed that he did not focus too much on academic subjects in high school and rather put less effort in high school mathematics. As a response to the question "what aspects of mathematics, if any, do you like the most?" Ken wrote, in the individual mathematics background survey, "applying what I know to solve problems I haven't seen before; specifically in classes such as chemistry and physics." When asked "what aspects of mathematics, if any, do you dislike the most?" Ken wrote "how much losing focus will affect your grade: careless mistakes can make a large difference." In the same survey, he self-rated the strength of his mathematics knowledge in arithmetic, algebra, geometry, upper level mathematics, and other mathematical topics such as discrete mathematics and statistics, on a scale of 1 to 5 (1 being the weakest and 5 the strongest), as 5, 5, 4, 5, and 4, respectively. In other words, other than geometry, discrete mathematics and statistics, he placed himself in the strongest category.

During both interviews, Ken remained calm and confident as he worked on the tasks. When he experienced puzzlement or difficulty, he offered explanations that were consistent with what he said before, and somewhat downplayed his struggles as simple forgetfulness. This seemed to have helped him avoid major frustrations. Attributing the inconsistencies in his results to not remembering well certain formulas or properties seemed to have helped him stay calm and confident. Below are the descriptions of Ken's responses to the tasks in the interview task instrument in APPENDIX B.

Task 1: Describing functions.

Ken started the first task: "In your own words, please explain what you think a "function" is. Feel free to write as much as you like. You may also draw graphs, diagrams or tables," by asking whether the researcher wanted him to write out what he thinks a function is. After receiving a positive response, he continued:

> Ken: When you have a function it's usually a function of something, and how I usually see it like $f$ of $x$ [*writes* $f(x)$] would be a function of $x$, which to explain that, it would be… and that would equal $y$ [*writes* $y = f(x)$], so $y$ is a function of $x$ and what that is saying to me that it like an $x$ value if $x$ is 1 that would equal $y$ [*writes* $y = f(1)$] and then $y$ would be equal some value; it could be 3 [*writes* $y = 3$]. And then a function would usually be some formula that would like for this one, $f$ of one, it would be $x$ plus 2 would be the function [*writes* $x + 2$ *next to* $y = f(1)$] and I use [*pointing at* $x + 2$] to plug in the $x$ to solve for $y$. [*A long pause*] That's about it.

After being prompted to feel free to draw graphs or diagrams, Ken first drew two axes and created a table with separate $x$ and $y$ columns and said: "Like for that function [pointing at $y = f(1)$, $y = 3$ and $x + 2$ in Figure 1.1], I can draw what that function is." He wrote $x + 2$, above the $x$ and the $y$ columns, and filled the table with numerical values he generated [Figure 1.2], and then went on to "draw it out" by plotting the $x$ and $y$ pairs $(1, 3)$ and $(2, 4)$ [Figure 1.1]. To generate these pairs, as he argued earlier, Ken used "the formula," $x + 2$, "to plug in the $x$ values to solve for the $y$ values."

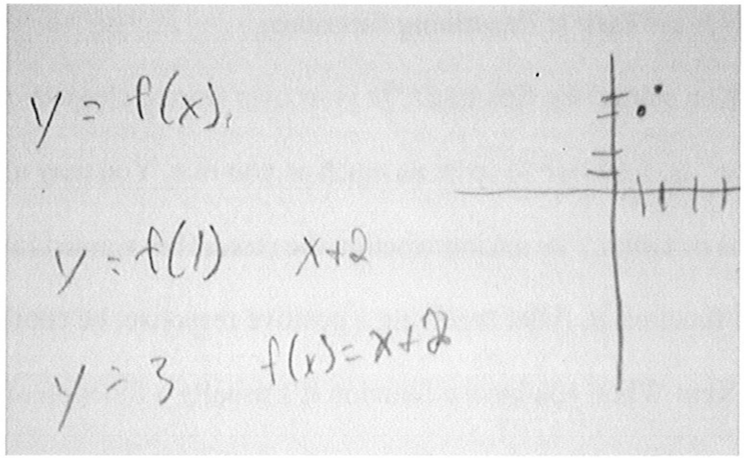



*Figure 1.1.* Ken's Response to Task 1

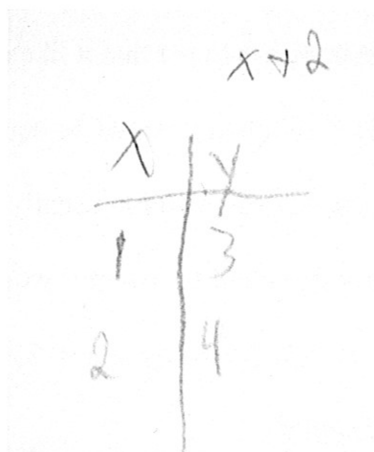



*Figure 1.2.* Ken's First Table

The following excerpt from Ken's first interview illustrates how Ken recapitulates what he thinks a function is:

R [Researcher]: So function is … It is a function of something.

K [Ken]: Uh-huh.

R: And what is that something?

K: What do you mean?

R: Like what is it a function of?

K: Of …

R: What kinds of things it is a function of?

K: The independent variable of a dependent variable.

R: And can you show me what the function is in here? [*Pointing at* $y = f(1)$, $y = 3$ *and* $x + 2$ *in Figure 1.1*]

K: It would be a function of $x$ equals $x$ plus 2. [*Writes* $f(x) = x + 2$ *in Figure 1.1*]

R: Uh-huh.

K: Where $y$ would be dependent variable and $x$ be the independent. Because $y$ depends on $x$.

In sum, according to Ken, a function is "usually a function of something" and "it would usually be some formula." Function is written as $y = f(x)$ where the dependent variable $y$ is a function of the independent variable $x$.

Task 2: Describing quadratic functions.

In the second task, Ken was asked to draw a quadratic function in a blank rectangular box and to (a) explain what makes his graph quadratic, (b) discuss what parts of the graph are important or special (and why); and (c) give an equation for the graph that he drew. His initial response to this task was to write $ax^2 + by + c = 0$ and assert that a quadratic function is usually in this form. After stating that this form can also be written in terms of some other variables, he changed his initial response $ax^2 + by + c = 0$ to $ax^2 + bx + c = 0$. Then he sketched a graph [Figure 1.3].

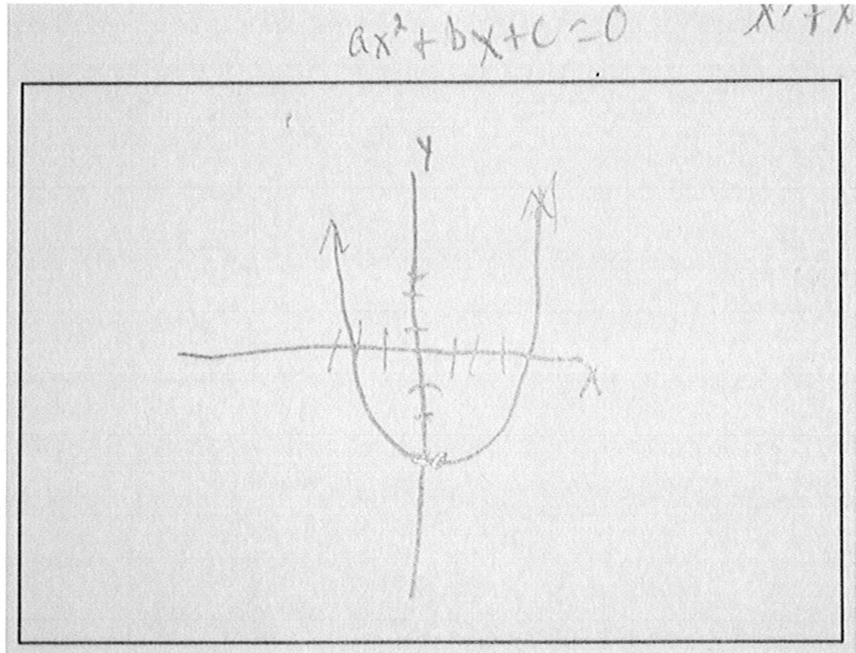



*Figure 1.3.* Ken's Self-generated First Quadratic Graph

When asked what makes this graph quadratic, he wrote: "because it fits the quadratic formula." He stated that usually when he sees something that is quadratic, it fits the formula. At this point, Ken's explanation suggests that he is thinking of some given "formula," not a graph, which somehow fits "the quadratic formula" $ax^2 + bx + c = 0$. He also calls this "the original formula."

K: And why that's quadratic, why the formula is quadratic, I don't really know to be honest.

R: Can I ask you to explain what this means? [*Pointing at Ken's response to Task 2a: "because it fits the quadratic formula"*]

K: I would say it is quadratic because it is in the form of that equation [*pointing at* $ax^2 + bx + c = 0$] of the quadratic equation. So that's why I would say it's quadratic. I don't really know what it means to be quadratic or not. I forgot the definition of it.

R: What is the difference between some equation like this [*pointing at* $ax^2 + bx + c = 0$] that looks like a quadratic and another equation that is not a quadratic? Because you know that one of them fits the quadratic function.

K: Uh-huh.

R: And the other doesn't fit. So how do you know which one fits?

K: Umm. I usually look at the variables, and it's just I have seen it so much it kind of sticks out to me whenever I see it. It usually cross, or umm [*tracing the graph that he drew in Figure 1.3 with his pencil in the air*] it usually looks like a parabola so, I just kind of recognized it. Or we are solving a problem I know it's a quadratic or not. It's just like umm basically memorized it. Like don't really understand it.

R: That's okay. When you look at the graph, umm how do you know if it's a quadratic graph? Because you drew that as an example of a quadratic function, right?

K: Uh-huh.

R: Why did you draw that way but not some other way? Some other curve.

K: Just because it did look like an example of something that I know it's a quadratic.

R: And what are the important pieces of the graph that are special you think?

K: I would guess that. [*Pause*] Can I ask you a question?

R: Yeah yeah.

K: If it is umm, would be like something $x$ squared plus like if it was in the form of [*writes* $x^2$ *and immediately changes it into* $x^3$] $x$ cubed plus $x$ squared plus $x$ plus $c$, would that be a quadratic? [*Writes the cubic equation in Figure 1.4*]

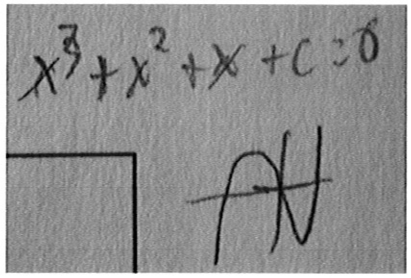


*Figure 1.4.* Ken's Cubic Equation

R: Hmm. What do you think?

K: I think not.

R: Okay. Why you think not?

K: Just because I'm thinking of follow the definition of a quadratic cause quadratic is usually just one [*tracing the graph that he drew in Figure 1.3 with his pencil in the air*], where is that it would look like [*pointing at the cubic equation in Figure 1.4, and drawing the cubic graph under the equation*] it would go up and a down and up. Which I don't think it is a quadratic.

R: Because it doesn't fit the shape? [*Pointing at the graph in Figure 1.3*]

K: Uh-huh. Just the model, which would be just one parabola. [*Tracing the graph that he drew in Figure 1.3 with his pencil in the air. Ken does not trace* $y = x^2$ *with a vertex at the origin. He traces the graph he drew*]

For Ken, the important parts of a quadratic graph are the minimum or the maximum points on the graph and "the ends" of the graph that "go to" positive or negative infinity. Besides these two characteristics of quadratic graphs, Ken stated that the "U-shaped" parabola can be upside down (the "inverse"), but it cannot be sideways. To provide a rationale for this claim, Ken argued that he remembers the sideways parabolas [Figure 1.5] have different equations; however he forgot what those equations were.

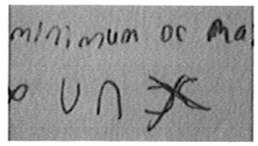


*Figure 1.5.* Sideways Parabolas

In sum, in this task, Ken drew a graph of a quadratic function [Figure 1.3] without paying close attention to the details of the graph. He simply sketched a curve that is somewhat of a "U-shaped," "one parabola" that has a minimum point and its ends extending towards infinity. He also argued that the equation of the graph will have the general or the original form $ax^2 + bx + c = 0$. In Task 2c, he wrote $f(x) = 3x^2 + 2x + 4 = 0$ as an example that could be an equation of a quadratic graph. Ken's responses to the questions in this task reveal somewhat of a weak connection, if not a disconnection, between the ways he thinks about functions in general and quadratic functions in particular. For example, in terms of functions, Ken sees dependent variables as functions of independent variables. Moreover, he thinks that one can find the values of a dependent variable—usually called $y$, which is a function of some independent variable called $x$, by "plugging in" the values of the independent variable in some formula. On the other hand, when asked about quadratic functions, he insists on a "quadratic formula" or "a general form" or "an original formula" $ax^2 + bx + c = 0$. And, he develops his arguments on quadratic graphs based on the notions of minimum and maximum points, end behaviors, the existence of only one parabola, and resemblance to the formula $ax^2 + bx + c = 0$. In other words, unlike in the first task on functions, he does not explicitly separate quadratic functions from their graphs. Only towards the end, when the researcher attempts to elicit the connections that he may make between the concepts of functions and quadratic functions, he reads off the minimum point on the quadratic graph in Figure 1.3 as $x = 1$ and $y = -3$ and writes $f(1) = -3$. Thus, to this point in the interview Ken does not indicate any conception of a quadratic relation between two variables, such as values of one variable being the squares of the values of another variable. There is no explicit mention of squares, or squaring.

Task 3: Graphing $y = -(x - 4)^2 + 16$.

Ken approached the third task by thinking out loud, asking himself if he "remembers how to do this." After circling the negative sign in front of $(x - 4)^2$ and calling it "the inverse over the x-axis," he drew the little curve at the top corner of his initial graph [Figure 1.6].

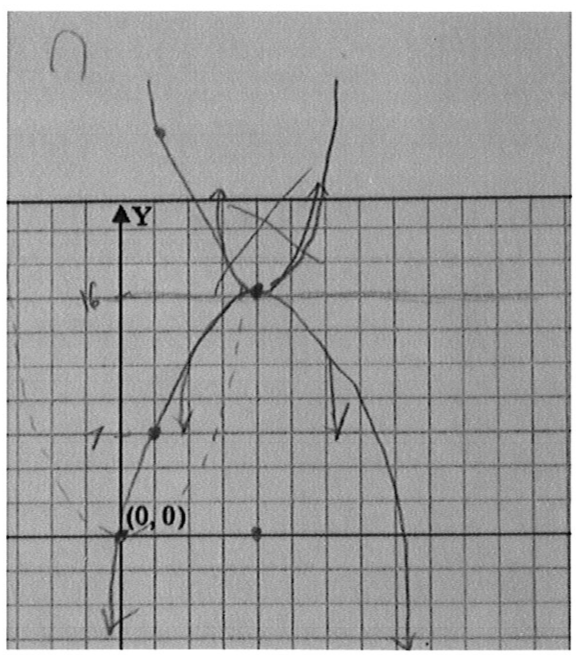



*Figure 1.6.* Ken's Response to Task 3

He continued with applying a series of transformations to $y = x^2$: he shifted it four units to the right, moved it up 16 units, and then drew the "inverse" of it by changing the direction in which the graph opens. Upon probing, he specified some of the points that the graph passes through by picking two $x$ values, namely $x = 0$ and $x = 1$, and computed the corresponding $y$ values by substituting the two $x$ values in the function equation.

K: So it would be shifted; the original graph would be just of a simple equation; would be something like that. [*Draws the dotted curve in Figure 1.6*]

R: What is that simple equation?

K: That's just the standard form of the quadratic. Just with the origin at zero, and it goes up from there.

R: Okay.

K: And then now it's gonna take that and shift it to the right by four units because it is $x$ minus four so it is one, two, three, four.

Ken used the terms reflection and inverse interchangeably, and he successfully coordinated function translations with reflection. Although he stated that a negative sign in front of a function indicates that its graph is reflected about the x-axis, he successfully reflected the graph of $y = (x - 4)^2 + 16$ about the line $y = 16$ (and not the x-axis, $y = 0$). The fact that he highlighted the line $y = 16$ [Figure 1.6] suggests that he might be thinking of applying the function translations to x- and y-axes as well. It is plausible that he might be thinking of the line $x = 4$ as "the new y-axis," the line $y = 16$ as "the new x-axis," and the point $(4, 16)$ as "the new origin." Furthermore, he somehow coordinates the multiple transformations that are applied to a particular function. These hypotheses will be further analyzed during the discussions of Ken's responses to Task 4 and Task 11. He offered more explanation to his notion of "inverse" of a graph in these tasks.

Task 4: Further discussion on quadratic functions.

When asked, in Task 4, how he would explain quadratic functions to a friend who missed class, Ken stated that one of the more important things that he hasn't mentioned yet is "the quadratic equation" $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$.

K: What this is solving for are the $x$ values on the quadratic.

R: Uh-huh.

K: So it's, and usually get two values because of the plus or minus.

R: Uh-huh.

K: So $x$ would equal something and $x$ would equal something else. [*Writes* $Xint =$ *and* $Xint =$]

It may be worthy of note that Ken first started writing the above expression while saying: "one of the more important things that I haven't mentioned yet [a very short pause]" and stopped at the step "$b \mp \sqrt{\ }$," and wrote $ax^2 + bx + c = 0$ while finishing his sentence with the phrase: "is the quadratic equation." It is evident that Ken refers to the equation $ax^2 + bx + c = 0$ by the phrases "general form of a quadratic" or "the quadratic formula." He used the phrase "quadratic equation" when discussing $ax^2 + bx + c = 0$ only once. It is rather a curious coincidence that he was writing his "general form" or "quadratic equation" $ax^2 + bx + c = 0$, while referring to $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$ as the quadratic equation.

Ken also wrote $y = -(x-4)^2 + 16$ as an example of a quadratic function, which he just worked on in the prior task, and stated that he would also show his friend this form because it's easier to graph it. He also asserted that if one factors this form out, they can obtain something that looks like the general form $ax^2 + bx + c = 0$.

K: You can graph it by this probably easier. Because this shows you the transformations or the things that are happening to the graph.

R: Hmm.

K: Like this would be, umm, that's the change in $y$ [*pointing at the number 16*] yeah.

R: Like the way you did before, you shifted up.

K: Yeah, that would shift the $y$ [*pointing at the number 16*] and that would shift the $x$ [*pointing at the number 4*] and that would inverse it [*pointing at the negative sign in front of* $(x-4)^2$]. If there is something in front of it [*hovering the tip of his pencil above the negative sign in front of* $(x-4)^2$].

R: What do you mean by inverse it?

K: Umm. Well because it's outside the parenthesis, it would be reflected over the x-axis. So it would look like that. [*Draws the dotted graph in Figure 1.7*]

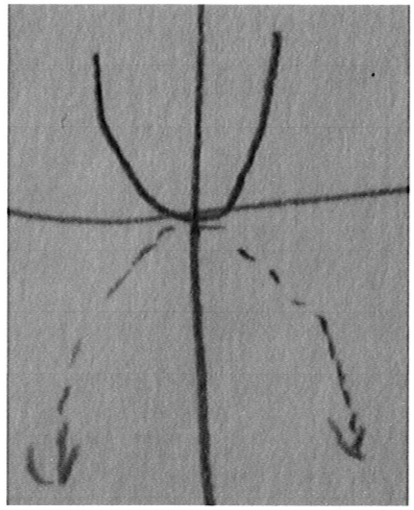


*Figure 1.7.* Ken's Inverse Graph

When the researcher asked Ken to further explain why a negative sign outside the parenthesis would reflect the graph over the x-axis, Ken said: "when you are solving it, it just comes out that the $y$ value it would be negative instead of where it would usually be positive." Then the researcher showed him the graph [the final graph that is opening downward] in Figure 1.6 and pointed to him the fact that although there is a negative sign in front of the parenthesis [in $y = -(x-4)^2 + 16$] the $y$ values between $y = 0$ and $y = 16$ are not negative. The below excerpt reveals how Ken reasons about the multiple transformations that are applied to a function:

K: Well yeah that's because they were shifted. By sixteen.

R: Hmm.

K: So it's not so much that the negative, it just, they are opposite of what they would be without it.

R: What do you mean by that?

K: Actually that is not correct. It's. [*Pause*]

R: Opposite of?

K: Well I guess it is to me. Like the way I'm thinking about it. Cause if I were to draw the original one without the negative [*draws a new larger graph around the small graph that is crossed out and opening upward in Figure 1.6*] it would look something like that [*pointing at the new larger graph that is opening upward in Figure 1.6*].

R: Uh-huh.

K: And, with it, so the point would be up there [*draws a point on the new larger graph that is opening upward in Figure 1.6, which represents the reflection of the point* $(1, 7)$ *that was drawn earlier*].

R: Uh-huh.

K: And if you were to just make a new like make a new axis I guess [*draws a line through* $y = 16$], so I guess you are flipping over that. [*Makes a flipping move with his right hand over the line* $y = 16$]

R: Hmm.

K: So that's what I meant by opposite. Which is not really the opposite but the way I am thinking about it; it is.

Task 5: Solving quadratic equations.

Ken approached Task 5 [Solve $2x^2 - 7x + 3 = 0$] by stating that whenever he sees a problem like this, he thinks of solving for the x-intercepts; and he would use the "quadratic equation" to solve for them. Note that, earlier in the interview, he explicitly stated that the x-intercepts of a quadratic graph are also the $x$ values or the zeros. Thus, Ken uses several different terms synonymously when referring to the x-intercepts. Before using the "quadratic equation," he first wrote the equation $ax^2 - bx + c = 0$ in Figure 1.8, which seemed to model the original equation $2x^2 - 7x + 3 = 0$.

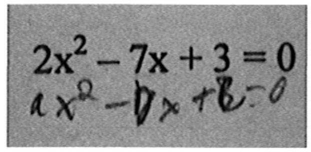



*Figure 1.8.* Ken's Initial Approach to Task 5

Then, he continued: "the opposite of *b*, which is positive seven, plus or minus the square root of *b* squared, which is negative seven squared, minus four times *a*, which is two, times *c*, which is three, divided by two times *a*, which is two" [Figure 1.9].

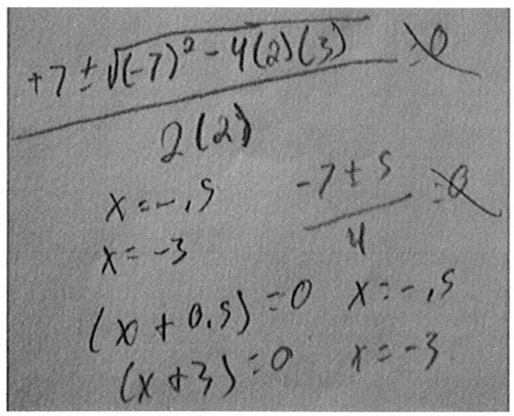



*Figure 1.9.* Ken's Solution to Task 5

He computed the expression $(-7)^2 - 4(2)(3)$ using a graphing calculator [that was provided by the researcher for this particular purpose], and he mistakenly rewrote his original "quadratic equation" $\frac{+7 \mp \sqrt{(-7)^2 - 4(2)(3)}}{2(2)}$ as $\frac{-7 \mp 5}{4}$. He found two $x$ values, $x = -0.5$ and $x = -3$. At this moment, he said:

K: Those are the values of $x$ and a lot of times you see them as $x$ plus zero point five [*writes* $(x + 0.5)$] and then $x$ plus three [*writes* $(x + 3)$].

R: What are those? The parentheses you just wrote? These two? [*Pointing at* $(x + 0.5)$ *and* $(x + 3)$]

K: Yeah, I am trying to remember. Umm, I wanna say that the values of $x$ in the equation. [*Pointing at* $2x^2 - 7x + 3 = 0$]

R: Uh-huh.

K: But I cannot really remember. I just know that it's like that. I can't think of why at the moment though.

R: Okay.

K: I think it's; $y$ equals. [*Speaking very quietly*]

R: Huh?

K: Yeah, I can't remember. It's because that equals something [*pointing at* $(x + 0.5)$]. Oh yeah, that equals zero [*writes* $(x + 0.5) = 0$], and then you would solve for $x$. They are like the zero [*writes* $(x + 3) = 0$], so $x$ equals negative point five and $x$ equals negative three [*writes* $x = -.5$ *and* $x = -3$ *for the second time*].

R: Uh-huh.

K: That's why. That's where it comes from. Because that [*pointing at* $(x + 0.5) = 0$ *and* $(x + 3) = 0$ *by connecting the two equations, in the air, with the tip of his pencil*] that part is, this is supposed to equal zero [*adds* "$= 0$" *in* $\frac{+7 \mp \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} = 0$]. That balances off the equation [*adds* "$= 0$" *in* $\frac{-7 \mp 5}{4} = 0$]. So once you get that [*pointing at* $\frac{-7 \mp 5}{4} = 0$], which is that [*pointing at*

$x = -.5$ *and* $x = -3$], you are solving for that [*pointing at* $(x + 0.5) = 0$ *and* $(x + 3) = 0$].

R: Hmm. Okay. When you subtract or add five to negative seven, that's negative two divided by four, that's zero?

K: Yeah.

R: Okay, let's.

K: I don't think that's right.

R: Negative two over four.

K: Which would be negative point five. I am not sure how that [*pointing at* $\frac{-7 \mp 5}{4} = 0$] relates to that [*pointing at* $(x + 0.5) = 0$ *and* $(x + 3) = 0$] really, I just know that there would be $x$ plus that [*pointing at 0.5*] equals zero. There [*pointing at* $x = -.5$]. Unless I solved it wrong. It could be me that mixed up I'm not sure. To check it I would just graph it on the calculator to make sure I did it right though.

R: How would you check it in the calculator?

K: I would go to $y$ equals and plug in the original equation and I would see where it crosses the x-axis to make sure it is right.

Ken correctly represented the -*b* term in the quadratic formula as "the opposite of *b*," which is - (-7) or + 7, but when he simplified it to $\frac{-7 \mp 5}{4}$ he mistakenly put -7 instead. It is unclear whether his two statements about, "*a*" being 2 and "*b*" being -7 and "*c*" being 3, and that $2x^2 - 7x + 3 = 0$ is $ax^2 - bx + c = 0$ might be playing two conflicting roles and therefore accounting for the mistake. He did not check whether

$x = -.5$ and $x = -3$ are the correct solutions or not. In addition, his interchangeable use of mathematical expressions and equations and his use of the name "quadratic equation" for $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$ seem to have led him to write $\frac{+7 \mp \sqrt{(-7)^2 - 4(2)(3)}}{2(2)} = 0$ and $\frac{-7 \mp 5}{4} = 0$ without computing the numerical expressions on the left hand sides of these two equations. There is no evidence that he differentiates, at least intentionally, expressions (algebraic or numeric) and equations.

Besides using the quadratic formula (or "the quadratic equation"), to solve $2x^2 - 7x + 3 = 0$, Ken argued that one can write $2x^2 - 7x + 3 = 0$ in a form $y = \_\,(x - \_)^2 + \_$, where "some value in front and it would be $x$ minus some value squared plus some value, so it would be in that form; equals $y$" [as illustrated in Figure 1.10]. One can then "graph the equation" and "solve for $x$ that way." Recall that Ken sees x-intercepts, zeros and solutions as the same. Furthermore, one can use a calculator to "graph the equation" and find the x-intercepts by using its 'calculate → zero' feature.

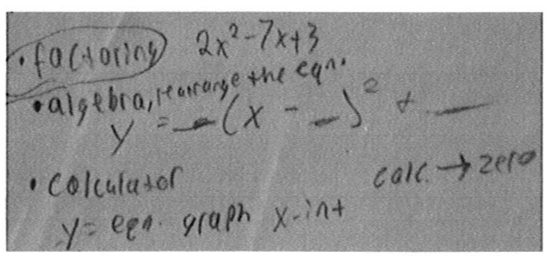



Figure 1.10. Ken's List of Strategies for Solving Quadratic Equations

Lastly, when asked to solve $x^2 - 5x + 6 = 0$ in a way that is different from the ways he used in the previous parts of this task, he said he would use factoring. Earlier, when Ken wrote $2x^2 - 7x + 3 = 0$ in the form $y = \_\,(x - \_)^2 + \_$, he mentioned that he forgot what this strategy was called and wrote the label "factoring" in front of $2x^2 - 7x + 3 = 0$ [as in Figure 1.10]. In this last part, however, he referred to "factoring" $x^2 - 5x + 6 = 0$ as writing it in the form $(x-2)(x-3)$. In other words, he corrected himself by going back to Figure 1.10 and circling the label "factoring" and linking it to $(x-2)(x-3)$ instead. He stated: "If you factor this out [$x^2 - 5x + 6$] it would equal that [$(x-2)(x-3)$] because $x$ times $x$, or if you 'FOIL' it [$(x-2)(x-3)$] back out, it would be $x$ squared minus five $x$ plus six [$x^2 - 5x + 6$]."

And then, he said:

K: To solve for $x$, you; I know how to do it, but I don't think I remember technically why it works, I just know like if you set one of the equations [*pointing at* $(x-2)$] equal to zero like $x$ minus two equals zero [*writes* $x - 2 = 0$] and solve $x$ that way by algebra [*writes* $x = 2$]. So $x$ equals two; and you can do the same thing to the other one [*writes* $x - 3 = 0$] zero and $x$ equals three [*writes* $x = 3$].

R: Okay.

K: Yeah. Now that I think about it, I am not sure why you can do that. But.

R: Which one, the zeros?

K: Yeah. Like why you can kinda neglect the $f$ of there [*pointing at the* $(x-3)$ *in* $(x-2)(x-3)$] and just say $x$ equals this [*pointing at* $x = 2$].

R: What do you mean $x$ equals this?

K: Well, like how you can just take only that one [*pointing at the* $(x-2)$ *in* $(x-2)(x-3)$] and bring it down here [*pointing at* $(x-2) = 0$] and say $x$ minus two equals zero and just kind of forgetting that part right there [*pointing at the* $(x-3)$ *in* $(x-2)(x-3)$] but it works, so.

Ken was not asked to further elaborate on this. Although the current study mainly focuses on students' understanding of quadratic functions; this task and the following task (i.e., Task 6, which asks students to explain the difference between a function and an equation) were designed to elicit students' ways of approaching to all elements of quadratic functions, including solving quadratic equations. Thus, it would have been beneficial if the researcher further probed Ken's thinking about solving equations in the form $(x-2)(x-3)=0$. In their study on students' understanding of quadratic equations, Vaiyavutjamai and Clements (2006) observed a misconception that students displayed regarding the meaning of variables in quadratic equations. Most of their interviewees thought that for example in $(x-3)(x-5)=0$, the two $x$'s have different values. The interviewees substituted $x=3$ in $(x-3)$ and $x=5$ in $(x-5)$, and argued that $(3-3)(5-5)=0$. Vaiyavutjamai and Clements argue that "at the post-teaching stage a minority of students [in the six Algebra 1 classes that they observed] grasped the concept of variable in the context of quadratic equations" (p. 72). As it will be clearer in the cross analysis of Ken's case with the next case, participants of this study also seemed to have such idiosyncratic conceptions of mathematical expressions and equations. Neither seemed to conceptualize or use a mathematical expression with multiple terms as a single entity. Whereas a mathematician or a mathematics educator may see $(x-3)(x-5)$ as a single expression, which is a product of two binomial expressions with the same variable $x$, and see $(x-3)(x-5)=0$ as an equation with one variable, students may perceive $(x-3)$ and $(x-5)$ in $(x-3)(x-5)=0$ as two 'equations' or two 'problems.' There is evidence that Ken views expressions such as $(x-3)$ and $(x-5)$ as two 'equations' or two 'problems.' He frequently refers to expressions as equations.

Task 6: Describing the difference between an equation and a function.

To Ken, there are several differences between a function and an equation. The main difference is that in an equation one solves for a variable (to find the value of that unknown variable) whereas in a function one uses the values of $x$ to "solve for $y$." In addition, functions have equations in them, and functions "always come out to be graphs."

K: Well like I said before a, well, in an equation, it's something equals something else, which is the same as function I guess but. [*Speaking quietly*]

R: What do you mean?

K: I can recognize the difference; I am just thinking of like a way to word the difference.

R: Give an example.

K: Yeah, well, a function will always be a function of something [*writes* $f(x)$] equals some equation [*writes* $f(x) =$ ______ *in Figure 1.11*], and the equation itself is I guess I can use a quadratic [*pause*], that's an equation [*writes* $2x^2 + 3x + 4 = 0$ *in Figure 1.11*] and for an equation you're usually just solving for a variable umm, in a function it's usually, it's, there are equations in functions and [*rewrites* $f(x) =$ ______ *in Figure 1.12 and writes "solve for variable" in Figure 1.13*] what makes a function different is that [*pause*] it's hmm it's usually like you like you're using an equation to solve for something else. That's not right.

R: What does that mean? What do you mean?

K: It's like with the function it's it would always come out to be a graph so a function of $x$ would be $x$ squared plus two [*writes* $f(x) = x^2 + 2$ *in Figure 1.12*].

R: Uh-huh.

K: And then, you're using you for this one [*pointing at* $f(x) = x^2 + 2$] you are basically plugging values for $x$ to come up with the values for $y$.

R: Uh-huh.

K: And a way I remember learning it is that you plug in put in the value, say $x$ [*draws the function box in Figure 1.12, writes the label x above the first arrow*], this is the function itself [*writes the word function inside the box*], and you come up with the value $y$ [*writes the label y above the second arrow*], so it's taking something to get another value whereas in an equation you're just solving for something.

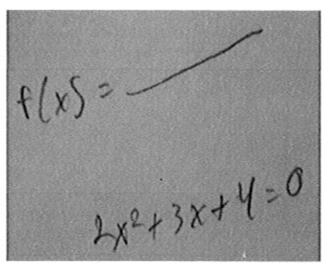



*Figure 1.11.* First Part of Ken's Response to Task 6

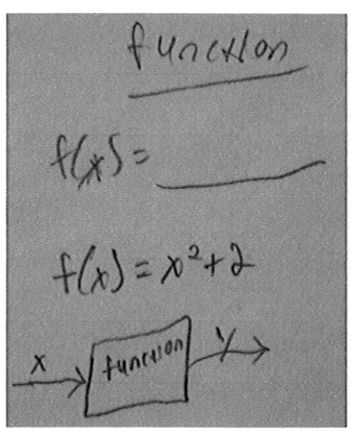


*Figure 1.12.* Second Part of Ken's Response to Task 6

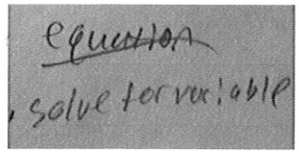



*Figure 1.13.* Third Part of Ken's Response to Task 6

Ken later added that if he takes a function like the one Figure 1.14, which he remembers as a piecewise-defined function, one can use different "equations" to find the $y$ values. When asked what he means by "equations," [Figure 1.14], Ken confidently and quickly circled the two expressions $x^2 + 2$ and $x$.

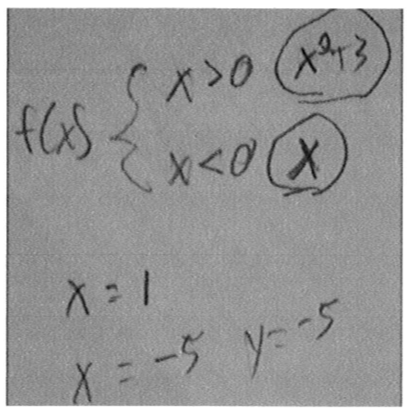



*Figure 1.14.* Fourth Part of Ken's Response to Task 6

K: That's for the first one and that's for the second one.

R: Okay.

[…]

R: In the equations $x^2 + 2$ and $x$, which variables are you solving for?

K: Oh, well it's, they actually are kind of more different because this is not just [*pointing at the circled x in Figure 1.14*] wait. It would have to be equal to something else for it to be an actual equation.

R: Oh, okay so they are not actual equations [*pointing at the circled* $x^2 + 2$ *and* $x$ *in Figure 1.14*].

K: No.

R: What are they?

K: They are functions. Well [*pause*], yeah. They are function of $x$ [*pointing at '*$f(x)$*' in Figure 1.14*]. Never thought about it before.

It is evident that Ken thinks of functions as some sort of processes, or collections, in which one uses $x$ values to solve for $y$ values, draws graphs out of those $x$ and $y$ values, or simply writes them in an equation form as "$f$ of $x$ equals something." In other words, $f(x) =$ ____, where a function $f$, of some independent variable $x$, equals some "equation." He summarizes the difference between an equation and a function:

K: I guess an equation it's like you are given an equation and you are solving for the variables, whereas a function, you are given a function of something and you're trying to [*a short pause*] graph it. Cause it's, a function usually comes out as a graph. Because it's, you like, you might be given the $x$ variables and you have to solve for the $y$ variables.

He also states that in an equation "a variable has discrete values, whereas in a function it changes depending on what you use." In other words, the value of an unknown in an equation doesn't change "because it is set to a discrete number" such as zero or, say, sixteen. In addition, because of equations' and functions' respective discrete and changing natures, Ken posits that he can turn an equation into a function by simply writing "$= f(x)$" at the end of an equation. He writes: $2x + 3x + 4 = f(x)$.

K: Two $x$ plus three $x$ plus four equals a function of $x$.

R: Uh-huh.

K: I just denote that it is a function.

R: Whereas if you replace that $f$ of $x$ with sixteen I think.

K: Then it's a set value and you are solving for the, yeah it's a set value, you are solving for $x$.

Task 7: Finding the vertex of $\mathrm{f(x)} = 6\mathrm{x} - \mathrm{x}^2$.

Ken first suggested that he can graph this function using a graphing calculator, but without it he would need $f(x) = 6x - x^2$ written in a different form. Earlier in Task 3, he graphed $y = -(x-4)^2 + 16$ by applying a series of transformations to the graph of $y = x^2$; therefore his approach to graphing quadratic functions is consistent with his suggestion to rearrange the given equation $f(x) = 6x - x^2$. After writing the function in the form $y = a\ (x-h)^2 + k$ he can graph it and look at its minimum or maximum point. He clearly stated that he does not know how to find the vertex by looking at $f(x) = 6x - x^2$. He also said: "If I really wanted to, I guess I would just plug in values of $x$ until I found the vertex [while hovering the tip of his pencil above the graph paper and making a swinging motion] it's kind of long way to do it." In his first attempt to rearrange the given equation, he "pulled out the $x$" and wrote: $x(6-x)$. Not pleased by this form, he then wrote $-x^2 + 6x$ and said:

K: Makes more sense. Oh, there is a formula you can use to find out the $\boldsymbol{x}$ the vertex of the $x$.

R: Uh-huh.

K: Forgot what it was though. I think it's opposite of *b* over two *a*?

R: Uh-huh.

K: I think that's right. So it would be negative six over two minus negative one, so it's negative six divided by negative two, which equals three. So $\boldsymbol{x}$ of it umm some three value [*marks the point* $(3, 0)$ *on the x-axis of the graph paper very lightly in Figure 1.15*], and for the y [*pause*], oh if you plug in the three because it's a function of $x$ [*pointing at* $f(x) = 6x - x^2$].

R: Uh-huh.

K: So by plugging in the three then you can solve for what the $y$ value is, so, negative $x$. I'm just going to use this [*pointing at* $f(x) = 6x - x^2$] when solving because it's the negative that's confusing me [*pointing at* "$-x^2 + 6x$"] six I know I don't wanna make a careless mistake.

R: Uh-huh.

K: Three squared, six times three would be eighteen minus nine, which would equal nine. So then that's the $y$ value.

R: Uh-huh.

K: And it's usually, the fact that that's a function [*pointing at* $f(x) = 6x - x^2$] so it's one, two, three, four, five, six, seven, eight, nine. That's the vertex. [*Plots the point* $(3, 9)$]

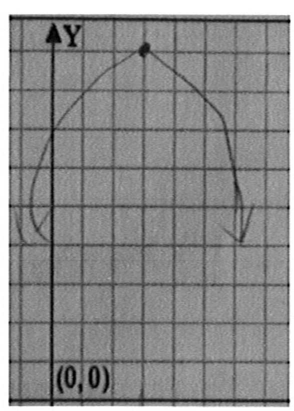


*Figure 1.15.* Ken's Response to Task 7

And, upon the researcher's request, he sketched the graph in Figure 1.15. Ken mentioned that it is "clearly inverted again" because of the negative sign in front of the $x$ squared. He also stated that he can "plug in some more points to find out some actual coordinates" since the more points "the more accurate it would be."

In response to the question: Can you make up a quadratic function with no vertex? Ken said: "no, it is part of the definition of what a quadratic is."

R: And what did you say about quadratic functions? I think we taught your friend there [*refers to Task 4*].

K: I think it is there but I used it when we [*picks up his Task 2 sheet*] here. That's what I did just something that made it special or important that it had a minimum or maximum value and the minimum or maximum is basically the vertex of the graph.

R: Umm.

K: Yeah.

R: And without a minimum or a maximum.

K: Then it's not a quadratic function anymore.

Task 8. Finding x-intercept, y-intercept, line of symmetry, and vertex.

Ken approached this task by comparing the four function equations for $f$, $g$, $h$ and $k$ to the functions that he worked on in the previous tasks. He stated that the $h$ function is very similar to the one he worked on in Task 7, "except for the negative sign." He then started the task by investigating the x-intercepts and said that in the $k$ function, one can set each of the factors equal to zero ($(x-3)=0$ and $(x+2)=0$), as he did previously, and easily solve for the x-intercepts. Ken asserted that $f$ and $h$, the two functions that are given in the form $y = ax^2 + bx + c$, are not too difficult in terms of finding the x-intercepts, but "he needs to do something." He offered the same explanation for the $g$ function as well. It is inferred that Ken is referring to setting the equations equal to zero and carrying out algebraic manipulations such as factoring. Recall that he was able to factor the expression $6x - x^2$ into $x\,(6 - x)$ in Task 7. In terms of finding the vertex, Ken wrote "$-b/2a$, plug $x$ to solve for $y$" next to the function $h$. Referring to $f$ as the first function, $g$ as the second function, $h$ as the third function, and $k$ as the fourth function, he then wrote: "vertex: 1st and 3rd easiest, 2nd and 4th most difficult." He offered the following explanation.

> K: Cause again I have to remember how to do it. I mean it's not that it's actually like complicated math or some abstract concept, it's just I forgot how to do it. So to solve, like if I was given a question like that tomorrow on my test.
>
> R: Hmm.
>
> K: I would have to remember how to do it. I would have to relearn it.

In terms of the y-intercept, Ken also wrote: "y-int: 1st and 3rd easiest, 2nd and 4th most difficult," arguing that one can find the y-intercept by setting the $x$ equal to zero. He also added that "wherever the graph crosses the y-axis, that's the y-intercept, where $x$ is zero." He said that in the first and third functions, y-intercept is zero. For the other ones, again, "he needs to do something."

Ken also discussed the line of symmetry, arguing that it is the line that goes through the vertex. He claimed that the line of symmetry is only the $x$ component of the vertex, and "$y$ values do not matter," because one cannot have sideways parabolas. When the researcher showed his graphs in Figure 1.5, Ken said:

K: Oh yeah, the vertical line test it's test of if it's actually a function. You could use that and explain why that's not a function [*pointing at one of the "sideways parabolas" in Figure 1.5*].

R: What is the vertical line test?

K: It tells you that, cause part of the definition of a function is that no one $x$ value you can have more than one $y$ value? I think that's right. Yeah.

He also moved his pencil along the graph from right to left, while mentioning that the vertical line has to cross the graph only once. It is assumed that he was referring to the vertical lines crossing the graph at each $x$ value only once. He also drew another graph in Figure 1.16 to demonstrate that one cannot have two $y$ values for a given $x$ value.

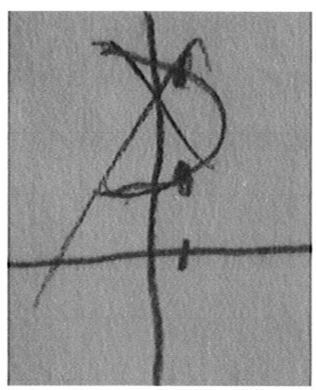


*Figure 1.16.* Ken's Explanation of the Vertical Line Test in Task 8

Ken also stated that the graph is symmetric about the line of symmetry.

Task 9. Finding the quadratic function that has a vertex $(-2, 5)$ and that passes through $(0, 9)$.

Ken's first reaction to this task was to ask: "Am I looking for the equation?" After receiving an answer of "yes," he decided to draw the graph first. He plotted the two given points, as in Figure 1.17, and sketched a parabola that passes through them.

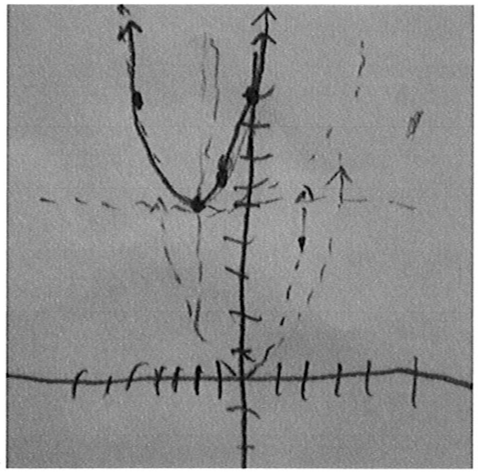


*Figure 1.17.* Ken's Response to Task 9

In terms of the equation for this graph, he first wrote $ax^2 + bx + c = 0$, and then stated that he would use the form in which it is easier to see the transformations [while writing the two equations in Figure 1.18].

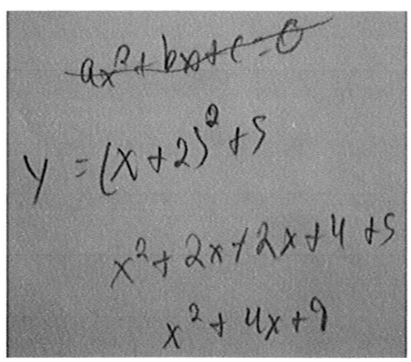


*Figure 1.18.* Ken's Equations in Task 9

He reiterated the rules for horizontal and vertical translations (as well as reflections), and said:

K: Now I am thinking about if it was either stretched or shrunk, which is if that's original graph it could be like that or it could be like that [*draws the graphs in Figure 1.19*]. And how exactly solve for that I don't know.

R: Can you manipulate that a little bit? In other words, umm what did we say that, rearrange the equation.

K: Sure. I can expand it. [*Writes the expressions in Figure 1.18*]

R: And how can we find if this is stretched or shrunk. [*Pointing at the graphs in Figures 1.18 and 1.19*]

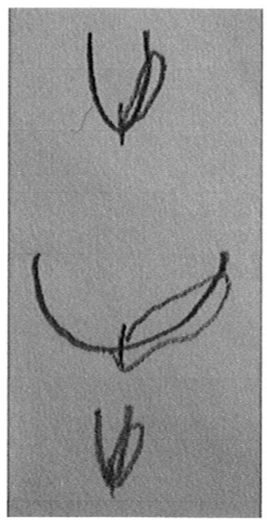


*Figure 1.19.* Ken's Demonstration of Dilations in Task 9

K: I am not sure that they are going out the same angle [*moving his pencil along the right hand side of the graph in Figure 1.17*]. So like a way of saying.

Ken argued that because the slope of the line that passes through the two points $(-2, 5)$ and $(0, 9)$ is $4/2 = 2$ and the slope of the line that passes through the two points $(0, 0)$ and $(2, 4)$ in the graph of $y = x^2$ is also $4/2 = 2$, there is no dilation. He offered this explanation while circling the parts of the graphs in Figure 1.19 and making triangles in the air on the graph in Figure 1.17 [from $(-2, 5)$ to $(0, 9)$ one would "go to the right two and up four," and from $(0, 0)$ to $(2, 4)$ one would also "go to the right two and up four"]. He also added that the two graphs are "proportional." After establishing that there is no dilation, and thus the leading coefficient is 1, Ken claimed that $y = (x + 2)^2 + 5$ is the correct equation of his graph in Figure 1.17.

Task 10. Choosing the easiest graph to represent with an equation.

Ken approached this task by discussing whether the three graphs were stretched or shrunk.

> K: The basic quadratic is a parabola pointing up [*draws the graph of* $y = x^2$ *in the air*]. And I can see that that one is pretty wide.
>
> R: Which one?
>
> K: The longest one [*pointing at the middle graph in Figure 1.20*]. It is expanded. And that one it's shrunk [*pointing at the graph on the right*] so there would be a half in front of that [*pointing at the a in the equation* $f(x) = ax^2 + bx + c$]. And that one [*pointing at the graph on the left*] it looks like a little smaller but would just use I guess not, I guess I would pick this one.

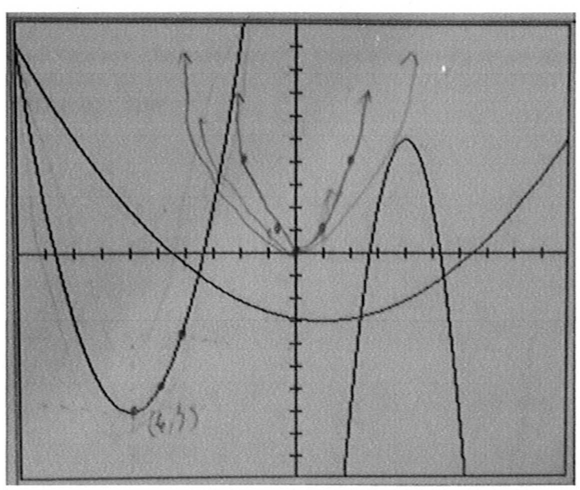


*Figure 1.20.* Ken's Response to Task 10

Ken chose the graph on the left in Figure 1.20 and used the same reasoning in the previous task about dilations. In other words, he compared the slopes of two lines passing through the points $(-6, -7)$ and $(-5, -6)$ and through $(0, 0)$ and $(1, 1)$. He identified the vertex of the graph as $(-6, -7)$ and concluded that there is no dilation. Thus he treated the graph the same as the shape of the parabola $y = x^2$, which is moved six units to the left and seven units down. He represented the equation of his graph [the one on the left in Figure 1.20] as in Figure 1.21.

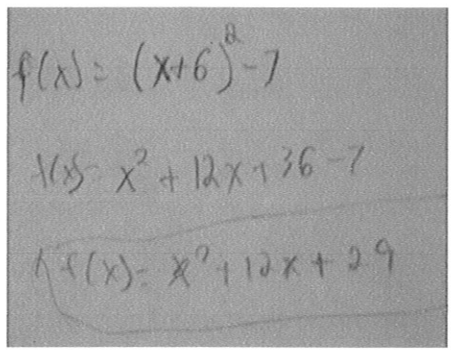


*Figure 1.21.* Ken's Equation in Task 10

Finally, after writing the equation in the form $f(x) = x^2 + 12x + 29$, Ken used the expression $-b/2a$ to compute the $x$ value of the vertex. He found the $x$ component of the vertex to be $x = -6$, which confirmed that his answer was correct.

Task 11. Creating a quadratic function with no x-intercept.

At first, Ken wrote $y = x^2 + 2$ as an example of a quadratic function with no x-intercept claiming that drawing the graph of $y = x^2 + 2$ would simply be moving the graph of $y = x^2$ up two units. Therefore there would be no x-intercepts. However, when the researcher asked if he can find another function that resembles the given $f$, $g$, $h$, or $k$ in Task 11, Ken started manipulating $(6 - x)^2$ [in the lower right corner of Figure 1.22].

After several attempts at finding out the vertex and the transformations applied to $y = x^2$, he decided to expand $(6 - x)^2$ as in Figure 1.23.

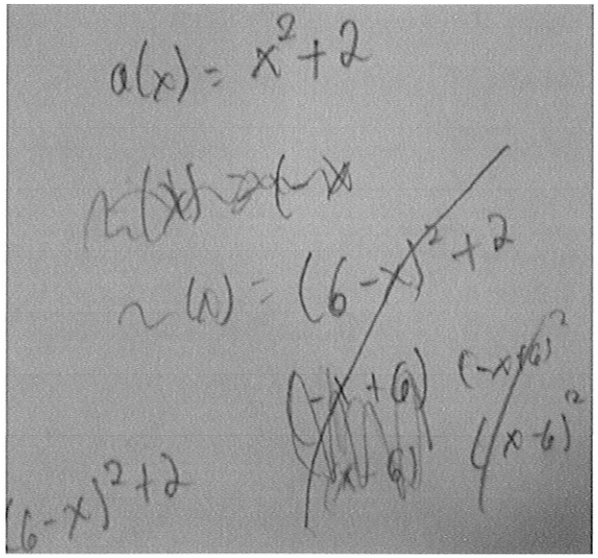



*Figure 1.22.* Ken's First Attempt to Visualize the Graph of $g(x) = (6 - x)^2$

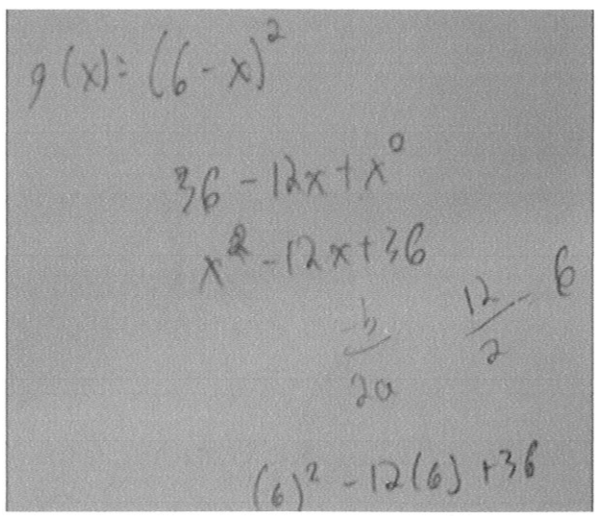



*Figure 1.23.* Ken's Second Attempt to Visualize the Graph of $g(x) = (6 - x)^2$

After finding the vertex [$(6, 0)$], he drew the graph in Figure 1.24, arguing that the graph is opening upward because there is no reflection about the x-axis. He also added that there is only a reflection about the y-axis. Without discussing this reflection, he drew the dashed graph in Figure 1.24 in order to represent the vertical translation of two units upward. He concluded that he found the function he was looking for [i.e., one without an x-intercept].

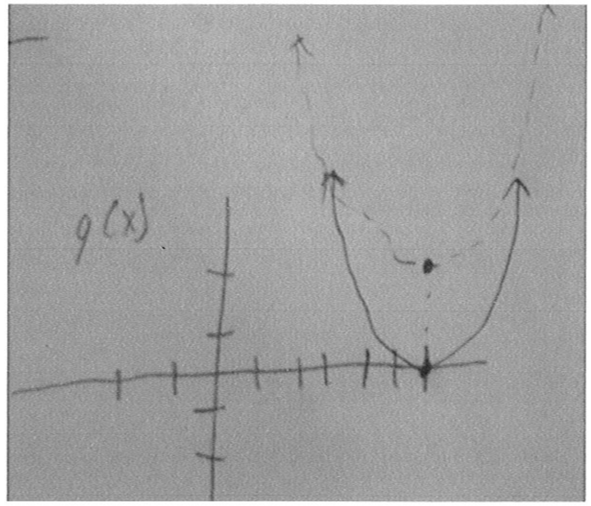



*Figure 1.24.* Ken's Sketch of the Graphs of $g(x) = (6 - x)^2$ and $y = (6 - x)^2 + 2$

Task 12. Comparing $\mathrm{px}^2 - \mathrm{qx} + 3$ and $\mathrm{px}^2 - \mathrm{qx} + 6$.

In this task, Ken first suggested that one can also write the given function equations in the form: $(x + \_\_)(x - \_\_)$. However, he added that he does not want to solve the problem "that way" and changed $(x + \_\_)(x - \_\_)$ to $(x - \_\_)(x - \_\_)$. He then wrote $y = 0$ above the phrase 'x-intercepts' on the task sheet, and stated that the graph of $g$ would be the same as the graph of $f$ shifted 3 units up. He illustrated this in Figure 1.25.

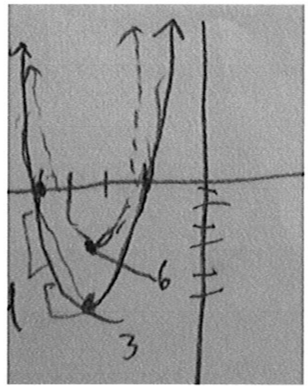


*Figure 1.25.* Ken's Response to Task 12

He chose arbitrary values for the vertices and the x-intercept to simply give examples and illustrate his point about the vertical translation. He said that the x-intercepts of $f$ and $g$ are not the same because:

K: By graphing that [*pointing at the graph in bold in Figure 1.25*] I can see that because it is an exponential growth [*pointing at the right hand side of the graph in bold in Figure 1.25, while tracing his pencil in the air along the curve*] it's not really. What I mean by that I mean like the distance between there is the same as up there [*drawing the small rectangular shapes on the left hand side of the graph in bold in Figure 1.25*].

R: Can you explain that?

K: Yeah, I guess the best way to explain that is by looking at the $x$ squared.

Ken drew the graph in Figure 1.26 and argued that the distances between $y$ values are getting larger as one moves to the next $x$ value. He pointed out the difference of 1 between the $y$ values of $(0, 0)$ and $(1, 1)$ and the difference of 3 between the $y$ values of $(1,1)$ and $(2, 4)$. Therefore, because of this exponential growth, Ken stated that the x-intercepts of the two graphs in Figure 1.25 would be difference since the dashed graph in

Figure 1.25 (i.e., the graph of $g(x) = px^2 - qx + 6$ "would not have enough time to get to" the x-intercepts of the graph of $f$.

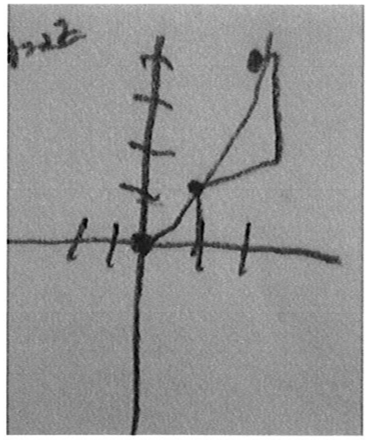


*Figure 1.26.* Ken's Explanation of the "Exponential Growth"

He concluded this task by stating that y-intercepts of $f$ and $g$ would also be different because $f(0) = 0 - 0 + 3 = 3$ and $g(0) = 6$. And for the vertex, he wrote: "$-b/2a$ same $x$ different $y$ by 3."

Case 2 Results: Sarah's Solution Activity.

The second case of this study is a college senior with a pseudonym Sarah. Sarah is a returning college student, who in high school, about twelve years ago, took Algebra, Geometry, Trigonometry and AP Calculus courses. Five or six years ago, she also took college algebra and statistics courses. She was enrolled in a college pre-calculus course during time of data collection. She characterizes her experiences in mathematics as: "I usually do very well in math classes and usually find the concepts easy to learn." In her individual survey comments, Sarah stated: "I like mathematics first of all, because there is always an exact answer. I am enjoying pre-calculus, and I enjoyed recently using the trigonometric identities to simplify expressions. I also remember enjoying postulates in geometry." When asked "what aspects of mathematics, if any, do you dislike the most?"

Sarah wrote that she remembers disliking some aspects of calculus. She remembers statistics and calculus "being a little difficult," but other than that she seems quite positive toward mathematics. In the same survey, she self-rated the strength of her mathematics knowledge in arithmetic, algebra, geometry, upper level mathematics, and other mathematical topics such as discrete mathematics and statistics, on a scale of 1 to 5 (1 being the weakest and 5 the strongest), as 5, 5, 4, 5, and 3, respectively. In other words, other than geometry, discrete mathematics and statistics, he placed herself in the strongest category.

During both interviews, Sarah worked very hard on each task in order to answer every question she was asked. When she experienced difficulty, she seemed only slightly frustrated and she continued to work hard on the task until her results made more sense to her. Below are the descriptions of Sarah's responses to the tasks in APPENDIX B.

Task 1: Describing functions.

Sarah started the first task by immediately writing on her paper "$f(x) =$" while stating:

> S [Sarah]: Umm, so I first think of a function I think of $f$ of $x$ equals something umm and if I draw a graph I think that definition of a function, I'm just going to draw a line [*draws the line in Figure 2.1*], umm will pass a vertical line test [*makes a horizontal motion with her pencil across the entire graph from left to right*] so that means it, I think it means it can't take, can't, let me draw one that isn't a function. [*Draws the curve in Figure 2.2*]
>
> R: Okay.

S: So this one [*pointing at the graph in Figure 2.2*] would not be a function cause it doesn't pass the vertical line test [*makes a horizontal motion across the graph with her pencil, holding it as a vertical line*]. So it can have no umm for every value of $x$ [*draws a table with two columns one labeled x, the other y in Figure 2.3*] umm there can't be the same, no, for every value of, okay hold on let me think here. This has the same $x$ value [*draws the dashed line in Figure 2.2*] so $x$ is three there [*writes 3 in the x column of the table in Figure 2.3*] umm see you can't have umm [*writes another 3 in the x column of the table in Figure 2.3*] more than one why can't I, okay, I'm sorry umm you can't have more than one $y$ value for a single value of $x$.

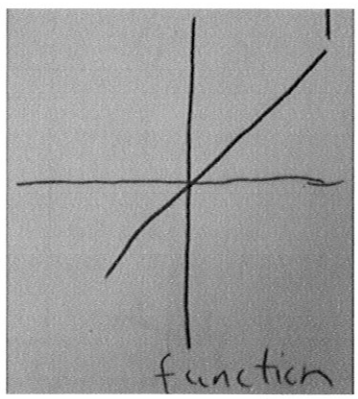



*Figure 2.1.* First Part of Sarah's Response to Task 1

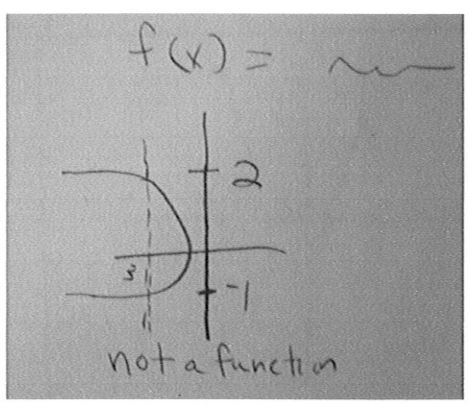



*Figure 2.2.* Second Part of Sarah's Response to Task 1

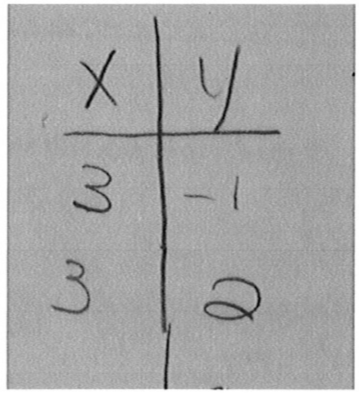



*Figure 2.3.* Sarah's First Table in Task 1

Similar to that of Ken, her explanation of what constitutes, or what does not constitute, a function included three different types of representations: numerical, graphical and algebraic. Unlike Ken, however, Sarah used 'the vertical line test' as an important tool in determining whether a given graph represents a function or not. When asked how the vertical line test works, she said: "Make a vertical line, and go across, and if the graph does not hit the vertical line in more than one place, then it's a function." For Sarah, every value of $x$ must have only one $y$ value—not more. When asked why for every value of $x$ there must be only one $y$ value, Sarah offered the following explanation:

S: $F$ of $x$ equals whatever that equation is [*draws the scribble next to "$f(x) =$" in Figure 2.2*]. $F$ of $x$ is the same as $y$ [*writes $f(x) = y$*] so if you have an equation [*pointing at the scribble in Figure 2.2*] umm that equals two different $y$'s wait I thought I was making some sense umm then they can't they won't equal two different $y$'s maybe? Or they won't be a function of $x$. But I'm still not. I think I am confusing myself now. Umm I just know that by definition I guess.

When the researcher asked how she would define a function in one sentence, Sarah stated:

S: An equation of something, a line or shape that you can graph that for every value of $x$ only has one $y$ value on the graph.

The way Sarah referred to a mathematical expression [in her case, the scribble in Figure 2.2] is similar to how Ken called expressions equations. In her discussion of functions, Sarah also defined the terms equation and graph: Equation is "more the numerical representation or a way to write it," and a graph is a "visual representation of the shape or whatever that equation makes." She gave two additional examples of a function, in Figures 2.4 and 2.5, besides the line in Figure 2.1. She wrote $y = 3x + 4$, drew a table with numbers 1, -1, -2, 0 in the first [$x$] column, and the corresponding 7, 1, -2 values in the second [$y$] column. She left the $y$ value that corresponds to $x = 0$ blank, but she computed $y = 3(0) + 4 = 4$ and plotted the point $(0, 4)$ on a coordinate plane. She sketched the graph of the line $y = 3x + 4$ in Figure 2.4 by plotting and connecting the points $(1, 7)$, $(-1, 1)$, $(-2, -2)$ and $(0, 4)$.

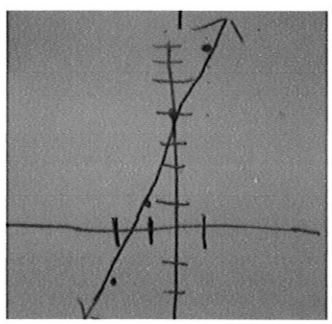


*Figure 2.4.* Sarah's First Additional Example in Task 1

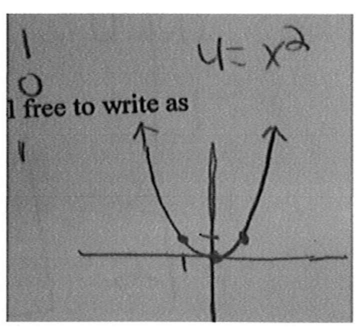


*Figure 2.5.* Sarah's Second Additional Example in Task 1

Her final example of a function was "a parabola," for which she argued that "the parent graph of a parabola" passes the vertical line test and is written as $y = x^2$ [while referring to her graph in Figure 2.5 and making a horizontal motion with her pencil across the entire graph from left to right]. She also generated a table of values in Figure 2.6 using the equation $y = x^2$. After she gave these two additional examples, the researcher summarized what Sarah has said until that moment about functions:

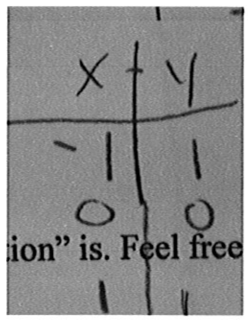



*Figure 2.6.* Sarah's Second Table in Task 1

R: Okay. And that's another example of an equation and the graph.

S: Yes.

R: Okay. Umm so in summary, if I understood it correctly, you mentioned about an equation and $y$ is equal to $f$ of $x$ and that's some equation.

S: Uh-huh.

R: And that equation has is producing some shape, which is represented visually by a graph.

S: Uh-huh.

R: And for each $x$ value in that graph there cannot be more than one $y$ value.

S: Yes.

R: And that's established by checking with vertical line test.

S: Uh-huh.

R: And so, what you call a function is? Is that the equation or the graph?

S: Umm I guess both really, the equation and the graph [*pause*] are functions.

R: What do they have in common?

S: Umm, what do they have in common? [*Speaks quietly*]

R: The equation and the graph. What are you thinking?

S: Umm I am trying to think of what they would have in common, I mean. If you take any point on the graph, and umm plug it into the equation so if you take any or if you take any $x$ value and plug it into the equation you will get the corresponding $y$ value that's on the graph [*pointing at the point* $(-1, 1)$ *in Figure 2.5*], if you take a $y$ value on the graph you'll get umm and plug it into the equation you will get the corresponding $x$ value. So they're umm I mean they're the same thing they are just different representations of it.

R: Okay. What is the thing? What is the thing that they both represent?

S: Oh they both represent the function.

R: Uh-huh. Okay. What is it? That we call function, that commonality. What kind of thing is that?

S: Umm. What kind of thing? I am not sure. I am trying to think. Umm, what kind of thing, is the relationship that they have? Or what kind of thing is a function? Umm.

Task 2: Describing quadratic functions.

Sarah approached the second task by stating that she is "trying to remember what a quadratic equation is." First, she wrote $ax^2 + bx + c = 0$ in Figure 2.7, and stated:

S: Umm, what is their shape usually? Umm, so I can just make up an equation so I can make a graph of it. Umm which I mean I do think [*draws two perpendicular lines, the* $x$ *and* $y$ *axes, in the box in Figure 2.7*] $y$ equals $x$ squared is a quadratic equation. Umm, but, umm that's almost too easy to really show this but umm maybe and what I think what makes it quadratic when I see quad [*underlines the first four letters "quad" in the word "quadratic" in the bottom of Figure 2.7*] I think of four, umm and I'm not really exactly sure why it's called that but the best thing that I can maybe think of is why it might mean four [*draws two empty parentheses* ( )( )] is because you when you take this equation [*pointing at* $ax^2 + bx + c = 0$] depending upon what it is and if you put it into its two factors.

R: Uh-huh.

S: You'll have something plus something plus something [*includes addition signs in the two empty parentheses:* ( + )( + )] umm.

R: Okay.

S: So there is one, two, three, four [*draws four small lines inside the two parentheses:* (_ + _)(_ + _)] of those that could be why it's called quadratic umm.

R: Can you draw? One quadratic function? Doesn't matter if it's.

S: If it's just $y$ is equal to $x$ squared?

R: If it's easy or difficult or doesn't matter.

S: Okay. I'll just draw this one cause that's just what I think of [*draws the graph of* $y = x^2$ *in Figure 2.7 in one move without lifting her pencil off from the paper*].

R: Graph of a quadratic function?

S: Uh-huh. Yeah.

R: Okay. What makes this graph quadratic?

S: Umm, well it I know it fits this equation [*pointing at* $ax^2 + bx + c = 0$] umm because if umm if $a$ was one and $b$ was zero and $c$ was zero, we would have umm $x$ squared equals zero? I don't know. Umm it would fit that definition I guess. Umm maybe because umm there is quadrants? [*Touches four points in the four distinct regions of the coordinate plane, Quadrant 1, 2, 3, and 4*] and it's the same on both sides of two of them? [*Touches the point on the y-axis in Figure 2.7, where* $y = 1$, *and two other points on the graph of* $y = x^2$ *in Figure 2.7, one point on the left, near* $(-1, 1)$ *and another on the right near* $(1, 1)$]

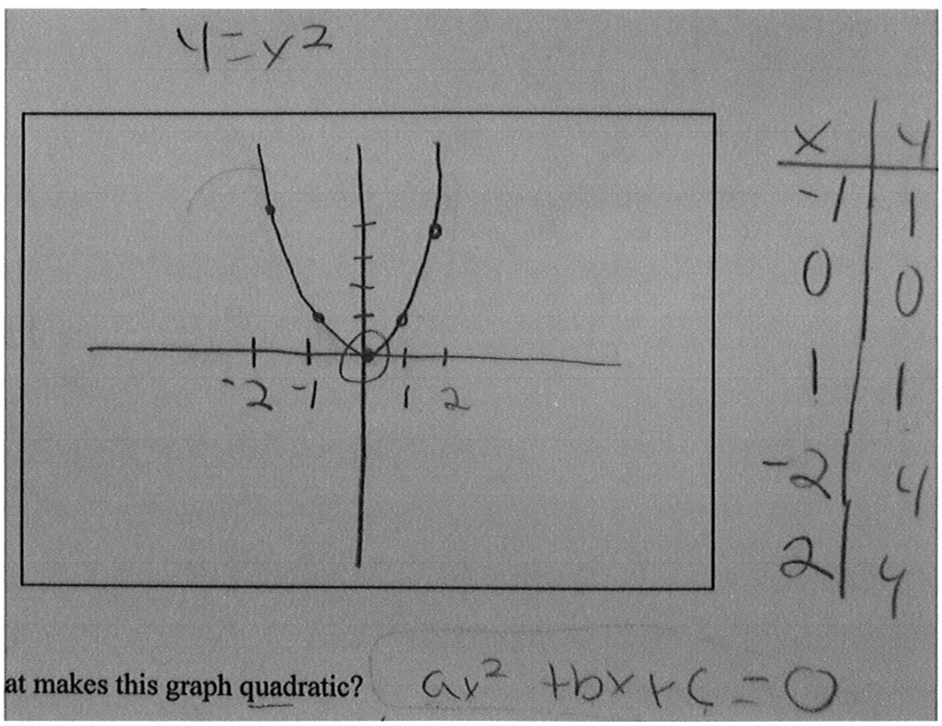



*Figure 2.7.* Sarah's Response to Task 2

When the researcher suggested that she may think of a quadratic function in terms of how it is different from other types of functions, and not so much in terms of how it is worded, Sarah said: "Because $x$ is squared?" while pointing at the $x^2$ in $ax^2 + bx + c = 0$. She continued: "So you will have more than one $y$ value for each $x$ maybe." To explain why that is so, Sarah generated the table in Figure 2.7 and plotted the five points $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(-2, 4)$ and $(2, 4)$ on the graph. She argued that $x = -2$ and $x = 2$ both have the same $y$ value of 4 (and that $x = -1$ and $x = 1$ both have the same $y$ value of 1), and therefore "the absolute value of any $x$ value will have the same $y$ value" "because it is squared." Although this assertion seems to contradict what she said about the definition of a function when using the vertical line test to ensure that there is only one $y$ value for each $x$ value, Sarah seems to be simultaneously operating in two different mathematical situations without the need to somehow connect the situations. While making her argument that $x = -2$ and $x = 2$ both have the same $y$ value of 4, Sarah was pointing at the two points $(-2, 4)$ and $(2, 4)$ one after the other on an imaginary $y = 4$ line. Therefore, she was correct in identifying two different things, or two different points with the same $y$ value on the quadratic graph, that are symmetric. Later, when she plotted the x-intercepts of $(x + 3)(x + 4) = y$, $(-3, 0)$ and $(-4, 0)$, she noticed that the $x$ coordinate of the vertex $x = \frac{-b}{2a} = -\frac{7}{2} = -3.5$ lies in the middle of the two x-intercepts. She introduced the term $\frac{-b}{2a}$ herself, using it to find the $x$ component of the vertex. Her sense of the existence of some sort of symmetry, which brings about "two $y$ values," one on the left and one on the right, seems to have been further confirmed in this later discussion. Thus, the need to identify and describe various cognitive structures (or lattices of acts of understandings, or bases of understandings) in all phases of construction using the clinical interview is clear in researchers' attempts to make inferences about "the structures that underlie behavior" (Noddings, 1990, p. 9). It is evident in her two separate behaviors that Sarah is not coordinating the vertical line test, which, for her, requires the existence of one $y$ value for each $x$ value, with her observation that there are two points on the quadratic graph that lie on the $y = k$ line and are symmetric about the line $x = h$, where vertex is at point $(h, k)$.

R: [*Sarah generated the table in Figure 2.7 and plotted the five points*] If you take one of the $x$ values, do you see two $y$ values or?

S: Uh-huh. Yeah. For the two and negative two [*pointing at the numbers -2 and 2 in the x column of the table in Figure 2.7*] they both have a $y$ value of four umm negative one and one [*pointing at the numbers -1 and 1 in the x column of the table in Figure 2.7*] both each have the $y$ value of one.

It is also plausible that because Sarah thinks that "the absolute value of any $x$ value will have the same $y$ value," and "because it is squared," $x = -1$ and $x = 1$ can somewhat be treated as a single $x$ value; and that because both $x = -1$ and $x = 1$ generate $y = 1$ simultaneously, side by side, $y = 1$ that corresponds to $x = -1$ and $y = 1$ that corresponds to $x = 1$ can somewhat be treated as two $y$ values. Also, note that Sarah did not mention the vertical line test anywhere in this second task. Thus far, she offered the quadratic equation $ax^2 + bx + c = 0$ as the general form of a quadratic function; sketched the "too easy" example of $y = x^2$ (the case where $a = 1$, $b = 0$, and $c = 0$ in $ax^2 + bx + c = 0$); and explored why the function might be named quadratic.

In terms of the important or special parts of the graph, Sarah asserted that the vertex and the origin are the most important parts of the graph. When asked, she explicitly stated that she refers to point $(0, 0)$ as the origin. She sees the origin as an important part of the graph because the graph of $y = x^2$ "is symmetric about the origin."

Furthermore, according to Sarah, the origin or the vertex in $y = x^2$ is also the minimum point and no $y$ value of the function can be negative or can "go below that." She confirmed this fact by stating that "anytime you square any number you will always get a positive number." When further asked if the origin is important only for the graph of $y = x^2$ in Figure 2.7 or for any graph, Sarah said she wishes she could draw and picture another quadratic graph in her head.

To find another example of a quadratic graph, she went on to fill the two parentheses she wrote earlier, as $(x + 3)(x + 4) = y$, and solved the two equations $x + 3 = 0$ and $x + 4 = 0$ and found the zeros of the function, $x = -3$ and $x = -4$.

> S: But that doesn't really tell me my $y$ value [*while solving the equations* $x + 3 = 0$ *and* $x + 4 = 0$ *and finding* $x = -3$ *and* $x = -4$]. What does that, oh, those are points for, okay, so for this equation [*pointing at* $(x + 3)(x + 4) = 0$] ... So $x$ is negative three, I believe that means those are points where it yeah where $y$ is zero [*marks two points on the x-axis in Figure 2.9,* $x = -3$ *and* $x = -4$]. So $y$ is zero at those points. And I can't remember how to convert this [*pointing at* $ax^2 + bx + c = 0$] to find what the vertex is.

She wrote $\frac{-b}{2a}$, and manipulated the expression $(x + 3)(x + 4)$ and wrote $x^2 + 4x + 3x + 12$ and $x^2 + 7x + 12$ in Figure 2.8. In order to find the $y$ value of the vertex, Sarah asserted that she could "plug in the $x$ value in that equation." [Pointing at the expression $x^2 + 7x + 12$ in Figure 2.8]

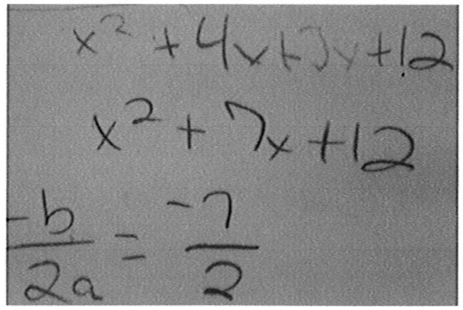



*Figure 2.8.* Finding the Vertex

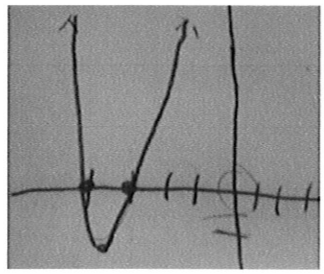


*Figure 2.9.* Graphing $(x+3)(x+4)=y$

Sarah remembered that there is a vertex form of the quadratic equation, $ax^2 + bx + c = 0$, but she couldn't recall what that was. However, she introduced $\frac{-b}{2a}$ as a way of finding the $x$ value of the vertex, and found $\frac{-7}{2}$, which she said makes sense for the vertex to be in between $x = -3$ and $x = -4$. When asked why, she responded: "Because, oh, yeah cause it's a quadratic so I guess quadratics are always parabolas." The vertex needs to be in between the two points where $\boldsymbol{y}$ is equal to zero because "it needs to be drawn like this"—she draws the graph in Figure 2.9. Sarah discussed about the symmetric property of parabolas earlier, thus it is reasonable to assume that she is thinking of some line of symmetry where the vertex lies. She did not make any reference to why the graph is facing upward or why the vertex was sketched in that particular location in Figure 2.9.

Before sketching this graph, while solving the two equations $x + 3 = 0$ and $x + 4 = 0$, Sarah stopped for a moment and asked herself: "Why am I setting these equal to zero? All that's going to do is get me an $x$ value." At first, she was looking at $(x + 3)(x + 4) = y$; but then she considered $ax^2 + bx + c = 0$ and said that "it should be equal to zero" because "that's the quadratic equation." She changed $(x + 3)(x + 4) = y$ into $(x + 3)(x + 4) = 0$. Recall that her original purpose was to generate another quadratic graph by choosing four terms in $(\_ + \_)(\_ + \_) = y$. After writing $(x + 3)(x + 4) = y$, she drew a blank table with $x$ and $y$ columns and mentioned that she would "solve for $y$ values." In other words, initially she seemed to be graphing the function $(x + 3)(x + 4) = y$ by generating $(x, y)$ pairs and plotting them in the coordinate plane—and not by finding x-intercepts. However, she did not do that. Her new focus on the quadratic equations $ax^2 + bx + c = 0$ and $(x + 3)(x + 4) = 0$ led her to work on solving $x + 3 = 0$ and $x + 4 = 0$ and finding $x = -3$ and $x = -4$. She later claimed that when $x = -3$ and $x = -4$, $y$ values are zero. And, she sketched the graph in Figure 2.9 by plotting these two points $(-3, 0)$ and $(-4, 0)$ and simply connecting them so that they form the shape of a parabola.

Sarah then tried to find the vertex by substituting $\frac{-7}{2}$ for $x$ in $x^2 + 7x + 12$ and tried to compute the corresponding $y$ value. After several arithmetic errors and subsequent corrections, she arrived at $y = \frac{-1}{4}$, which made sense to her in terms of her graph in Figure 2.9. She closed this task by adding the phrase "$y$ values are zero" to her initial response to the question in Task 2b about important or special parts of a quadratic graph. The parts of the graph where $y$ values are zero, the x-intercepts, are important to

Sarah because they helped her draw the graph. As a response to Task 2c, she wrote $x^2 + 7x + 12 = y$ and $(x + 3)(x + 4) = y$ to indicate her graph in Figure 2.9.

Task 3: Graphing $y = -(x - 4)^2 + 16$.

S: So, I'll probably just start by plugging in values of $x$ because that's the easiest way to do that.

When the researcher asked if there is any other way she can graph the equation, Sarah said that she can do what she just did: She "could set the equation equal to zero and find the $x$ values and draw them." She then wrote $0 = -(x - 4)^2 + 16$ and solved the equation as in Figure 2.10.

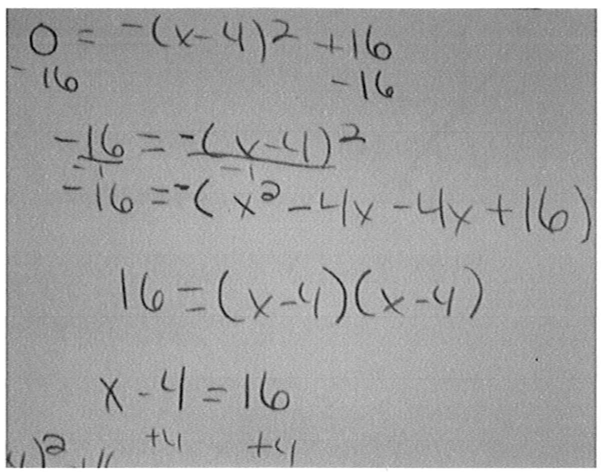



*Figure 2.10.* Finding the x-intercepts of $y = -(x - 4)^2 + 16$

At the solution step "$-16 = -(x - 4)^2$," Sarah "factored out" the term $(x - 4)^2$ by uttering "first, outer, inner and last." And, after she wrote "$-16 = -(x^2 - 4x - 4x + 16)$," she said: "Is that what I wanna do? Wait a minute." She then decided that she is not going to do "that," and divided both sides of the equation by negative one and wrote "$16 = (x - 4)(x - 4)$." This choice of strategy for solving quadratic equations will become clearer in a later task on solving quadratic equations.

S: And since those are both the same [*pointing at* $(x-4)(x-4)$], I'm only going to do once I'm only gonna have one zero value for $x$ [*writes* $(x-4)=16$ *and* $x=20$]. When $y$ is zero, $x$ is twenty [*draws a single point on the x-axis of a blank coordinate plane*].

She checked her solution in Figure 2.10 by mentally computing all the steps one more time, and said: "So all I have is the vertex really." It is unclear at this point why she called $x = 20$ as the vertex. Immediately after mentioning the vertex, Sarah claimed that she could also factor the equation out [while pointing at the right hand side of the original equation $y = -(x-4)^2 + 16$] and find the *'a'* and *'b'* values in order to find where the vertex is.

S: But is it going to be the same anyway?

R: Hmm.

S: Umm, $b$ over two $a$ equals, forget how you do that; I'm just going to plug in some values because.

R: Okay.

S: I am not remembering right now.

Sarah then counted by twos on the x-axis up to 18 and stopped at the point on the axis adjacent to $x = 20$. Later in the interview, Sarah will mention that she is drawing a quadratic graph by connecting the $x$ values that are next to each other. By "next to each other," it is understood that she is thinking of consecutive integer values, $n-1, n, n+1$, or $n-2, n, n+2$, etc., depending on the scale used. After finding a large $y$ value, 196, by substituting $x = 18$ in $y = -(x-4)^2 + 16$ Sarah decided to start from a smaller $x$ value instead. Figure 2.11 shows the table of $x$ and $y$ values that she generated; and

Figure 2.12 shows the points that she plotted and the graph that she drew—again, by connecting the points one by one such that the shape come out as that of a parabola.

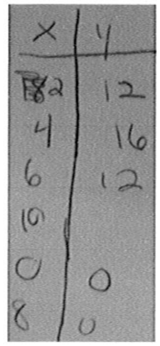



*Figure 2.11.* Sarah's Table of Values in Task 3

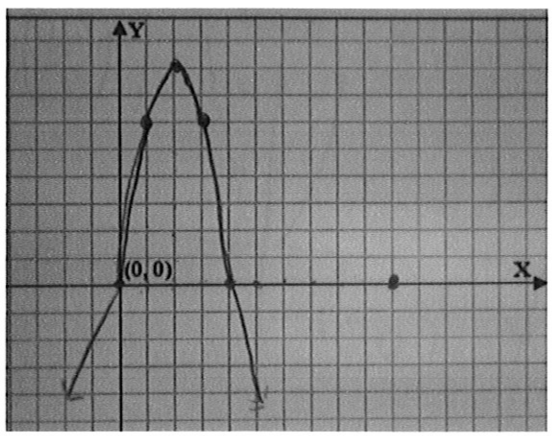



*Figure 2.12.* Sarah's Response to Task 3

At a certain moment when she had only three points plotted, i.e., the initial point $(20, 0)$ on the far right, $(2, 12)$ and $(4, 16)$, Sarah seemed puzzled about the way the three points looked on the graph paper.

S: What am I doing wrong here? [*Computes the y values for* $x = 2$ *and* $x = 4$ *again*]

[...]

S: This is not what I expected this graph to look like? Umm.

R: Why is that the case?

S: Umm I thought it would be a parabola. Umm.

R: What is wrong with that scenario here?

S: Umm, oh! [*Exclamation*] I see, it's negative [*pointing at the negative sign in front of the term* $(x-4)^2$ *in the original function equation* $y = -(x-4)^2 + 16$]. So it's gonna be, it's gonna be this way [*draws a very small parabola opening downward, in a blank space on the paper*]. That's why. Okay. I believe that's what's happening. Umm, so six minus four is two, squared, which is four, sixteen minus four is twelve again [*plots the fourth point on the graph paper,* $(6, 12)$]. Okay, there's my vertex. [*Pointing at* $(4, 16)$]

Sarah then counted the small squares on the graph paper along the x-axis, and chose $x = 10$ as the next $x$ value to use. She computed the corresponding $y$ value, $y = -20$, but then, because her graph "is going to be way down there," [somewhere at the lower portions of Figure 2.12] and because she doesn't know "how she got that" [the point $(20, 0)$ on the far left], she changed her choice of $x$ value to $x = 0$ and stated that she is not sure how she found that "zero value" [the point $(20, 0)$ on the far right]. After she plotted the point $(0, 0)$, the researcher asked how many such points she would need to graph the equation. She said:

S: Once you have enough that you see a shape; you can kind of connect the lines. Umm, I don't know if there is set number that you would need.

At this moment, she indicated that she got that point, on the far right, wrong. When the researcher further asked her what kinds of ideas or strategies she follows when connecting the points (or the lines) to draw the graph, she stated:

S: I guess the strategy would be the next closest value of $x$. Umm, because I would draw to there [*connects the points* $(0, 0)$ *and* $(2, 12)$] and then I would draw to that one [*connects* $(2, 12)$ *and* $(4, 16)$] and then this is the next $x$ [*connects* $(4, 16)$ *and* $(6, 12)$] that's the next $x$, so [*connects* $(6, 12)$ *and* $(8, 0)$ *so that the curve resembles the shape of a parabola*]. I don't know how I got that one [*pointing at* $(20, 0)$] so anyways, so yeah.

R: Okay.

Upon researcher's request to reconsider what parts of this new graph [Figure 2.12] are important or special, Sarah posited that the vertex is now the maximum point instead of the minimum point, "because there is a negative sign in front of the quadratic."

R: Can you explain why, when there is a negative, the graph looks like this one? [*Pointing at the graph in Figure 2.12*]

S: Umm. I just know it means you flip it over umm.

R: Over what?

S: From what it would be if that was positive. Umm, let me think. Well it's making you subtract from the sixteen rather than add, so you are subtracting umm so you would be going the same number of spaces in this direction [*traces the tip of her pencil along the graph in Figure 2.12 on the right side of the vertex*] that you would be going in this direction if it was positive [*moves the tip of her pencil in the air along a vertical line upward, starting from the vertex*] therefore it makes it curve down. So, the negative and positive direction of the opening of the parabola.

R: Okay. Are you flipping it about a certain thing or?

S: Umm. Two, four, six, eight, ten, twelve, fourteen [*counts by two on the y-axis with the tip of her pencil in the air*]. Yeah, about the sixteen, which is the maximum. So, yeah. I can't remember.

R: What is that?

S: I am trying to like, usually I don't have problems with remembering these things but now I am having trouble like what the *c* value is. Umm, I guess that is the, that's where the lowest or highest umm $y$ value will be of the graph.

R: The *c*?

S: Yeah, when you have a $x$ squared plus $b$ $x$ plus $c$. Yeah.

Task 4: Further discussion on quadratic functions.

When asked, in Task 4, how she would explain quadratic functions to a friend who missed class, Sarah added that if one starts with the basic graph of $y = x^2$ one can have different transformations of that basic graph [while drawing the two graphs in Figure 2.13 in a row]. She also argued that the graph of $y = -x^2$ represents a reflection of the graph of $y = x^2$ over the x-axis. It is unclear if she simply remembers the rules of transformations she learned in her pre-calculus class or if she is thinking about $y = -x^2 + 0$ as 'making her subtract from the zero rather than add to it' (as in the previous task) and that she is "flipping" the graph over the vertex, $(0, 0)$.

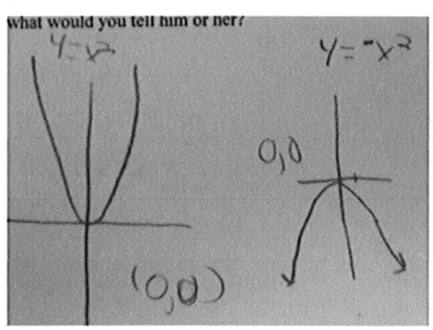



*Figure 2.13.* Graphing $y = -x^2$ and $y = x^2$

S: What are the other ways you can? Trying to remember the other ways that you can graph these. Umm, there is a way to predict the vertex. Umm, but I guess I could tell them also that if you have umm let's see $y$ equals and this is with the let's see $x$ squared minus four.

She generated the quadratic function $y = x^2 - 4$ and factored out the term "$x^2 - 4$" and wrote $y = (x - 2)(x + 2)$. Then, as in previous tasks, she solved the two equations $x - 2 = 0$ and $x + 2 = 0$ [Figure 2.14] and plotted two points where $x = -2$ and $x = 2$ and "the y-values would be zero" [Figure 2.15].

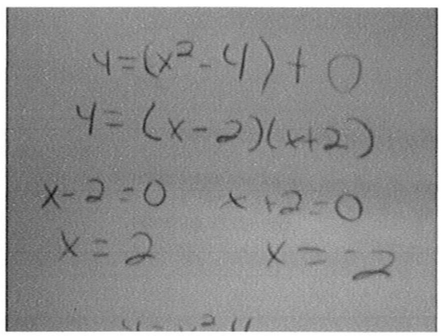



*Figure 2.14.* Finding the x-intercepts of $y = x^2 - 4$

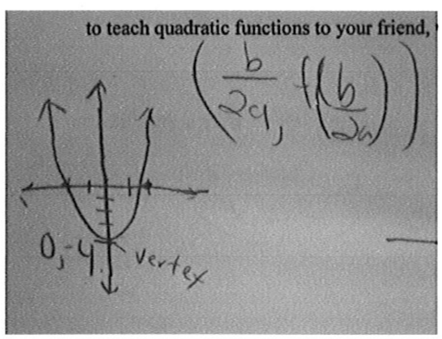



*Figure 2.15.* Graphing $y = x^2 - 4$

S: I don't, I honestly can't remember how to find the vertex right now, or. But that's a way to help graph it. [*Draws the graph in Figure 2.15*]

Sarah also added a zero to $y = x^2 - 4$ in Figure 2.14 [after enclosing the "$x^2 - 4$" term inside parentheses] and argued that, in $y = (x^2 - 4) + 0$, "whatever number you add here or subtract here [the part '$+ 0$'] will move the vertex up or down on the y-axis" [while tracing the tip of her pencil in the air along the y-axis upward and downward].

S: So it will translate up or down.

Lastly, when asked to explain again what vertex means, Sarah clearly stated that she is referring to a point or a "coordinate" with an $x$ and a $y$ value, which is "either the minimum or the maximum of the graph" "depending on where it [the graph] points." She also went back to all three graphs in Figures 2.13 and 2.15 and wrote $(0, 0)$ for the two graphs in Figure 2.13 and wrote the word "vertex" in Figure 2.15. Then she stated that she couldn't remember how to find the vertex of $y = (x^2 - 4) + 0$. Immediately after stating this, however, she added that "on that one" [on the graph in Figure 2.15] the vertex would be "whenever $x$ is zero." As she demonstrated earlier, Sarah seems to somehow pairs up points, here the x-intercepts $(-2, 0)$ and $(2, 0)$, and looks for the vertex in between those two points. After substituting zero for $x$, she found the $y$ value to be negative four and wrote $(0, -4)$ next to the vertex. She also argued that because $b = 0$ in $y = x^2 - 4$, the $x$ component of the vertex will be zero in $x = \frac{b}{2a}$ [while writing $\left(\frac{b}{2a}, f\left(\frac{b}{2a}\right)\right)$ without any negative sign]. Note that Sarah introduced the term "$\frac{-b}{2a}$" as the $x$ component of the vertex in Task 2, therefore it is possible that she does not remember this "formula" well.

Task 5: Solving quadratic equations.

Sarah approached Task 5 [Solving $2x^2 - 7x + 3 = 0$] by saying:

S: So I know I want to factor the $x$ and then I'm gonna have two linear equations multiplied by each other [*writes* ( )( ) = 0].

After checking the equations $(2x + 1)(x + 3) = 0$ and $(2x + 1)(x - 3) = 0$ to see if they represent $2x^2 - 7x + 3 = 0$, using the 'FOIL' (First, Outer, Inner, and Last) method; she settled on the equation $(2x - 1)(x - 3) = 0$ and expanded it into $2x^2 - 6x - x + 3 = 0$ and then to $2x^2 - 7x + 3 = 0$. She then solved the two equations: $2x - 1 = 0$ and $x - 3 = 0$, and wrote $x = ½$ and $x = 3$.

Besides using this method of factoring a quadratic expression into two binomials, Sarah also mentioned that she could use the "quadratic formula" but did not pursue or write this second method. In the second part of this task however, she did write: $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$ and called it the quadratic formula. Without doing any computations, she explained that she would get two different answers from this formula, "because plus or minus accounts for that" ['$\mp$' in $\frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$]. One of the answers would include a minus sign and the other would include a plus sign. When asked what the *a*, *b* and *c* values would be, Sarah wrote $a = 2$, $b = -7$ and $c = 3$ by looking at the original quadratic equation $2x^2 - 7x + 3 = 0$.

S: I really don't think of any other ways to solve for it. Honestly. Umm, those are the two ways I would use.

R: Alright.

In the third part of this task [solving $x^2 - 5x + 6 = 0$ for $x$ in a different way than the way she solved the original equation], Sarah said she would use the "quadratic" and wrote her solution in Figure 2.16.

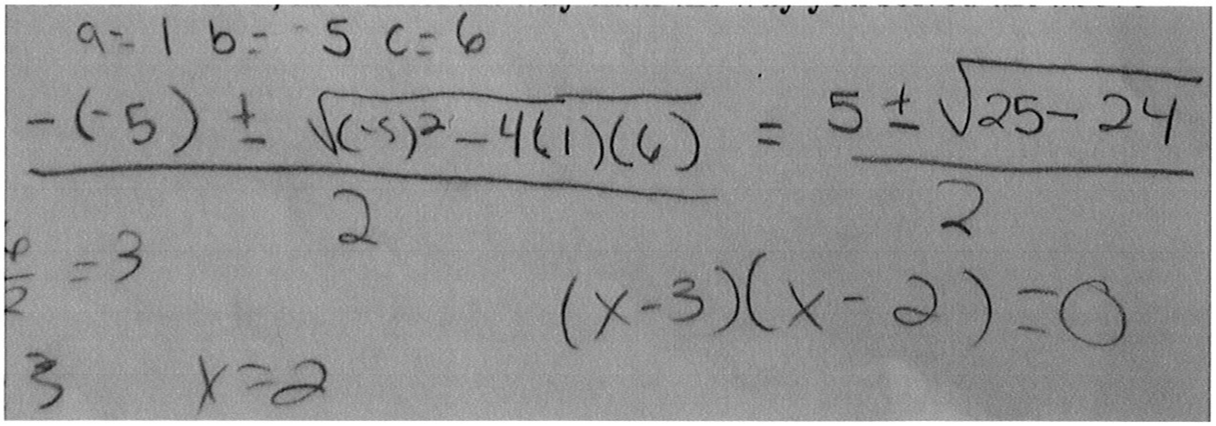



*Figure 2.16.* Sarah's Solution to the Third Part of Task 5

After finding the two solutions $x = 3$ and $x = 2$, Sarah quietly uttered: "That doesn't seem right; because, negative *b*, which would be, cause one of them should be negative." When asked why, she said: "Or they both should be negative."

> S: Because I have negative five there [*pointing at -5 in* $x^2 - 5x + 6 = 0$]. So, I am not sure what I did but it should be $x$ minus three times $x$, oh, [*writes:* "$(x - 3)(x$" *and stops writing*] oh yeah that's right. I'm sorry, because once you do $x$ minus three equals zero [*pointing at* $x = 3$], so yeah that's right [*writes* "$(x - 3)(x - 2) = 0$"] equals zero. So that's right.

Unlike Ken, for Sarah, setting both of the factors of $x^2 - 5x + 6$ equal to zero and solving for $x$, as in $x - 3 = 0$ therefore $x = 3$, and in $x - 2 = 0$ therefore $x = 2$, was not something problematic. Throughout the two interviews, she did not mention this; she set the two factors equal to zero and solved for the $x$ values.

Task 6: Describing the difference between an equation and a function.

To Sarah, an equation is not always a function; she said: "I know that." Her second reaction to this task was to write in words: "A function always has an equation that can represent it."

S: So there like an equation is a representation of the function, but the function is actually umm the graph of it [*in question tone*]. Umm I guess … But every equation is not a function.

R: Can you give me an equation that's not a function?

S: Umm, well the like, a, umm I'm trying to think how. What I think like the entire graph of like $y$ is equal to sine of $x$ is not a function [*writes* $y = Sin\, x$]. No, it just doesn't have an inverse; that is a function. Umm, I can I mean I can draw one, I just can't think of how to write that equation. Umm, but I suppose [*draws the graph in Figure 2.17*] like that, umm well I mean I can think $y$ squared equals $x$ isn't a function. I'm pretty sure [*writes* $y^2 = x$]. Cause I think that's [*pointing at the graph in Figure 2.17*] what that graph would be if you interchange your $y$ and $x$ values [*draws a small graph of* $y = x^2$] umm that'll be the same but when $x$ is one, square root of $x$, is that right? So this would be $y$ equals the square root of $x$ [*writes* $y = \sqrt{x}$] umm so [*makes a mark in Figure 2.17 that represents the point* $(1, 0)$] that's one, yeah so that wouldn't be a function [*pointing at the equation* $y = \sqrt{x}$]. That's an equation that's not a function.

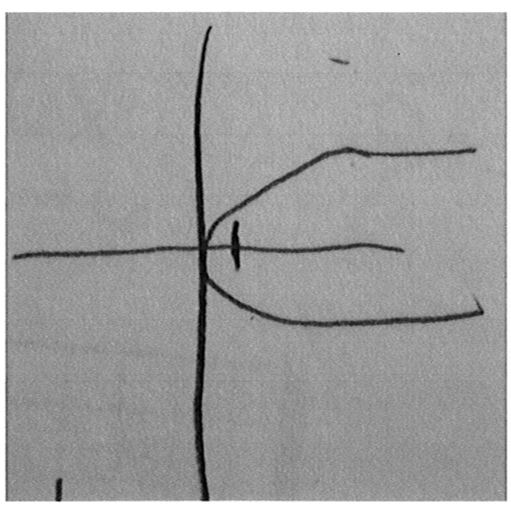


*Figure 2.17.* Sarah's First Graph in Task 6

When the researcher asked if she can give another example of an equation that is not a function, Sarah wrote $y = x^3$.

S: Is that a function? Umm, I don't think so. If $x$ is negative one [*pause*]. Negative one [*pause*] would be negative one. Umm, that doesn't seem right though.

R: Why you say that?

S: I am trying to remember what shape this graph should be [*draws the point* $(-1, 1)$ *in Figure 2.18*]; and [*long pause*] this doesn't seem right [*speaking silently*]. If it's negative two, two times would be eight [*draws the point* $(-2, 8)$ *in Figure 2.18*] but it would be negative eight [*lightly marks a point in Figure 2.18 to represent* $(-2, -8)$]. Yeah, I'm, cause that seems obviously that's not, wouldn't just be a line. But for some reason that seems to be what.

R: How would you connect those dots?

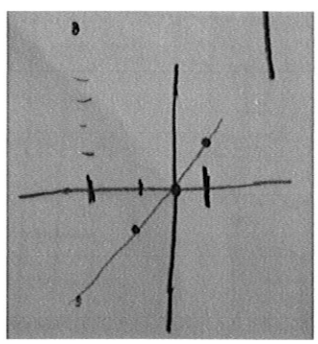


*Figure 2.18.* Sarah's Second Graph in Task 6: Attempting to Graph $y = x^3$

After connecting the three points in Figure 2.18 with a line, Sarah decided to redraw the points in Figure 2.19. In Figure 2.18, the plotted points were not drawn on scale; and it is unclear what prompted her to redraw the points, however, she seemed quite doubtful that the graph of $y = x^3$ "wouldn't just be a line." These episodes offer significant insight into how Sarah thinks about graphs of functions. It is evident that she is somehow looking for particular shapes that she can recognize. There is no evidence of her seeing $x$ and $y$ values varying according to a certain relation among them. For example, instead of looking at, say $(-2,-8)$, $(-1,1)$, $(0,0)$, $(1,1)$, $(2,8)$, where the $y$ values are the cubes of the $x$ values, and where the rate of change in $y$ values and change in $x$ values is not constant, Sarah only looks at how the plotted points look like or what recognizable shape they form.

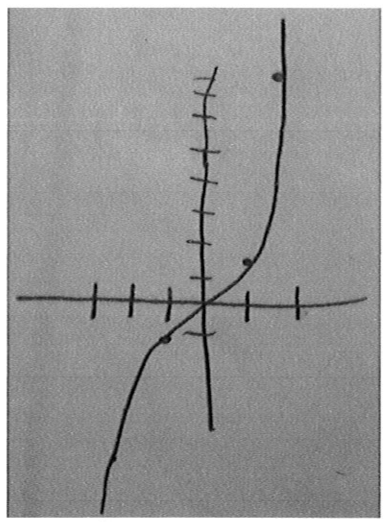


*Figure 2.19.* Sarah's Third Graph in Task 6: Graphing $y = x^3$

When she sketched the graph in Figure 2.19, Sarah stated that it is still a function because the graph passes the vertical line test. She applied the vertical line test by moving her right index finger across the graph, showing that the vertical lines cross the graph only once at each value of $x$. She concluded her discussion of the function $y = x^3$ by saying: "I'm not really sure if I drew that right though." The researcher then introduced another example of an equation by asking: "What about the equations like five is equal to three plus two?" According to Sarah, because there is no $x$ or $y$ value, an equation like

"$5 = 3 + 2$" cannot be a function. She also asked herself whether "$5 = 3 + 2$" is an equation, but quickly affirmed that it is. Thus, for her, a function must have $x$ and $y$ variables in them. Her requirement that functions will have ($x$ and $y$) variables in them prompted her to generate another example, $y = 5$, which has only one variable in it. She drew the graph of $y = 5$ in Figure 2.20 and applied the vertical line test and declared that this equation is a function.

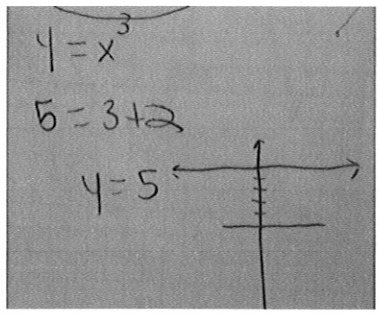



*Figure 2.20.* Sarah's Fourth Graph in Task 6: Graphing $y = 5$

The researcher continued to explore Sarah's understanding of the difference between a function and an equation by asking her to compare, without looking at or thinking about graphs, the below three equations that have $x$ and $y$ variables in them:

- $y^2 = x$
- $y = \sqrt{x}$
- $y = x^2$

Sarah stated that $y = \sqrt{x}$, in particular, is easy to tell that it is not a function "because the square root of any number can have a positive value or a negative value," and therefore there are more than one "answer," or more than one $y$ value for a given $x$. It is apparent that Sarah treats $y = \sqrt{x}$ as $y = \mp\sqrt{x}$, and she is using the idea of functions having one $y$ value for each value of $x$. She claimed that $y = x^2$ is a function, "because any value of $x$ that you are gonna have, it's gonna give you a different value of $y$."

R: For example if you choose $x$ is one, you get one,

S: Uh-huh.

R: $x$ is negative one, you get the same $y$?

S: Yeah. Wait. No I'm. Yeah, you can have the same $y$ [*pointing at the* $y$ *in the equation* $y = x^2$] umm, but you can't have one $x$ [*pointing at the* $\sqrt{x}$ *in* $y = \sqrt{x}$] and have two different $y$ values [*pointing at the* $y$ *in* $y = \sqrt{x}$].

Sarah further supported her argument that $y = \sqrt{x}$ does not represent a function—by choosing an $x$ value, i.e., 49, and asserting that there could be two $y$ values, -7 or 7.

Task 7: Finding the vertex of $\mathrm{f(x)} = 6\mathrm{x} - \mathrm{x}^2$.

Sarah's approach to this task was identical to Ken's in that they both wanted to first change the form of the equation. She stated: "Just because it is easier, I am gonna change the way it is formatted first of all," and wrote "$-x^2 + 6x$" and "$a = -1, b = 6$." She claimed that she does not have a "*c*" value. When Zaslavsky's (1997) research participants dealt with functions with an equation of the form $y = ax^2 + bx$, Zaslavsky found that most of the participants claimed that the quadratic function does not have a y-intercept because it does not have a "*c*" value. Based on this finding, Zaslavsky (1997) suggested that *the seeming change in form of a quadratic function whose parameter is zero* acts as an obstacle that causes difficulties for students when working with quadratic functions that "don't have" all the three terms of the standard form $y = ax^2 + bx + c$. Although Sarah did not investigate the y-intercept of the given function, and therefore did not experience any difficulty in correctly interpreting the y-intercept of a quadratic function given in the form $y = ax^2 + bx$ that has a zero $c$ parameter, her confidence in not having a "c" value at least supports the finding that students understand $y = ax^2 + bx$ as having only two parameters, $a$ and $b$, as opposed to $a$, $b$, and $c$, where $c = 0$.

After writing "$-x^2 + 6x$, $a = -1$, and $b = 6$.", Sarah also wrote $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, as she did in Task 4, and computed the vertex as $(-3, -27)$ by first finding its $x$ value using "$\frac{b}{2a}$" and then substituting it in "$-x^2 + 6x$." She plotted this point in Figure 2.21, and quickly moved on to the next question, "Can you make up a quadratic function with no vertex?" When the researcher asked her if she could speculate a little bit on the graph of "$-x^2 + 6x$," she started writing "$0 = -x^2 + 6$" and rearranged this equation into $0 = x(-x + 6)$, and found $x = 0$ and $x = 6$ to be the two x-intercepts of the function. When she plotted the points $(0, 0)$ and $(6, 0)$ in Figure 2.21, she seemed puzzled by seeing the three points, $(-3, -27)$, $(0, 0)$ and $(6, 0)$ on the graph paper.

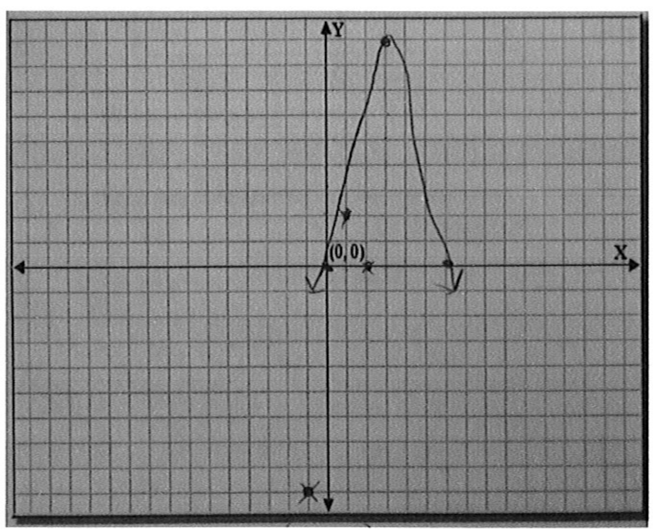



*Figure 2.21.* Sarah's Response to Task 7

Sarah's puzzlement seemed to be due to the fact that before computing and plotting the two x-intercepts, she briefly described the graph as having a vertex or a maximum at $(-3, -27)$ and opening downward from this maximum point—because of the negative coefficient of the $x^2$ term. She responded to this inconsistency between her expectation (based on her understanding of reflections) and the result of her computations (for finding the x-intercepts) by expressing that maybe the graph does not open downward.

S: I thought that meant that it would open downward. That's if the entire, [*pause*] yeah, I'm still not sure.

While she was trying to make sense of drawing a quadratic graph through those three points, Sarah articulated the following chain of reasoning:

S: That point cannot be the vertex. This, it's squared [*pointing at the* $x^2$ *term in the function*] so you know it is going to be a parabola, which is a quadratic function. So, the vertex should be in between those points at least [*draws an imaginary vertical line with the tip of her pencil in air halfway between the points* $(0, 0)$ *and* $(6, 0)$]. I thought I did the math right.

After a long pause, the researcher showed Sarah what she wrote in Task 2, $\frac{-b}{2a} = \frac{-7}{2}$, i.e., the $x$ component of the vertex of $y = x^2 + 7x + 12$. After seeing this, she seemed relieved.

S: Oh. Did I say *b* over two *a*? It should be negative *b* over two *a*.

She computed the vertex again, using $\frac{-b}{2a}$ this time, and found the point $(3, 9)$ as the vertex. She also changed the scale of the graph and made the $x$ and $y$ intervals equal to one unit, and drew the graph in Figure 2.21. Once this inconsistency was resolved, the researcher asked Sarah to elaborate on the parameter $c$ in $ax^2 + bx + c$ being the maximum or the minimum point, which she mentioned in an earlier task.

S: $c$ is zero here, but nine is actually the highest value [*pointing at* $(3, 9)$].

R: What would the meaning of $c$ be?

S: Umm. I can't remember now [*speaking very silently*]. If I add the $c$ in the equation, how would it relate to it?

R: You mentioned it earlier.

S: Uh-huh.

R: Like goes up and down.

S: Uh-huh. So I guess $c$ doesn't represent the highest or the lowest value. But, if we were to have a $c$ value [*pointing at* $y = 6x - x^2$] I believe what it would do would just move that [*pointing at the vertex* $(3, 9)$] down or up. So if it was a positive $c$ it would move it up and if it was a positive $c$ it would move that down.

R: Negative $c$?

S: Yeah, negative $c$. I'm sorry.

In response to the question, in Task 7b, "Can you make up a quadratic function with no vertex?," she said: "I'm not sure." Then she argued that if $a = 0$, then that wouldn't be a quadratic function. She wrote $y = 0x^2 + 2x + 3$ and $y = 2x + 3$ and concluded that a quadratic function without a vertex would not be a parabola, and instead, it would be a line.

Task 8. Finding the x-intercept, y-intercept, line of symmetry, and vertex.

Recall that in Task 8 participants are asked to compare four quadratic functions with respect to their x-intercepts, y-intercepts, vertices and lines of symmetry. In particular, students are asked to determine the easiest and the most difficult function among $f(x) = 4x - x^2$, $g(x) = (6 - x)^2$, $h(x) = -6x - x^2$ and $k(x) = 5(x -$

$3)(x + 2)$ in terms of finding their x-intercepts, y-intercepts, vertices and lines of symmetry. Sarah started comparing the functions with regard to the y-intercepts and stated:

S: Well, all of them would end up being zero. Umm for the y-intercept so, I feel like I am not doing something right [*pause*]. Umm let me think umm, but I mean that is right because it is gonna intercept the y-axis and the y-axis is whenever $x$ is zero.

R: Uh-huh. Okay.

S: So, really any of them.

Sarah also mentioned that she knows that the $x$ value of the y-intercept is zero. However, without substituting zero for $x$ in the four function equations and computing the corresponding $y$ values, she somehow thought that setting $x$ equal to zero would make all the equations equal to zero. No further explanation was provided on this reasoning.

In terms of finding the x-intercepts, Sarah stated that the $k$ function would be the easiest because when you set the function equation equal to zero, you can easily solve for $x$. However:

S: You set it equal to zero umm I almost think. Because you can, I am not, because you still have that five there I am not sure exactly what I should do properly. I don't know if you can just say five equals zero and just get rid of that [*writes* $5(x - 3)(x + 2) = 0$]. Umm. Because you still have that five there, I can't remember at the moment. What, how that would affect the x-intercepts. Maybe I won't say that one is the easiest.

Therefore she was puzzled by the constant coefficient 5 and changed her answer to $g$ being the easiest function in terms of finding the x-intercepts. She wrote the equation $(6 - x) = 0$ found $x = 6$ by solving it. Note that similar to her way of solving the equation $16 = (x - 4)(x - 4)$ in Task 3, Sarah again only considered one of the factors of the quadratic equation. Recall that in Task 3, she said: "Since those are both the same, I am only going to do once." This time she also added:

S: You don't need to write it out twice because the $x$ is, you're gonna get the same each time.

In terms of finding the vertex points, Sarah said that $f$ and $h$ are easier because one can easily identify the parameters $a$ and $b$ and find the vertex using $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

S: The line of symmetry is gonna go through the vertex. So essentially, once you found the vertex, you found the line of symmetry.

Task 9. Finding the quadratic function that has a vertex $(-2, 5)$ and that passes through $(0, 9)$.

In Task 9, when asked to find the quadratic function with a vertex at $(-2, 5)$ and that passes through the point $(0, 9)$, Sarah decided to draw these two points in order to visualize what the graph might look like. She plotted the points $(-2, 5)$ and $(0, 9)$ in Figure 2.22 argued that graph will go upward:

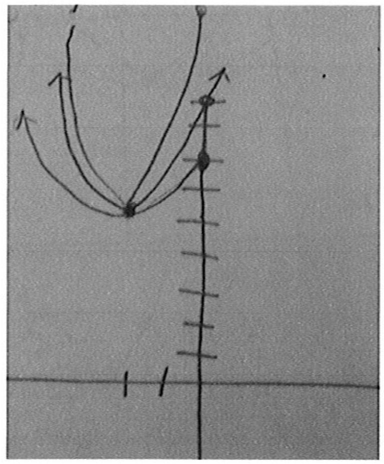


*Figure 2.22.* Sarah's Response to Task 9

S: I can tell that it's gonna have to go up because this is the vertex [*pointing at the point* $(-2, 5)$] so if there is a point up here it's gonna have to be umm pointing up like that [*draws a curve that starts at* $(-2, 5)$ *and that stops at* $(0, 9)$].

R: Why is that again?

S: Umm. Because, the vertex is either the lowest or the highest point. If the graph were turning down, it wouldn't hit that point [*pointing at the point* $(0, 9)$].

R: Okay.

S: So I know that $x$ squared is gonna be positive as well.

After drawing the graph in Figure 2.22 that passes through the two given points, Sarah directly started writing the equation $y = (x + 2)^2 + 5$ based on what she remembered about the vertex form of a quadratic function where $y = a(x - h)^2 + k$. She did not mention however that the standard graph of $y = x^2$ moves 2 units to the right and 5 units up. She simply remembered that in the vertex form one substitutes the $h$ and $k$ values of the vertex $(h, k)$. She did not write the form $y = a(x - h)^2 + k$ anywhere in this task; she only wrote $y = (x + 2)^2 + 5$.

She then checked whether her equation was correct by substituting $x = 0$ and $x = -2$ in her equation. Because she found the corresponding $y$ values to be $y = 9$ and $y = 5$, respectively, she confirmed her solution as correct.

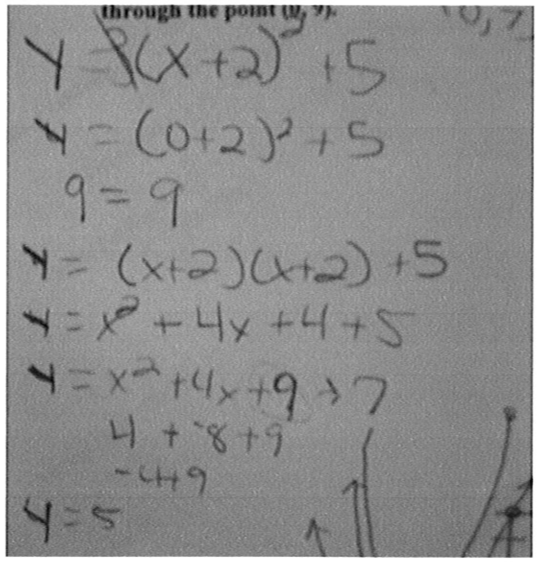


*Figure 2.23.* Sarah's Equation in Task 9

After Sarah was convinced that she was finished with the problem, and did not see the need to further explore the leading coefficient of the quadratic function, the researcher posed another question. The researcher asked Sarah to find the equation of a quadratic function that has a vertex $(-2, 5)$ and that passes through the point $(0, 7)$.

She then sketched a curve starting from the vertex and ending at $(0, 7)$. She then argued that the equation would be the same because she still would have $y = (x + 2)^2 + 5$. However, when the researcher asked her if she could check it as she did earlier, she then stated that the y-intercept should be 7 and not 9.

S: Hmm. What else could I do to solve for it? Umm.

Then she completed the initial curves that she sketched in Figure 2.22 into parabolas and asserted that the graph that passes through $(0, 7)$ would be wider, and thus "there needs to be a number in front of the $x$," "an $a$ value." She used the equations $y = x^2 + 4x + 4 + 5$ and $y = x^2 + 4x + 9$ in developing an argument that she needs a 7 at the end of the equation because the y-intercept is $(0, 7)$. After thinking about how to obtain "a 7 at the end," she wrote the equations in Figure 2.24.

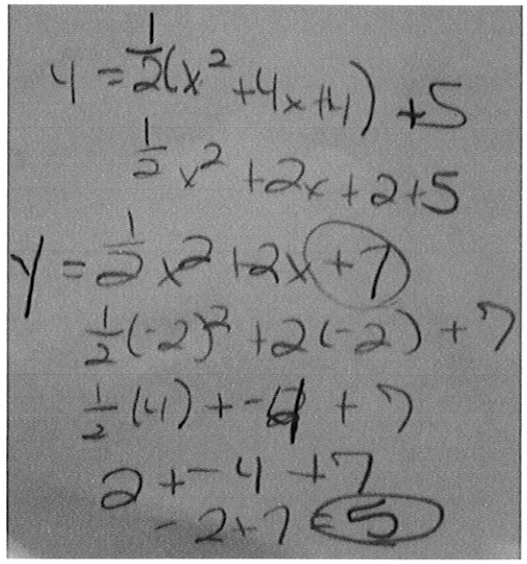


*Figure 2.24.* Sarah's Solution to the Alternative Question in Task 9

Sarah's strategy for obtaining a 7 was to divide the constant term 4 by 2 in the equation $y = x^2 + 4x + 4 + 5$. She completed her solution by checking her answer, i.e., the equation $y = \frac{1}{2}x^2 + 2x + 7$, by substituting $x = -2$. She confirmed that her answer was correct.

Task 10. Choosing the easiest graph to represent with an equation.

Sarah approached this task by stating:

S: First of all, the first thing I'm looking at the umm is the vertex, the x-intercepts umm and really that's about it. Umm and then I'm trying to look at which points hit on the x-intercepts [*pointing at the x-intercepts of the graph labeled f in Figure 2.25*] and which points of the vertex are most like whole numbers.

She also claimed that none of the x-intercepts [of the three graphs in Figure 2.25] are "on a number." Sarah then picked the graph of $f$ in Figure 2.25 because its x-intercepts "looked like half" to her. Her first attempt to find the equation of $f$ involved identifying the vertex point and the $c$ value [in the lower portion of Figure 2.26]. Recall that she also found the equation of the alternative function in Task 9 [with y-intercept at $(0, 7)$] using the vertex and $c$.

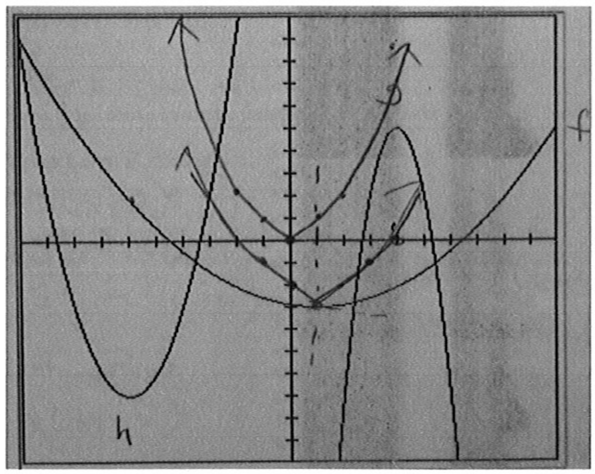



*Figure 2.25.* Sarah's Response to Task 10

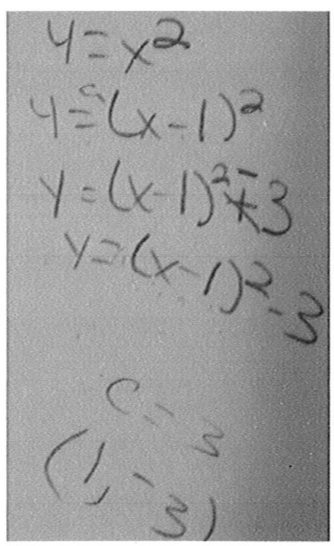



*Figure 2.26.* Sarah's Initial Strategy for Finding the Function Equation in Task 10

She wrote $c = 3$ and $(1, -3)$ to denote the y-intercept and the vertex respectively. It is assumed that she mistakenly wrote $c = 3$ instead of $c = -3$. Because the y-intercept "looks like -3" it is taken as -3. Note that the $y$ values of the vertex and the y-intercept are the same, i.e., $y = -3$. Then, instead of using the vertex form, as she did in the previous task, Sarah changed her strategy to exploring transformations. Starting from $y = x^2$ [Figure 2.26] she applied a horizontal translation of 1 unit to the right and a vertical translation of 3 units down.

S: Then the tricky part is figuring out umm the width here. And how that.

R: What do you mean the width?

S: Well now I need I know there is going to be a number in front of this [*pointing at* $y = (x - 1)^2 - 3$]. Because it is wider I know it is going to be a fraction. Umm what I need to figure out is how far it's stretched.

At this point Sarah drew the graph in Figure 2.25 that represents $y = (x - 1)^2 - 3$ and identified the x-intercepts of the two graphs, i.e., of $f$ and $y = (x - 1)^2 - 3$, to be $x = -4.5$ and $x = 6.5$ and $x = -2$ and $x = 4$ respectively. She then suggested that because the difference between the x-intercepts $x = 4$ and $x = 6.5$ is 2.5 or 5/2, the number in front of the equation [$a$] has to be 2/5. As an explanation, she said:

S: So it is two and a half wider [*pointing at the graph of* $f$]. I think if it was two wider it would be, that value of a would be one half.

She said she doesn't know why that is true but it seems to work. Figure 2.27 shows her work.

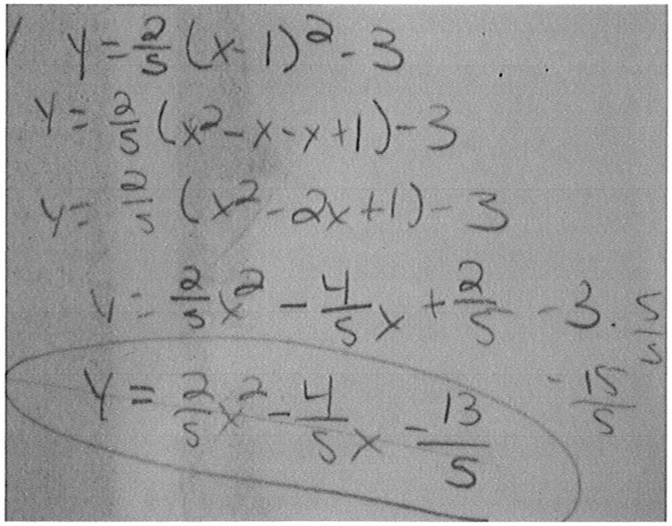



*Figure 2.27.* Sarah's Final Equation in Task 10.

To check if this equation was correct, Sarah tried to test a point on the graph of $f$, $(6, 2)$, and found that the equation was not correct.

## Cross Case Analyses

### Cross Case Analysis: Ken and Sarah.

In the previous two sections, the cases of Ken and Sarah were presented in detailed descriptions. Ken's and Sarah's responses to the tasks, their drawings, and verbatim statements provide a context for their cases (Merriam, 1988). The two interviews with each participant, the written artifacts of the students' mathematical solutions and drawings, and documentation of their self evaluations, are the study's main data sources; and the data generated in each case is analyzed holistically as an entire case (Yin, 1989). In order to address the issue of lack of depth inherent in a multiple case study (Creswell, 1998), as compared to a single case study, these two initial cases were treated as illustrative cases that were described and analyzed more in-depth. The resulting descriptions, themes and assertions guided the analysis of the subsequent cases.

In the initial analysis of the results presented in the previous two sections, Ken's and Sarah's mathematical behaviors in quadratic function situations were interpreted using two theoretical constructs from the existing literature on understanding in mathematics and on students' understanding of functions. First, the notion of *act of understanding* (Ajdukiewicz, 1974; Sierpinska, 1994) is used to interpret how and what the two students were thinking about various aspects of quadratic functions while solving the tasks. Ajdukiewicz (1974) defined understanding as "an act of mentally relating the object of understanding to another object" (cf., Sierpinska, 1994, p. 28). He viewed *act of understanding* as directing ones thoughts about a given word or expression, or relating these givens, to some other object.

While Ajdukiewicz's (1974) definition included only words and verbal expressions as the objects of understanding, Sierpinska (1994) generalizes this definition to include any type of object, and not merely verbal expressions, which, in mathematics, can be "[mathematical] concepts, relations between concepts (sometimes stated in forms of theorems), problems, arguments (proofs), methods, theories, mathematical symbolism, mathematical representations such as diagrams, graphs etc" (p. 42). Furthermore, she adds that an *act of understanding* is an actual or potential mental experience, and calls the two objects in Ajdukiewicz's definition the *object of understanding* and the *basis of understanding*. In other words, Sierpinska refers to *act of understanding* as an act of mentally relating the *object of understanding* to another object called the *basis of understanding*.

Sierpinska (1994) contends that basis of understanding can be viewed as mental representations or mental models. Ajdukiewicz defined mental "representations as instantaneous mental experiences of an individual: definite experiences at a given moment in a given person's mind. In an act of understanding based on a representation of the object that is being understood, the subject [the individual] does not take any position toward this object and does not evaluate or judge it. The object of understanding is only being matched with some mental image and/or description" (Sierpinska, 1994, p. 49). Thus, the collection or the fabric of such mental images (or basis of understanding) of a given mathematical word or concept (or object of understanding) could also be viewed as what Vinner and Dreyfus (1989) called concept image. Vinner and Dreyfus (1989) differentiated these images from concept definitions that are comprised of conventional and formal mathematical knowledge.

Finally, a basis of understanding, or a fabric of related mental images and representations, can also be viewed as conceptual structures from the theoretical framework of cognitive constructivism discussed in Chapter 1. The object of understanding and the basis of understanding are also compatible with the parts of Steffe's (1994, 2002) notion of *scheme*. Steffe's notion of scheme, which is similar to an act of understanding, is composed of three parts: (1) the triggering experiential situation, (2) the student's activity, and (3) the results of that activity from the perspective of the student (von Glasersfeld, 1991). Thus, while the object of understanding is compatible with the triggering experiential situation, and the act of relating the object to the basis is with the student's activity, the basis of understanding is similar to the cognitive structure that is being accommodated to results of activity from the perspective of the student (von Glasersfeld, 1991).

To justify the use of the theoretical constructs of act of understanding and basis of understanding in mathematics education research and pedagogy, Sierpinska (1994) writes: "…sometimes understanding is confused (or deliberately merged) with knowing, and … this is perhaps not a desirable thing to do in education. Unfortunately, institutionalized education is framed to develop students' knowledge rather than thinking" (p. 68). Evaluative qualifications of understanding, such as "good," "deep," "robust," "rich," or "weak," "poor," "incomplete," can indeed give mathematics educators guidance in finding ways to help students develop the bases of understanding that are compatible with the conventional knowledge in the discipline, or resemble a conceptual understanding (Hiebert and Lefevre, 1986) or a relational understanding (Skemp, 1976). "… When we speak not of understanding in general, but of good or deep understanding, for example, in mathematics, then we think of the possible activities that a student could engage in, indeed, what actions could he or she perform on the object of understanding" (Sierpinska, 1994, p. 103). However, by separating understanding from true, correct, or conventional knowledge, or correct chain of inferences from premises to conclusions, or correct reasoning, or successful actions or systematic cognitive structures that explain why the actions are successful (as in Piaget, 1978), educators can better assess where students are in their level of learning or development. Recall that Piaget (1978) considered understanding as having conceptualizations that are based on knowledge of why a certain action was successful and why certain other possible actions wouldn't. In other words, he contended that there must be some reasoned understanding in order for it to be called understanding at all.

In Sierpinska (1994), understanding is defined very differently. As an alternative to reducing understanding to knowing or knowledge, as Sierpinska and Ajdukiewicz propose, it is more beneficial in education to explore students' *ways of understanding* or *understandings* that are their *basis of understanding*. While the current literature on students' understanding of quadratic functions provide insight into students' misconceptions and obstacles, common errors, and their difficulties in quadratic function situations, it is the aim of this analysis to contribute to the body of knowledge on students' understanding of quadratic functions by explicating holistic pictures of students' conceptual structures or collections or fabrics of understandings (or bases of understanding) that they relate to with regard to different aspects of quadratic functions.

According to Sierpinska (1994), there are four mental operations involved in an *act of understanding* that either determine the *object of understanding* or relate or link the *object* and *basis of understanding*: Identification, discrimination, generalization and synthesis. In identification, a student isolates (or singles out) some object and recognizes it as something that he or she intends to understand. He or she may or may not name or classify the object. "Identification is the main operation involved in acts of understanding called *einsicht* by *Gestalt* psychologists: acts that consist in a re-organization of the field of consciousness so that some objects that, so far, have been in the background, are now perceived as the 'figure'" (Sierpinska, 1994, p. 57). Categorization is not seen as part of identification because categorization involves generalization: a class of objects is considered to be a particular case of another class of objects. Discrimination is the identification of two objects as two (or different), and not as one. In the mental operation of generalization, a class of objects, situations, events, problems, theorems or theories is thought of as a particular case of another class of objects, situations, events, problems, theorems or theories. And synthesis is the search for a common link or a unifying principle that results in recognition of a collection of objects or generalizations as a coherent whole (Sierpinska, 1994). Abstraction is not included as a separate operation of an act of understanding, because it does not link the *object* and *basis of understanding*. Instead, it is the detachment of certain features from the *object of understanding*. As Sierpinska points out, abstraction is involved in all the above operations.

One might argue, from experience, that generalization and synthesis are too difficult for most high school or college students, who for instance, when faced with a task or a problem, may only identify what is given, what is to be found and what category of problems does it belong to (so that they can carry out the memorized routine solutions); as opposed to carrying acts of understanding that "may result in its reformulation, in the discrimination between the essential and the superfluous assumptions, its generalization, or discovery of an important analogy" (Sierpinska, 1994, p. 45). Although students may lack the mathematical sophistication of research mathematicians, they may still use the above operations at varying levels in relation to their existing understandings.

Unlike the 'activity theory' of Leont'ev (1981) and Davydov and Radzikhovskii (1985), which asserts that in order to understand an object, a learner must act upon it to transform it into a new object, Sierpinska's *act of understanding* does not involve an a priori voluntary aim to change the object. While acting on the object of understanding, the learner maps the object onto his or her basis of understanding (or collection of understandings), according to some criterion, without having any intention of doing anything to do object other than understanding it—or relating it to other objects through identification, discrimination, generalization or synthesis. Sierpinska acknowledges that the object may be transformed into a new object during the act of understanding; however, she opposes the premise that the learner has a prior aim of changing it. Dewey (1971), in his book *How We Think*, also writes: "An increase of the store of meanings makes us conscious of new problems, while only through translation of the new perplexities into what is already familiar and plain do we understand or solve these problems. This is the constant spiral movement of knowledge" (p. 140). For Sierpinska, this translation could be only the identification of the object as an existing understanding or understandings. In other words, the new problem or the new object that the learner attends to is mapped onto what is already familiar and plain—as in Dewey's words. This identification does not involve an intention to change what is being identified.

In order to clarify the distinction between meaning and understanding, it may be sufficient to articulate how Sierpinska (1994) differentiates the two. According to Sierpinska, most of the traditional philosophical thought explains understanding with meaning, positing that once a subject (or learner) 'grasps' the meaning of an object he or she understands it. There is an assumed essence or core meaning of objects; and the possession (or knowledge) of that meaning constitutes understanding. As discussed earlier, knowledge, understanding, correct reasoning, logical thought, or the 'grasp of meaning' are used synonymously in most theories of traditional philosophy. Sierpinska, however, suggests that meaning should be explained with understanding, or acts of understanding. In explaining meaning with understanding, one defines meaning of an object (for a learner) as a collection of understandings.

In terms of conditions of understanding and criteria for relating the object of understanding with its basis, Sierpinska argues that attention, intention and question are three necessary conditions of an act of understanding. She notes that, however, it is difficult to identify the sufficient conditions of an act of understanding. In terms of the criteria for relating the object and the basis, family resemblance, recall of a similar experience, internal consistency or finding a fit based on some existing internal order are considered.

Sierpinska claims that attention to an object is necessary in an act of understanding; even if some idea or thought just come to our minds, without experience, or if we directly relate it to our basis of understanding without attending to it, it must still come from a prior act of understanding of it that was rooted in an earlier experience. Intention is necessary also, because it enables acting and thus identifying, discriminating, generalizing and synthesizing. Although the existence of a question is not always necessary for an act of understanding to occur, as in relating a word in mother language to some meaning, "it seems that any act of understanding that brings about a substantial change in what we know, or think, or believe, is preceded by a question. A sensible and interesting question seems to be absolutely necessary in maintaining both the attention that allows us to notice that there is something to understand, and the tension that is required in conducting long reasoning that only can promise the reward in understanding" (Sierpinska, 1994, p. 64).

Finally, according to Sierpinska's (1994) theory of understanding that is espoused here, an *act of understanding* should be distinguished from a *process of understanding*. An act of understanding is an actual single experience occurring in an individual's mind at a given time. On the other hand, process of understanding is "regarded as lattices of acts of understanding linked by reasoning [inference or deduction]" (p. 72).

The second theoretical construct used in the cross analysis of the two cases of Ken and Sarah, is that of *action view of functions*. Dubinsky (1992) and Carlson et al. (2010) explain a certain way that students approach to function situations as *action view of functions*:

> Students with an action view of function often think of the graph of a function as being only a curve (or fixed object) in the plane; they do not view the graph as defining a general mapping of a set of input values from the independent axis to a set of output values on the dependent axis. They then interpret the location of points, the vertical line test, and the "up and over" evaluation of the slope of functions as geometric properties of the graph and not as properties of a more general mapping or relationship between two varying quantities. (Carlson et al., 2010, p. 116)
>
> A student with an action view of function tends to rely solely on computational reasoning. For the real valued function, $f$, defined by $f(x) = 2x^2 + 1$, students with an action view are confined to seeing the defining formula as a computational procedure for finding a single answer for a specific value of $x$. They view the formula as a set of instructions: square the value of $x$, multiply this number by two, and then add one to get the answer. (Carlson et al., 2010, p. 116)

The rationale for using these two theoretical constructs, *act of understanding* and *action view of functions*, is that they are most compatible with the researcher's initial interpretations of Ken's and Sarah's mathematical behaviors. To recall, for Ken, a function is some formula where one uses an $x$ value to solve for a $y$ value. It is a function of something, of the independent variable, and it is represented as '$y$ equals $f$ of $x$' or '$y = f(x)$.' Quadratic functions on the other hand, are usually in the form $ax^2 + bx + c = 0$, which Ken calls the "quadratic formula" or "the original formula." Quadratic functions, like any other function, have graphs and their graphs look like a parabola. The important aspects of graphs of quadratic functions, for Ken, are the minimum or the maximum point (the vertex), and the end behaviors of the parabola that "go to positive or negative infinity."

Ken drew quadratic graphs either by first drawing the prototypical $y = x^2$ graph and applying a series of transformations to it, or simply sketching a singular shape that looked like a parabola. When the given quadratic function was not in the form $y = a(x-h)^2 + k$, he used the "formula"$-b/2a$, that he remembered, to find the $x$ component of the vertex, and sketched a shape around this vertex point which looked like a parabola—without specifying other points on the graph unless it was explicitly asked for. Ken found the $y$ values of points by substituting the $x$ values in the function equation. He always looked at the leading coefficient of the quadratic function equation to see if the graph is "regular" or facing upward, or "inverse" or facing downward. A negative leading coefficient meant the graph was "inverse," of what it would have been without the negative sign, or facing downward.

Sarah, on the other hand, sketched her quadratic graphs by either generating and connecting several points one by one such that the shape come out as that of a parabola, or by trying to find two x-intercepts and a vertex point (using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$) and connecting these three points so that the shape come out as that of a parabola. In the first strategy, she also looked for symmetry between pairs of $x$ values so that she can locate the $x$ component of the vertex. Similar to Ken, she found the $y$ values (of points on the graph) by substituting the $x$ values in the function equation. Her strategy choice did not depend on the form in which the quadratic function was given. If it was given in the form '$y = a(x-h)^2 + k$,' she tried to find the x-intercepts by solving the equation '$0 = a(x-h)^2 + k$,' and locating the $x$ component of the vertex by finding the midpoint between the two x-intercepts. Then she connected the three points one by one such that the shape came out as that of a parabola. If it was given in the form $y = ax^2 + bx + c$, again she tried to find two x-intercepts by solving $0 = ax^2 + bx + c$ and the vertex point this time by using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$, and connected these three points so that the shape come out as that of a parabola. If on the other hand she was not able to find two x-intercepts with which she can draw a symmetric parabola shape (as in Task 3), she went back to the first strategy of generating and connecting several points one by one such that the shape come out as that of a parabola.

For Sarah, a function is "an equation of something, a line or shape that you can graph that for every value of $x$ only has one $y$ value on the graph." For her, a function seems to be somewhat of a collection of two things: an equation and a graph. It is an equation with two variables where one uses an $x$ value to solve for a $y$ value—just as in Ken. It is also a graph where for every value of $x$ there is only one $y$ value, as confirmed by the vertical line test. This graph must also be a certain recognizable shape. Function equation is represented as '$y$ equals $f$ of $x$' or '$y = f(x)$.' Quadratic functions are represented by the quadratic equation '$ax^2 + bx + c = 0$.' Quadratic functions, like any other function, have graphs and their graphs look like a parabola. The important aspects of graphs of quadratic functions, for Sarah, are the minimum or the maximum point (the vertex), and the points where $y$ values are zero. The origin is also important, because as in '$y = x^2$', the graph is symmetric about the line that contains the origin. In other words, line of symmetry is an important part of quadratic function graphs.

Let's examine more closely, how these two students acted in some of the problem situations, which generated significant findings. In Task 1, Ken identified the word 'function' as the object of his act of understanding, and related it to the following understandings: 'a function of something,' 'a formula,' 'the equation $y = f(x)$,' 'the dependent variable $y$ is a function of the independent variable $x$,' 'solve for $y$,' and 'plug in $x$ values in the formula.' Although he did not mention explicitly, he seemed to relate the object of his act of understanding, the word function, to the understandings 'a function of something,' and 'the equation $y = f(x)$,' based on what he recalls from his pre-calculus class: that a function "$f$" is "$f$ of $x$" or $f(x)$. The other understandings, 'a formula,' 'solve for $y$,' and 'plug in $x$ values in the formula,' seem to be based on an internally consistent view of functions as formulas, which is compatible with Dubinsky (1992) and Carlson et al.'s (2010) action view of functions where the defining formula [$y = f(x)$] is a computational procedure for finding a single answer [$y$] for a specific value of $x$.

Furthermore, when prompted to feel free to draw graphs or diagrams, Ken drew a table and a graph for the example of a function that the generated, $f(x) = x + 2$ in Figures 1.1 and 1.2. This prompt, to which Ken attended, was his object of understanding. He also discriminated the words 'graphs' and 'diagrams' as two objects to which he related what he seemed to recall from his past experiences the existence of tables and graphs that are parts of functions. Therefore, 'a table of $x$ and $y$ values' and 'a set of plotted coordinate points on a graph paper' can also be considered as parts of his basis of understanding of function. The mental representations that constitute Ken's relevant conceptual structure are included in the following basis of understanding—which can be called the fabric of his function understandings.

Ken's basis of understanding of the word *function* can be summarized as follows:

- A function of something
- The equation $y = f(x)$
- The dependent variable $y$ is a function of the independent variable $x$
- A formula
- Solve for $y$
- Plug in $x$ values in the formula
- A table of $x$ and $y$ values
- A set of plotted coordinate points on a graph paper

Sarah, on the other hand, identified the word 'function' as the object of her act of understanding and related it to the following understandings: '$f$ of $x$ is equal to something,' 'the equation $f(x) =$ ... ... ...,' 'a graph,' 'a line or shape that you can graph,' 'the vertical line test,' 'a graph that passes the vertical line test,' 'can't have more than one $y$ value for a single value of $x$.' She also drew a table of $x$ and $y$ values and sketched two graphs, one representing a function and the other one not. Thus, 'a table of $x$ and $y$ values' will also be considered as part of her basis of understanding the word function. When asked why for every value of $x$ there must be only $y$ value, Sarah first attempted to relate this understanding of hers [now the object of this act of understanding] to some other understandings on the bases of reasoning and internal consistency. However, she stopped at a certain point in her reasoning, and instead mapped this object to the phrase 'by definition' based on memory.

> S: $F$ of $x$ equals whatever that equation is [*draws the scribble next to "$f(x) =$" in Figure 2.2*]. $F$ of $x$ is the same as $y$ [*writes $f(x) = y$*] so if you have an equation [*pointing at the scribble in Figure 2.2*] umm that equals two different$y$'s wait I

> thought I was making some sense umm then they can't they won't equal two different $y$'s maybe? Or they won't be a function of $x$. But I'm still not. I think I am confusing myself now. Umm I just know that by definition I guess.

Sarah also offered a definition of function: "An equation of something, a line or shape that you can graph that for every value of $x$ only has one $y$ value on the graph." She gave two additional examples of functions, $y = 3x + 4$ and $y = x^2$, and articulated the following additional ways of understanding the word function: 'equation as the numerical representation of function or a way to write it,' 'a graph as a visual representation of the shape or whatever that equation makes,' 'both equation and graph are functions,' 'take any $x$ value and plug it into the equation and you will get the corresponding $y$ value,' 'take a $y$ value on the graph and plug it into the equation and you will get the corresponding $x$ value.' When asked, what the two representations, the equation and the graph, have in common, or what they both represent, Sarah acted on this question by relating it to her understandings that they both are functions and they both represent the same thing, i.e., the same function. Upon further questioning, she seemed to link different acts of understanding and reason that the commonality is 'the relationship that they have.' This process of understanding seems to be the coordination of the acts of understanding that were based on the understandings of 'equation as the numerical representation of function or a way to write it,' 'a graph as a visual representation of the shape or whatever that equation makes,' 'both equation and graph are functions,' 'take any $x$ value and plug it into the equation and you will get the corresponding $y$ value,' 'take a $y$ value on the graph and plug it into the equation and you will get the corresponding $x$ value.' Because the researcher did not pressure Sarah further to explain what she referred to when she said: "the relationship that they have," it is unclear how these understandings are linked. Below is a summary of Sarah's understandings.

Sarah's basis of understanding of the word *function* can be summarized as follows:

- $f$ of $x$ is equal to something
- The equation $f(x) = \dots\dots\dots$
- A graph
- A line or shape
- The vertical line test
- A graph that passes the vertical line test
- Can't have more than one $y$ value for a single value of $x$ ("by definition")
- A table of $x$ and $y$ values
- Equation as the numerical representation of function or a way to write it
- Graph as a visual representation of the shape or whatever that equation makes
- Both equation and graph are functions
- Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value
- Solve for $y$ values
- Take a $y$ value on the graph and plug it into the equation and you will get the corresponding $x$ value
- The relationship that they have

Similar to Ken, Sarah also demonstrated an action view of functions_where, in her case, the graph of a function is viewed as being only a curve (or fixed object) in the plane. Moreover, the vertical line test is viewed as a geometric property of the graph of a function.

The theme that emerged from the above analysis of Ken's response to Task 1 was that Ken seems to understand *function as a unique type of equation where one solves for* $y$. In Sarah's case, on the other hand, her frequent use of the vertical line test as a property of the function graph led to the emergence of the theme: *function as a unique type of graph where every value of* $x$ *has only one* $y$ *value on the graph.* And finally, both Ken and Sarah seemed to have a way of understanding *function as a collection of things.* This theme emerged after the cross analysis of the two cases, in which both students seemed to refer to a collection of graphs, tables, equations, and $x$ and $y$ variables, without paying attention to what they may have in common. Sarah called equations and graphs as different representations of functions, and, upon extended probing, she referred to a relation between things; however she did not make it explicit whether she sees a relation between equations and graphs or between two co-varying variables.

The analysis of the results of this study supports Thompson's (1994) hypothesis about students' use of multiple representations of functions: "Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead maybe areas of representational activity among which, as Moschkovich, Schoenfeld and Arcavi (1993) have said, we have built rich and varied connections, that our sense of 'common referent' among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of 'function' is not represented by any of what are commonly called multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance" (p. 24). Although Sarah uttered the phrase "both equations and graphs represent functions," neither Ken nor Sarah, acted on the different representations of a function on the basis of an understanding of invariance among them. To these students, the representations were simply parts of somewhat of a collection, the totality of which was seen as function.

Thompson's (1994) assertion is also consistent with Sierpinska's theory of understanding, which makes it clear that the subject (or the person), who is engaged in an act of understanding, attends to and identifies the object of *their* understanding and relates it to *their* basis of understanding. Without the knowledge of what the students are attending to or identifying as objects of their understanding, or what ways of understanding they relate these objects to, mathematics educators may not effectively address the appropriate use of the pedagogical tools such as multiple representations. As Thompson (1994) suggests, the pedagogical implication of this observation is that mathematics educators can better address this issue by "finding situations sufficiently propitious for engendering multitudes of representational activity, and [orienting] students toward drawing connections among their representational activities in regard to the situation that engendered them" (p. 24). Mathematics educators know from their own experiences that the representations are often presented to students as "different representations of functions," and they often emerge neither from situations nor from students' own activities.

Three hypotheses were thus generated from these emergent themes: (1) Students with similar sets of experiences may understand function as *a unique type of equation where one solves for* $y$ or as *a unique type of graph where every value of* $x$ *has only one* $y$ *value on the graph*; (2) Students with similar sets of experiences may have a way of understanding *function as a collection of things*; and (3) Tables, graphs, and expressions might be multiple representations of functions to mathematics teachers, but they are not representations of anything to their students.

In Task 2, recall that participants were asked to draw a quadratic function in a blank rectangular box and to (a) explain what makes their graph quadratic, (b) discuss what parts of the graph are important or special (and why); and (c) give an equation for the graph that they drew. Ken's initial act of understanding_the object of 'drawing a quadratic' was to recall what he named 'the quadratic formula,' or 'the original formula,' or 'the quadratic equation,' $ax^2 + bx + c = 0$. He then sketched the graph in Figure 1.3 in one move without lifting his pencil off from the paper. In other words, he drew this graph as rather a single picture that somehow resembled that of a parabola—he did not plot points, or followed any systematic method for drawing the graph. Ken again demonstrates an action view of functions where the graph of a function is viewed as being only a curve (or fixed object) in the plane. Ken's response to the first part of this task suggests that he relates quadratic functions to the following understandings: 'a graph or an equation is quadratic if it fits the quadratic formula: $ax^2 + bx + c = 0$,' 'the graph of a quadratic is the shape of a parabola,' 'there is only one parabola,' [whereas he suggests that a cubic function might have two parabolas and three x-intercepts], 'the graph has either a minimum or a maximum point,' the ends of the graph either go to positive infinity or negative infinity,' 'the parabola is a U-shaped parabola,' 'the U-shaped parabola can be upside down, i.e., it can be "inverse" if there is a negative sign in front of the equation,' 'the U-shaped parabola cannot be sideways because it would then have a different type of equation,' and '$f(x) = 3x^2 + 2x + 4 = 0$ is an example of an equation of a quadratic graph.'

Ken's criteria for relating the object of his understanding, quadratics, to his basis of understanding that 'the graph of a quadratic is the shape of a parabola,' is that he simply recalls similar experiences in his classes and that recognizes problems involving the quadratic equations, $ax^2 + bx + c = 0$, and parabolas as quadratic problems. The following excerpt illustrates this point:

R: Why did you draw that way but not some other way? Some other curve.

K: Just because it did look like an example of something that I know it's a quadratic.

Thus, in his acts of understanding in all three parts of this task, Ken generated somewhat of a fixed shape of a parabola in a single drawing, related the word quadratic to a fit or a match with the "original formula," $ax^2 + bx + c = 0$, and related the example $f(x) = 3x^2 + 2x + 4 = 0$ to the phrase: equation of a quadratic function. While the task explicitly stated the phrase "quadratic function," Ken somehow did not identify the word function as his object of understanding in the above acts of understanding. For example, no part of this task included any use of most of his basis of understanding of functions in Task 1. Specifically, he did not map any object onto the understandings of dependent variable $y$ being a function of the independent variable $x$, or function as a formula, or solving for $y$, or plugging in $x$ values in the formula, or a table of $x$ and $y$ values being part of a function. It rather seemed that he only attended to the word quadratic and identified this word as his only object of understanding. This was also evident in the fact that he never uttered the word function in this task. The inference that Ken's object of understanding was only the word quadratic could have a significant implication for teaching quadratic functions, which usually starts in high school algebra where students mostly only manipulate expressions, solve equations and explore transformations of prototypical graphs (e.g., that of $y = x^2$). Below is Ken's fabric of understandings related to quadratic functions (or simply 'quadratics').

Ken's basis of understanding of *quadratic functions* can be summarized as follows:

- The quadratic formula, the original formula, or the quadratic equation: $ax^2 + bx + c = 0$
- A parabola
- A graph or equation is quadratic if it fits the quadratic formula: $ax^2 + bx + c = 0$
- The graph of a quadratic is the shape of a parabola
- There is only one parabola
- The graph has either a minimum or a maximum point
- The ends of the graph (the parabola) either go to positive infinity or negative infinity
- The parabola is a "U-shaped" parabola
- The U-shaped parabola can be upside down, i.e., it can be inverse if there is a negative sign in front of its equation
- The U-shaped parabola cannot be sideways because it would then have a different type of equation

- $f(x) = 3x^2 + 2x + 4 = 0$ is an example of an equation of a quadratic graph

These quadratic function understandings can be further organized into: equation understandings, graph understandings, and connections.

1) Equation Understandings:

- The quadratic formula, the original formula, or the quadratic equation: $ax^2 + bx + c = 0$
- $f(x) = 3x^2 + 2x + 4 = 0$ is an example of an equation of a quadratic graph

2) Graph Understandings:

- A parabola
- The graph of a quadratic is the shape of a parabola
- There is only one parabola
- The graph has either a minimum or a maximum point
- The ends of the graph (the parabola) either go to positive infinity or negative infinity
- The parabola is a "U-shaped" parabola
- The U-shaped parabola can be upside down, i.e., it can be inverse if there is a negative sign in front of its equation
- The U-shaped parabola cannot be sideways because it would then have a different type of equation

3) Connections:

- A graph or equation is quadratic if it fits the quadratic formula: $ax^2 + bx + c = 0$

Sarah's initial act of understanding was also, as in Ken's case, to recall what she called 'the quadratic equation,' $ax^2 + bx + c = 0$. After she wrote this equation, she asked herself: "what is their shape usually?" She recalled $y = x^2$ and stated that it is a quadratic equation. The facts that she wrote the equation $ax^2 + bx + c = 0$ next to the question in part (a), what makes this graph quadratic, and that she spent some time discussing about "what makes it quadratic" (without drawing any graphs), lead to the inference that the object of her initial act of understanding was either the word "quadratic" or the question "what makes something (maybe an equation) quadratic." This object of understanding is very similar to Ken's object of understanding in which the word quadratic was also central. Sarah's reasoning about the question "what makes it quadratic" revealed the following ways of understanding this question: 'quad means four,' 'the reason it might be four is that the quadratic equation $ax^2 + bx + c = 0$ can be factored into ( )( ),' '( )( ) can be written as ( + )( + ),' and '( + )( + ) can be written as (_ + _)(_ + _).'

After being prompted to draw a graph, Sarah sketched the graph of $y = x^2$ in Figure 2.7 in one move without lifting her pencil from the paper. Ken also drew his first quadratic graph the same way, but, unlike Ken's, Sarah's drawing indicated some kind of systematic approach, where the left hand side of the graph ends at a minimum or a bottom point and the right hand side begins from that point and is drawn by following some symmetry to the one previously drawn on left. As did Ken, when asked what makes her graph quadratic, Sarah related this question to the existence of a fit or a match with what she called quadratic equation: $ax^2 + bx + c = 0$. Unlike Ken, however, Sarah explicitly stated that $y = x^2$ fits $ax^2 + bx + c = 0$ when $a = 1$, $b = 0$, and $c = 0$.

R: Okay. What makes this graph quadratic?

S: Umm, well it, I know it fits this equation [*pointing at* $ax^2 + bx + c = 0$] umm because if umm if $a$ was one and $b$ was zero and $c$ was zero, we would have umm

$x$ squared equals zero? I don't know. Umm it would fit that definition I guess. Umm maybe because umm there is quadrants? [*touches four points in the four distinct regions of the coordinate plane, Quadrants 1, 2, 3, and 4*] and it's the same on both sides of two of them? [*Touches the point on the y-axis in Figure 2.7, where* $y = 1$, *and two other points on the graph of* $y = x^2$ *in Figure 2.7, one point on the left, near the point* $(-1, 1)$ *and another on the right, near* $(1, 1)$]

Recall that Ken gave $f(x) = 3x^2 + 2x + 4 = 0$ as an example of an equation of a quadratic graph. When Sarah was stating that $y = x^2$ fits $ax^2 + bx + c = 0$, when $a = 1$, $b = 0$, and $c = 0$, she identified the resulting equation $x^2 = 0$, and acted on this equation. She said "I don't know," and then in her act of understanding $x^2 = 0$ she related this result to the understanding: 'a fit with the definition of quadratic equation.' Seemingly unsatisfied with this result, she went back to her exploration of the meaning of the word quadratic and suggested that the reason it might be four (or that there are four things in quadratics) is that there are quadrants and "it's the same on both sides of two of them." This assertion further supports the earlier inference (in the analysis Sarah's mathematical behavior in Task 1) that she demonstrated an action view of functions. In other words, whereas in Task 1, the vertical line test was a geometric property of the graph of a function; here symmetry is used as a property of only the graph and "not as [a property] of a more general mapping or relationship between two varying quantities" (Carlson et al., 2010, p. 116).

Also recall that, from the previous section, the description of Sarah's case, when the researcher suggested that she may think of a quadratic function in terms of how it is different from other types of functions, and not so much in terms of how it is worded, Sarah said: "because $x$ is squared?" while pointing at the $x^2$ in $ax^2 + bx + c = 0$. She continued: "So you will have more than one $y$ value for each $x$ maybe." To explain why that is so, Sarah generated the table in Figure 2.7 and plotted the five points: $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(-2, 4)$ and $(2, 4)$ on the graph. She argued that $x = -2$ and $x = 2$ both have the same $y$ value of 4 (and that $x = -1$ and $x = 1$ both have the same $y$ value of 1), and therefore "the absolute value of any $x$ value will have the same $y$ value" "because it is squared." Although this assertion seems to contradict what she said about the definition of a function when using the vertical line test to ensure that there is only one $\boldsymbol{y}$ value for each $\boldsymbol{x}$ value, Sarah seemed confident about her reasoning and did not display any puzzlement. This finding confirms Vinner and Dreyfus' (1989) notion of *compartmentalization of knowledge*, which seems to account for the fact that Sarah did not use her understandings of the concept of function, such as the vertical line test, to solve this problem. Here, compartmentalization is viewed as treating related concepts as independent, as in the sense of Gerson (2008).

These acts of understanding indicate that she is relating the term quadratic to the idea of symmetry between points on two sides of the graph (left and right) that lie on the same horizontal line. These acts of understanding, however, do not involve reference to her basis of understanding of functions such as her central notion of the vertical line test. Ken's acts of understanding quadratic function situations also did not involve most of his understandings that were central to him in the function situations of the first task. The omission of the vertical line test in Sarah's above acts also strengthen the inference that she demonstrates an action view of functions in which she uses the vertical line test as a geometric property of the graph of a function—not as a property indicating a special relationship between two quantities, which applies to quadratic function graphs as well. In sum, Sarah seemed to relate only graphs of 'functions' to the vertical line test, and not quadratic function graphs (or quadratics, as she sees it). Therefore a separation was observed in both cases between the students' acts of understanding functions and quadratics.

Sarah's act of understanding of the object, the parts of the graph that are important or special, included the following ways of understanding: 'the graph of $y = x^2$ is symmetric about the origin $(0, 0)$ and therefore origin is important for this graph,' 'the vertex of the graph is either the minimum point (and no $y$ value can go below that point) or the maximum point (and no $y$ value can go above that point),' 'points where $y = 0$ are important because they help in graphing the equation,' and 'points where $y = 0$ are zeros.'

As an additional example, beside $y = x^2$, Sarah generated another quadratic by choosing four terms and filling in $y = (\_ + \_)(\_ + \_)$ as $y = (x + 3)(x + 4)$. While attending to this equation and isolating it as the object of her understanding (with the goal of graphing it), Sarah related this equation to her understanding 'the quadratic equation $ax^2 + bx + c = 0$,' and said that "it should be equal to zero" because "that's the quadratic equation." She changed $(x + 3)(x + 4) = y$ into $(x + 3)(x + 4) = 0$. At first, immediately after writing $(x + 3)(x + 4) = y$, she drew a blank table with $x$ and $y$ columns, and stated that she would "solve for $y$ values." In other words, initially she seemed to be graphing the quadratic $(x + 3)(x + 4) = y$ by generating $(x, y)$ pairs and plotting them in the coordinate plane, as she would do to any function. It seems that initially she may have related the quadratic to her basis of understanding of functions such as 'generating a table of $x$ and $y$ values,' and 'taking any $x$ value and plug it into the equation and you will get the corresponding $y$ value.' But instead, her act of understanding $(x+3)(x+4)=y$ by relating it to the way of understanding: 'A graph or equation is quadratic if it fits the quadratic equation, or the definition $ax^2+bx+c=0$,' seemed to have led Sarah to change $(x+3)(x+4)=y$ into $(x+3)(x+4)=0$ and solve the equations $x+3=0$ and $x+4=0$ and find $x=-3$ and $x=-4$. And she sketched the graph in Figure 2.9 by plotting these two points $(-3,0)$ and $(-4,0)$ and simply connecting them so that they form the shape of a parabola. Before sketching the graph, which resembled a parabola, Sarah found the x-coordinate of the vertex using $-b/2a$ (which she remembered from her class) and said that $-b/2a=-7/2$ makes sense to her because it is in between $x=-3$ and $x=-4$. However, she did not consider finding the y-coordinate of the vertex in order to graph it. She found the y-coordinate of the vertex by referring to her understanding of 'taking any $x$ value and plugging it into the equation to get the corresponding $y$ value.' She then found $y=-1/4$ to be the y-coordinate of the vertex and confirmed that her graph was correctly drawn.

Sarah's basis of understanding of *quadratic functions* can be summarized as follows:

- The quadratic equation: $ax^2+bx+c=0$
- $y=x^2$ is a quadratic equation
- Quad means four
- The reason it might be four is that the quadratic equation $ax^2+bx+c=0$ can be factored into ( )( )
- ( )( ) can be written as ( + )( + )

- $(\ +\ )(\ +\ )$ can be written as $(\_ + \_)(\_ + \_)$
- A graph or equation is quadratic if it fits the quadratic equation, or the definition: $ax^2 + bx + c = 0$
- The reason it might be four is that there are quadrants and it's the same on both sides of two of them
- The graph is quadratic because $x$ is squared
- In $ax^2 + bx + c = 0$, when $a = 1$, $b = 0$, and $c = 0$, the resulting equation $x^2 = 0$ fits the definition of quadratic equation
- The graph of $y = x^2$ is symmetric about the origin $(0, 0)$ and therefore origin is important for this graph
- The points on the left hand side of the vertex are symmetric with the points on the right hand side of the vertex
- The vertex of the graph is either the minimum point (and no $y$ value can go below that point) or the maximum point (and no $y$ value can go above that point)
- To generate an equation of a quadratic, one could choose four terms and fill in $y = (\_ + \_)(\_ + \_)$
- $-b/2a$ gives the $x$ value of the minimum or the maximum value
- There is a vertex form of the equation $ax^2 + bx + c = 0$ ("but cannot remember")
- Quadratics are always parabolas
- Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value
- Points where $y = 0$ are important because they help in graphing the equation
- Points where $y = 0$ are zeros

- In order to graph a quadratic equation one could find two x-intercepts and a vertex point (using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$), which is the midpoint between the two x-intercepts, and connect these three points so that the shape come out as that of a parabola
- $(x + 3)(x + 4) = y$ or $x^2 + 7x + 12 = y$ is an example of an equation of a quadratic graph

These quadratic function understandings can be further organized into: equation understandings, graph understandings, understanding of related concepts, and connections.

1) Equation Understandings:

- The quadratic equation: $ax^2 + bx + c = 0$
- $y = x^2$ is a quadratic equation
- In $ax^2 + bx + c = 0$, when $a = 1$, $b = 0$, and $c = 0$, the resulting equation $x^2 = 0$ fits the definition of quadratic equation
- To generate an equation of a quadratic, one could choose four terms and fill in $y = (\_ + \_)(\_ + \_)$
- There is a vertex form of the equation $ax^2 + bx + c = 0$ ("but cannot remember")
- Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value
- $(x + 3)(x + 4) = y$ or $x^2 + 7x + 12 = y$ is an example of an equation of a quadratic graph

2) Graph Understandings:

- The graph of $y = x^2$ is symmetric about the origin $(0, 0)$ and therefore origin is important for this graph

- The points on the left hand side of the vertex are symmetric with the points on the right hand side of the vertex
- The vertex of the graph is either the minimum point (and no $y$ value can go below that point) or the maximum point (and no $y$ value can go above that point)
- Quadratics are always parabolas
- Points where $y = 0$ are important because they help in graphing the equation
- Points where $y = 0$ are zeros
- In order to graph a quadratic equation one could find two x-intercepts and a vertex point (using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$), which is the midpoint between the two x-intercepts, and connect these three points so that the shape come out as that of a parabola

3) Understanding of Related Concepts:

- Quad means four
- ( )( ) can be written as ( + )( + )
- ( + )( + ) can be written as (_ + _)(_ + _)

4) Connections:

- The reason it might be four is that the quadratic equation $ax^2 + bx + c = 0$ can be factored into ( )( )
- A graph or equation is quadratic if it fits the quadratic equation, or the definition: $ax^2 + bx + c = 0$
- The reason it might be four is that there are quadrants and it's the same on both sides of two of them
- The graph is quadratic because $x$ is squared
- $-b/2a$ gives the $x$ value of the minimum or the maximum value

As in the case of Ken, Sarah never uttered the word function in this task. She also seemed to be only attending to the word quadratic (as opposed to the phrase quadratic function) and identifying it as her object of understanding. In addition, similar to Ken, Sarah made very little use of her basis of understanding for functions.

The two participants' above acts of understanding in Task 2, in summary, suggest the following similarities and differences in their understandings of quadratics. Both students identified the graphs of quadratics as having the shape of a parabola, which is a self-attribute of quadratic functions. And both students referred to the attribute that quadratic graphs (or parabolas) have only one minimum or one maximum point. Only Sarah however, made reference of this point being related to the minimum or maximum $y$ value as well. Thus, the two students also demonstrated certain understandings that were unique to them. For example, to Ken, the end behaviors of the graph of a quadratic function were important. These end behaviors are also connected to reflection in that if the leading coefficient is negative the graph opens downward and it opens upward if the leading coefficient is positive. On the other hand, Sarah used the attribute of line of symmetry in quadratic graphs while graphing quadratic functions. She also considered a product of two binomials having four terms, such as $(\_ + \_)(\_ + \_)$, as an important form of quadratic functions. And, whereas Sarah related the quadratics to the observation that '$x$ is squared,' Ken made no reference to the notion of squaring.

The most important commonality however, which was observed in both cases on several occasions, led to the emergence of a theme. Both students somewhat used the equation $ax^2 + bx + c = 0$ *as a prototype quadratic function*. It should be recalled that the students sometimes replaced the term 'quadratic function' with the word 'quadratic.'

The use of $ax^2 + bx + c = 0$ *as a prototype quadratic function* thus seemed to have contributed to the compartmentalization of their knowledge of functions (in general) from their understandings of quadratic [functions] in particular (Vinner and Dreyfus, 1989) in that they both attended to and operated on the concepts of quadratic equations, quadratic formulas, quadratics and parabolas—as opposed to the concept of function. In other words, incompatible conceptions coexisted without the students being aware of them. The understanding of $ax^2 + bx + c = 0$ *as a prototype quadratic function* further supports Thompson's (1994) hypothesis about multiple representations of functions. It suggests that because both Ken and Sarah were relating expressions to equations and sets instructions to carry out (as opposed to single entities—as in Sfard's (1991) notion of object understanding), and equations to functions and formulas, while operating within their action view of functions, they were not engaged in acts of understanding that were based on representational activities (Thompson, 1994) that relate the given equations or graphs to the underlying invariant relationships. Instead, they both identified the term quadratic only with the shape of a parabola and the equation $ax^2 + bx + c = 0$; and they conducted computations that aimed at the geometric or the algebraic properties of the respective types of function representations.

The centrality of the equation $ax^2 + bx + c = 0$ in their basis of understanding of quadratic functions, coupled with their seeming understandings of expressions as equations (as observed in the previous section) or as sets of instructions or somewhat as processes (Sfard, 1991) also caused difficulties for both students in handling situations where they had to find $y$ values, x-intercepts, or vertex coordinates. Recall that Ken offered $f(x) = 3x^2 + 2x + 4 = 0$ as an example of an equation of a quadratic graph;

and Sarah went back and forth among $ax^2 + bx + c = 0$, $(x + 3)(x + 4) = 0$ and $(x + 3)(x + 4) = y$ while finding the x-intercepts and the $y$ coordinate of the vertex in Task 2.

The basis of understanding $ax^2 + bx + c = 0$ *as a prototype quadratic function* is also compatible with the understandings of *functions as unique types of equations or graphs* and *functions as collections of things*. For both Ken and Sarah, it was somewhat unproblematic to simultaneously and consistently act on equations in the form $ax^2 + bx + c = 0$ and on shapes of parabolas in tandem without having the need to look for some coherence or unity between different representations that they generated. While this theme may be named '*functions as collections of representations,*' the lack of unity, commonality among representations, or invariance has been studied in the literature before. For instance, Ken's and Sarah's acts of understanding and bases of understanding quadratic functions seem to be incompatible with a *covariational view of functions* (Thompson, 1994; Carlson, 1998, Carlson et al., 2010). In referring to this type of functional reasoning, Carlson et al. (2010) state that "the ability to interpret the meaning of a function modeling a dynamic situation also requires attention to how the output values of a function are changing while imagining changes in a function's input values" (p. 115). Although the task instrument of this study did not involve a quadratic function task that models a dynamic quantitative situation, neither Ken's nor Sarah's understandings involved the use or mention of variables changing in relation to one another. Neither of them related symbolic situations such as $y = x^2 + 2$ to two variables with two sets of values, e.g., $x = 0, 1, 2, 3, \ldots$ (where values are increased by a constant difference), and $y = 2, 3, 6, 11, \ldots.$ (where values are two more than the squares of the corresponding $x$ values, and where they change according to a special quadratic relationship).

In Task 3, Ken's act of understanding the prompt: Graph $y = -(x-4)^2 + 16$ involved his immediate reference to: the "simple form" or "standard form" or the graph of $y = x^2$ and his understanding that 'the U-shaped parabola can be upside down, i.e., it can be "inverse" if there is a negative sign in front of its equation.' After relating the quadratic function equation, which is given in the form $y = a(x-h)^2 + k$ to these two understandings, Ken translated the graph of $y = x^2$ 4 units to the right horizontally and 16 units upward vertically. He also identified the need to do the "inverse," which meant to him reflection of the graph of $y = (x-4)^2 + 16$ about the line $y = 16$.

K: Well yeah that's because they were shifted. By sixteen.

R: Hmm.

K: So it's not so much that the negative, it just, they are opposite of what they would be without it.

R: What do you mean by that?

K: Actually that is not correct. It's. [*Pause*]

R: Opposite of?

K: Well I guess it is to me. Like the way I'm thinking about it. Cause if I were to draw the original one without the negative [*draws a new larger graph around the small graph that is crossed out and facing upward in Figure 1.6*] it would look something like that. [*Pointing at the new larger graph in Figure 1.6*]

R: Uh-huh.

K: And, with it, so the point would be up there [*draws a point on the new larger graph in Figure 1.6, which represent the reflection of the point* $(1, 7)$ *that was drawn earlier*].

R: Uh-huh.

K: And if you were to just make a new like make a new axis I guess [*draws a line through* $y = 16$], so I guess you are flipping over that [*makes a flipping move with his right hand over the line* $y = 16$].

R: Hmm.

K: So that's what I meant by opposite. Which is not really the opposite but the way I am thinking about it; it is.

Ken's clear coordination of function translations and reflections in quadratic function situations involving the activity of graphing functions given in the form $y = a(x - h)^2 + k$ further demonstrate the compartmentalization of his bases of understanding for functions, quadratic functions and transformations of functions that are given in the form: $y = a(x - h)^2 + k$ (Vinner and Dreyfus, 1989). Thus, in this task, Ken fluently and successfully applied all the necessary function transformations on to the graph of $y = x^2$ without discussing any other aspect of quadratic functions. Below is list of Ken's basis of understanding quadratic graphs in the form $y = -(x - h)^2 + k$.

Ken's basis of understanding of *the graph of* $y = -(x - h)^2 + k$ can be summarized as follows:

- The parabola is a "U-shaped" parabola
- The simple form or standard form $y = x^2$
- The graph of $y = x^2$

- The U-shaped parabola can be upside down, i.e., it can be inverse if there is a negative sign in front of its equation
- Horizontal translation of $h$ units to the right
- Vertical translation of $k$ units upward
- The graph of $y = -(x-h)^2 + k$ as the reflection of $y = (x-h)^2 + k$ about the line $y = k$

Sarah on the other hand, did not relate this question to such a 'transformational view' as did Ken. Rather, she seemed to identify the question as that of graphing a given equation by 'taking any $x$ value and plugging it into the equation and getting the corresponding $y$ value,' generating a table of those $x$ and $y$ values, plotting them on the graph paper, and by connecting all the points. She referred to this method as "the easy way." Upon the researcher's suggestion that she may use a different method instead, Sarah recalled her understanding that 'points where $y = 0$ are important because they help in graphing the equation,' which was demonstrated earlier as a part of her basis of understanding of quadratic functions. Then she tried to solve the equation $0 = -(x-4)^2 + 16$ so that she can identify the two x-intercepts. Because she also related this graphing situation to her understanding that 'in order to graph a quadratic equation one could find two x-intercepts and a vertex point (using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$), which is the midpoint between the two x-intercepts, and connect these three points so that the shape come out as that of a parabola,' there is evidence that the object of her understanding was specifically graphing a quadratic equation and not any type of function equation. It was also demonstrated earlier that 'the graph is quadratic because it is squared' is a way of understanding quadratics for Sarah.

Sarah's solution included the following steps: (1) $0 = -(x-4)^2 + 16$; (2) $-16 = -(x-4)^2$; (3) $16 = (x-4)(x-4)$; (4) $16 = (x-4)$; (5) $20 = x$. When her expectation, that she should find two x-intercepts for this quadratic equation, was not met, Sarah said: "So all I have is the vertex really." The following excerpt illustrates how she reasoned about the above solution steps, which suggest a unique way of understanding quadratic equations.

> S: And since those are both the same [*pointing at* $(x-4)(x-4)$], I'm only going to do once I'm only gonna have one zero value for $x$ [*writes* $(x-4) = 16$ *and* $x = 20$]... When $y$ is zero, $x$ is twenty.

Sarah's understanding of the equation $16 = (x-4)(x-4)$, as $16 = (x-4)$, 'because they are both the same' is a finding about how students think about quadratic equations, adding to the findings of Vaiyavutjamai and Clements (2006). Such way of understanding of the product of two identical binomial expressions ($(x-4)(x-4)$) also suggests that students may not relate the expression $(x-4)(x-4)$ to the concept of product. Ken's puzzlement about why when solving $(x-2)(x-3) = 0$ one sets both $(x-2)$ and $(x-3)$equal to zero to solve for$x$ also indicates a similar way of understanding the product $(x-2)(x-3)$ as somewhat 'two problems written side by side.'

Because Sarah found only one x-intercept instead of two, she identified the point $(20, 0)$ and the vertex. Then, she chose an $x$ value consecutive to 20, $x = 18$, and tried to plot points and connect them in order to generate a parabolic graph. When she found that the corresponding $y$ value for $x = 18$ was too large, she decided to choose smaller $x$ values starting from $x = 2$. When she couldn't relate her object of understanding, the three points $(20, 0)$, $(2, 12)$ and $(4, 16)$, to her basis of understanding, that connecting the points should form the shape of a parabola, she decided to call the point $(20, 0)$ a mistake and continued generating more points in the table in Figure 2.11. She also identified a reflection (when the noticed the negative leading coefficient of the equation), which accounted for the fact that the graph looked like it was facing downward.

R: Can you explain why, when there is a negative, the graph looks like this one? [*Pointing at the graph in Figure 2.12*]

S: Umm. I just know it means you flip it over umm.

R: Over what?

S: From what it would be if that was positive. Umm, let me think. Well it's making you subtract from the sixteen rather than add [*in* $y = -(x - 4)^2 + 16$], so you are subtracting umm so you would be going the same number of spaces in this direction [*traces the tip of her pencil along the graph in Figure 2.12 on the right side of the vertex*] that you would be going in this direction if it was positive [*moves the tip of her pencil in the air along a vertical line upward, starting from the vertex*] therefore it makes it curve down. So the negative and positive direction of the opening of the parabola.

R: Okay. Are you flipping it about a certain thing or?

S: Umm. Two, four, six, eight, ten, twelve, fourteen [*counts by two on the y-axis with the tip of her pencil in the air*]. Yeah, about the sixteen, which is the maximum.

Ken demonstrated almost an identical way of understanding multiple transformations in quadratic function situations in which function translations and reflections are coordinated.

Sarah's basis of understanding of *the graph of* $y = -(x-h)^2 + k$ can be summarized as follows:

- The graph is quadratic because $x$ is squared
- Quadratics are always parabolas
- Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value
- Points where $y = 0$ are important because they help in graphing the equation
- Points where $y = 0$ are zeros if there are two of them, and they are vertices if there is one of them
- In order to graph a quadratic equation one could find two x-intercepts and a vertex point (using$(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$), which is also the midpoint between the two x-intercepts, and connect these three points so that the shape come out as that of a parabola
- The graph of $y = -(x-h)^2 + k$ is the reflection of $y = (x-h)^2 + k$ about the line $y = k$

Thus, unlike in Task 2, this time Sarah related the given quadratic graphing situation to her basis of understanding of functions. As was in the case of Ken, she did not relate any of her objects of her understanding within this task to the prototypical quadratic $ax^2 + bx + c = 0$. In addition, they both understood the graph of $y =$

$-(x-h)^2+k$ as the reflection of $y=(x-h)^2+k$ about the line $y=k$, demonstrating coordination of function translations and reflection.

As further discussion on quadratic functions in Task 4, Ken offered the general solution to the quadratic equation $ax^2+bx+c=0$, $\frac{-b\mp\sqrt{b^2-4ac}}{2a}$ as the 'quadratic equation,' and identified it as an important aspect of quadratic functions. Specifically, he related $\frac{-b\mp\sqrt{b^2-4ac}}{2a}$ to finding the x-intercepts of a quadratic graph. He also defined transformations of functions, which he successfully applied to the parent graph of $y=x^2$, as "things that are happening to the graph." The way he named the expression $\frac{-b\mp\sqrt{b^2-4ac}}{2a}$ as the 'quadratic equation' further strengthens the earlier inference that Ken operates on expressions as if they are equations, or problems, or sets of instructions to carry out.

Sarah also articulated a few additional understandings that she has about quadratic functions. She repeated that the graph of $y=-x^2$ would be a reflection of the original graph $y=x^2$ over the line on which the vertex lies (the x-axis). She further clarified that she does understand reflections of quadratic graphs the way that Ken understands. In other words, 'if the leading coefficient of the quadratic equation is negative then the original graph reflects about the horizontal line on which its vertex lies.' Therefore, this task helped clarify some of the earlier ways of understanding certain aspects of quadratic functions and graphs. She also added that if one adds a certain number $k$ in an equation such as $y=(x^2-4)+k$, then the graph of $y=(x^2-4)$ would move $k$ units up. If one subtracts a positive number $k$ as in$y=(x^2-4)-k$, then the graph of $y=(x^2-4)$ would move $k$ units down. There is no evidence at this point how she relates the addition or subtraction of such term in vertical translation situations involving the quadratic function forms $y = a(x-h)^2 + k$ and $y = ax^2 + bx + c$. It is plausible that she may operate on $y = a(x-h)^2 + k$ in a very similar way (as in $y = (x^2 - 4) + k$), however, there is no evidence at this point how she thinks about the constant $c$ in $y = ax^2 + bx + c$.

Thus, additions to Sarah's and Ken's bases of understanding of *quadratic functions* can be summarized as follows:

Ken:

- The quadratic equation $\frac{-b \mp \sqrt{b^2-4ac}}{2a}$ can be used to find the x-intercepts or zeros of a quadratic graph
- Transformations of a quadratic function graph are things that are happening to the graph

Sarah:

- If the leading coefficient of the quadratic equation is negative then the original graph reflects about the horizontal line on which its vertex lies
- If one adds a certain number $k$ in an equation such as $y = (x^2 - 4) + k$, then the graph of $y = (x^2 - 4)$ would move $k$ units up
- If one subtracts a positive number $k$ as in $y = (x^2 - 4) - k$, then the graph of $y = (x^2 - 4)$ would move $k$ units down

In Task 6, Ken's acts of understanding the difference between equations and functions revealed more details about how he understands quadratic functions. Earlier it was argued that that Ken understands functions as unique type of equations where one solves for $y$. He also understands the existence of a collection of representations for functions. His understanding of functions consists of a process of computing values of the dependent variable $y$ from the given values of the independent variable $x$. If an equation allows such computation, then he sees it as a function. Sarah on the other hand, acted on this difference based on her central conception of the vertical line test. She drew graphs of several candidate equations and checked if their graphs passed the vertical line test. If they did, then she called them functions. These findings are also consistent with what Gerson (2008) writes: "A student's concept image of function might include: "you plug a number to it" or "it passes the vertical line test." It might include a mental picture of "$f(x)$" or the graph of the function $f(x) = x^2$" (p. 28).

In Tasks 7 and 8, both students related what they understood as their object of understanding with regard to various attributes of quadratic functions such as line of symmetry, vertex, x-intercepts and y-intercept. Both students seemed to be fluent procedurally in finding the $x$ component of the vertex using $-b/2a$ when the quadratic function equation was in the form $y = ax^2 + \text{bx} + \text{c}$. They demonstrated a mixture of algebraic manipulation skills in converting one form of quadratic function equation to another. In Task 8, Ken's act of understanding the line of symmetry in a quadratic graph involved recalling and relating it to the vertical line test. He introduced his additional understandings of the vertical line test and the function definition of "no one $x$ value you can have more than one $y$ value.

In Task 9, using a strong sense of function transformations, Ken related the question of "how can we find if a quadratic graph is stretched or shrunk?" to a basis of understanding about dilations. His understanding of quadratic growth is observed to be somewhat broad and unspecified; this understanding became apparent in Task 12

[Figures 1.24 and 1.25] where he discussed exponential growth in parabolas. In terms of how to determine if a graph is dilated, Ken used a unique conception of proportionality. Recall that in Figure 1.19, Ken argued that because the slope of the line that passes through the two points $(-2, 5)$ and $(0, 9)$ in the graph of $y = (x + 2)^2 + 5$ is $4/2 = 2$ and the slope of the line that passes through the two points $(0, 0)$ and $(2, 4)$ in the graph of $y = x^2$ is also $4/2 = 2$, there is no dilation. Also recall that he offered this explanation while circling the parts of the graphs in Figure 1.9 and making triangles [i.e., from $(-2, 5)$ to $(0, 9)$ one would "go to the right two and up four," and from $(0, 0)$ to $(2, 4)$ one would also "go to the right two and up four"]. This way of understanding dilations was not observed in Sarah's case. Sarah on the other hand, seemed to be able to find dilation factors through decomposing numbers. Recall that she looked at $c$ values in Task 9 [Figure 2.24] and tried to find out what number she needed to multiply the $c$ values in order to preserve the given points on the quadratic graph.

These idiosyncratic ways of understandings of Ken and Sarah and the themes that were developed in this cross analysis are compared and contrasted with the subsequent two cases, Seth and Joseph.

Cross Case Analysis: Case 3 (Seth) and Case 4 (Joseph).

Seth is a university freshman, who identified his mathematics background as "very vague and primitive." He also stated in his background survey (Appendix C) that he is "artistically minded and [he has] trouble wrapping [his] mind around numbers." He took Algebra II, Advance Functions and Modeling, Discrete Math, Pre-calculus and Statistics courses in high school and received a mixture of grades (from F to A) in them. Despite the number of mathematics courses that he took, Seth identified himself as having difficulty in upper level mathematics courses. He also wrote: "I am mostly a visual and tactile learner," and "I hate theoretical math because I can't see it."

Joseph is a recent graduate from a local public high school, who is accepted to a major university in his home state. He took Algebra II, Geometry, Pre-calculus, and AP Calculus courses in high school and received A's and B's in all of them. He was less articulate than the other three cases and he provided additional insights to his answers to the interview tasks only when the researcher asked for them. He characterized his mathematical knowledge as "good, but not perfect because there is plenty of topics out there that [he has] not yet learned about." He also stated that until pre-calculus, he has "been able to pretty much breeze through [his] math classes."

In this section, Joseph's and Seth's acts of understanding quadratic functions are analyzed together. Their mathematical problem solving behaviors and their articulations of their thoughts are also compared to those of Ken's and Sarah's (as well as the themes that emerged therein).

Seth's act of understanding the word function involved recalling, "off of top of [his] head," the understandings: '$f$ of $x$ equals,' 'a function of a particular equation equals something,' 'the graph of the line $y = x$ or $f(x) = x$' [in Figure 3.1], '$f(x) = y$,' and '$f$ of $x$ is another way of saying $y$.' He stated that he cannot think of what a function is or what it might be, but if there were a particular problem that asks him to do something to the given function, then he would be able to solve it. When asked to give examples of functions, he wrote: $f(x + 1) = 7$ and $f(x) = 6$.

SE [Seth]: Alright, a function. The most I know about a function off the top of my head is $f$ of $x$ equals. So, the function of a particular equation equals something.

So, that's really all I've got at this point. So, I'm gonna write that $f$ of $x$ equals. That's all I know about a function. So, umm.

R: Ok, what are you thinking?

SE: Umm, $f$ of $x$ equals $y$ [*writes* $f(x) = y$]. I'm trying to figure out how to attack this problem; how to write it in a definition sense, as opposed to just $f$ of $x$ equals. So, I'm searching my mind for any words that I might have associated with a function. And none are coming to mind. So, I'm kinda worried a little bit. But, I could draw a graph $f$ of $x$ equals that [*pointing at* $y$], you know, that's not, umm, $f$ of $x$ equals $y$. At this point I've kind of stopped working on the problem.

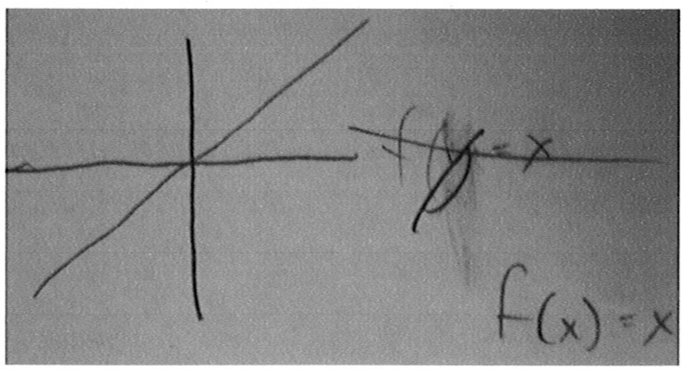



*Figure 3.1.* Seth's Response to Task 1

Seth's basis of understanding the word *function* can be summarized as follows:

- $f$ of $x$ equals
- A function of a particular equation equals something
- The graph of the line $y = x$ or $f(x) = x$
- $f(x) = y$
- $f$ of $x$ is another way of saying $y$
- $f(x + 1) = 7$ and $f(x) = 6$ are examples of functions

Joseph's act of understanding the word function, on the other hand, involved identification and recall of the existence of an equation, "its own graph," and a "table of coordinates."

> J [Joseph]: Whenever I think of a function, it's more like you have an equation and it's, it always has its own graph and table of coordinates. So, umm whenever I think of a function I always think of things like [*writes* $ax^2 + bx + c = 0$ *in Figure 4.1, which he later rewrote as* $f(x) = ax^2 + bx + c$ ] let this be$x$, umm of quadratic equations and they always have their own tables [*draws the table in Figure 4.2*] like I said with several coordinates and graphs [*draws the graph in Figure 4.1*]. Umm and also with functions you can umm I do remember there was one thing I was taught that the definition of a function. I just don't quite remember it.

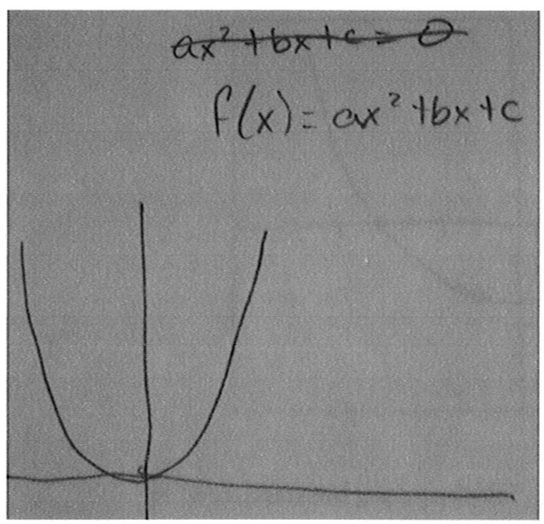



*Figure 4.1.* Joseph's Response to Task 1

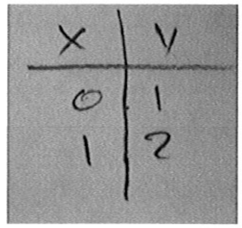



*Figure 4.2.* Joseph's First Table

R: What comes to your mind when you are trying to remember it? Besides the graph you said, and a table.

J: Uh-huh. Well, umm with functions also there are several different types of forms for functions that you have to solve for some type of variable.

R: Uh-huh.

J: That's pretty much all I can remember right now.

R: Ok, can you tell me more about those zero, one, one, two? Those points? [*Pointing at the table in Figure 4.2*]

J: These coordinates would be used to graph the actual line.

R: Alright, so let's recap what you said about functions. What is a function?

J: Well, basically a function is something that you, it's a, umm, an equation that you are able to plot on the coordinate plane and the whole side has its own set of coordinates on table. Umm.

Joseph's basis of understanding the word *function* can be summarized as follows:

- You have an equation, such as $f(x) = ax^2 + bx + c$, and it always has its own graph and table of coordinates
- There are several different types of forms for functions that you have to solve for some type of variable
- Coordinates would be used to draw the graph

Thus, Both Seth and Joseph used graphs and equations in their acts of understanding the word function. However, only Joseph seemed to display what was inferred as a way of understanding *functions as a collection of things*, held by Ken and Sarah. To Seth, the equations and graphs were also related, but they did not form some collection called function.

In the second task, Seth related the prompt, draw a quadratic function, to the understandings that 'a quadratic function is $x$ squared,' 'it's a parabola,' 'the $x$ value is squared so all $y$ results are positive resulting in a parabola shape,' and 'the $x$ value is squared so all $y$ are in a parabola shape.'

SE: Ok, a quadratic function is $x$ squared it's a parabola [*sketches the graph in bold in Figure 3.2*]. So that would be wider than that [*pointing at the graph he drew, without making any changes to it*]. So, that's about it.

R: Ok.

SE: What makes this graph quadratic? The $x$ value is squared so all $y$ results are positive resulting in a parabola shape.

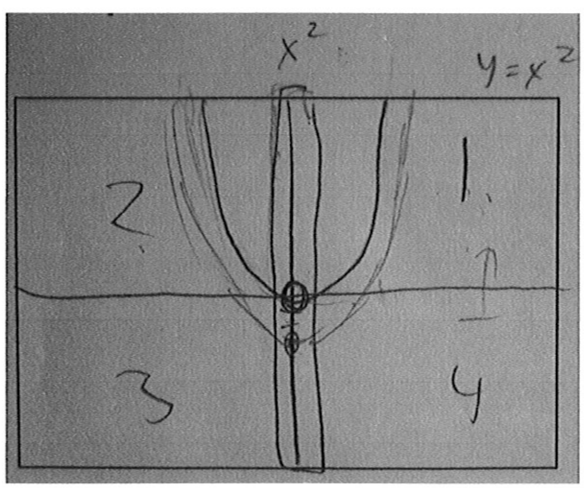



*Figure 3.2.* Seth's Response to Task 2

When he read the question about the important or special parts of the graph, Seth stated:

SE: Which parts of the graph are important or special and why? Umm, at this point I'm wondering what, which parts of the graph means. What I would do as a student, would say: "What do I know about parts of a graph?" Well, this side is positive and this side is negative. This is quadrant one, two, three and four [*writes 1, 2, 3, 4 in the four quadrants in Figure 3.2*]. So, quadrants one and two are where the entire graph takes place. I would say quadrants one and two are the most important parts of the graph.

R: For this particular graph?

SE: For this particular graph. Because that's where the entire graph takes place.

R: What do you mean by the entire graph takes place?

SE: Umm, everything stops here at $y$ is $x$ squared [*places his pencil horizontally on the x-axis*]. So, this is the graph, this upper area here [*pointing above the x-axis*].

R: Ok.

SE: Down here, there's nothing [*pointing below the x-axis*].

R: Ok.

SE: I mean, that's my thought process there.

R: Can you think of a quadratic function that would start.

SE: Lower?

R: In the third or fourth?

Then Seth drew the lighter colored graph in Figure 3.2, and discussed what he remembers about function translations that would move graphs up or down, or, left or right. He recalled that when you subtract a number from the "end," as in $y = x^2 - 3$, the original graph moves down. Upon reflecting on this result of his act of understanding vertical translation, Seth went back to his response to Task 2a, what makes this graph quadratic, and changed the sentence "the $x$ value is squared so all $y$ results are positive resulting in a parabola shape" to "the $x$ value is squared so all $y$ are in a parabola shape." However, he maintained that the $x$ values are still squared and therefore all $y$ results are still positive, but, as in the above example, those $y$ values may move up or down depending on the equation.

Seth's behavior confirms Zaskis et al.'s (2003) finding about horizontal translations of quadratic functions that students "just remember the rule" that "if there is a positive sign inside the parenthesis, as in $y = (x + 2)^2$, the original graph moves to the left, and that "if there is a negative sign inside the parenthesis, as in $y = (x - 2)^2$, the graph moves to the right." In other words, they relate their object of understanding, the question of what transformation to apply, to their basis of understanding of these rules about horizontal and vertical shifts on the grounds of memorization—and not logical reasoning, generalization or synthesis. Identification of rules based on memory suffices for most students. It was for Seth.

Lastly, when asked how he would write the equations for the two graphs in Figure 3.2, Seth wrote $y = x^2$ for the bolder one and $y = x^2 - 3$ for the lighter one. The researcher also asked Seth to explain what the function is in the second situation, $y = x^2 - 3$, represented by the lighter graph in Figure 3.2. Seth stated that he is not sure what "function" means, and wrote: $f(x^2 - 3) =?$ in Figure 3.3. He also put the term $f(x^2 - 3)$ in a rectangular box indicating that it is the $y$ value of the function.

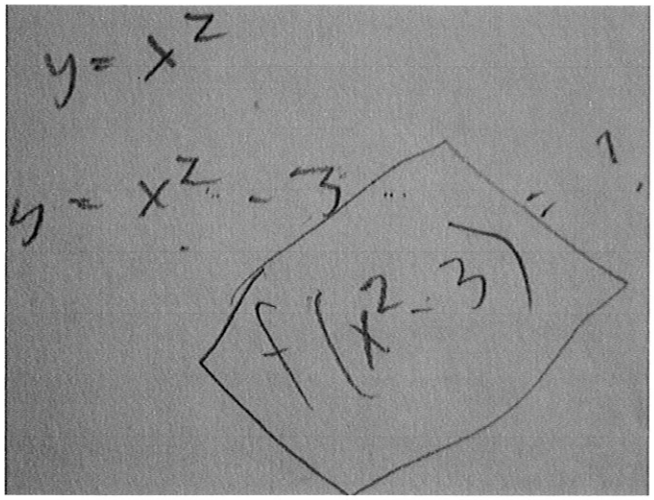


*Figure 3.3.* Seth's Representation of $y = x^2 - 3$ as a Function

Thus, Seth's basis of understanding *quadratic functions* can be summarized as follows:

- $x$ squared is a parabola
- $y$ equals $x$ squared
- The $x$ value is squared so all $y$ results are positive, resulting in a parabola shape
- In the graph of $y = x^2$ the first and second quadrants are the most important because there are more values (or points) in those quadrants

These quadratic function understandings can be further organized into: equation understandings and graph understandings.

1) Equation Understandings:

- $y$ equals $x$ squared

2) Graph Understandings:

- $x$ squared is a parabola

- The $x$ value is squared so all $y$ results are positive, resulting in a parabola shape
- In the graph of $y = x^2$ the first and second quadrants are the most important because there are more values (or points) in those quadrants

Joseph acted on Task 2 by drawing a coordinate plane and choosing a "vertex at $(-1, -2)$." He also chose "an x-intercept of negative four and zero" $[(-4, 0)]$ in order to "keep the actual function balanced" so he "can come up with what a good estimate of the equation would be." After connecting these two points and sketching a half of the shape of a parabola [Figure 4.3], he identified $(3, 0)$ as his other x-intercept and drew the other half of the graph [i.e., the half that is on the right hand side of the vertex $(-1, -2)$ in Figure 4.3].

> J: So, it would cross at (3,0) for its other x-intercept. Ok, I'm just trying to straighten it out. Ok, so with this function what makes it a quadratic is because it has umm two roots, which is both the x-intercepts here. [*Pointing at (-4,0) and (3,0)*]

After claiming that the graph is quadratic because there are two x-intercepts (or roots), Joseph then asserted that these two x-intercepts and the y-intercept are the most important or special parts of the graph because "they help find the equation." Thus, he related the graph of a quadratic function to the following basis of understanding: 'keep the actual function balanced so that you can come up with what a good estimate of the equation would be,' 'the graph is quadratic because it has two roots,' and 'the x-intercepts and y-intercept are important because they help find the equation of the graph.'

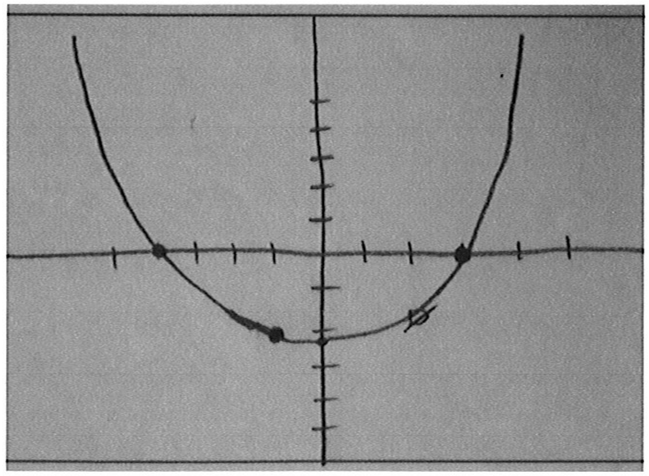


*Figure 4.3.* Joseph's Quadratic Graph in Task 2

After observing Joseph making several references to the x-intercepts as being important in finding the equation of the graph, the researcher asked:

R: What if your graph did not have an x-intercept, or? Is that possible? Or y-intercept? Then what would it be? What part would be special?

J: I think it would be possible to find the equation with umm finding estimates where the $x$ and y-intercepts would be like, umm this y-intercept is a little bit below negative two. So, I'd probably say it's somewhere around negative two point two zero. So, you just try to give estimates to see if you come close to what you are looking for.

R: So, basically, you are saying $x$ and y-intercepts are.

J: Very important to find the equation.

Thus, Joseph seemed to be confident about the importance of the intercepts. When he is given a certain graph, he argued that it is sufficient to use estimates if the exact coordinates of the intercepts are unknown. When he carried out an act of understanding the equation of his graph in Figure 4.3, however, he had difficulty using the intercepts—to find the equation. He attributed his difficulty to the fact that he has not tried to find equations of given graphs before; instead he has always found graphs of given equations. In Zaslavsky's (1997) study, students also preferred translating from equations to graphs over translating from graphs to equations. For Seth, finding the equations of his graphs in Figure 3.2 was not problematic because he started with an equation, $y = x^2$ and recalled its graph. He also recalled that when you subtract a number "from the end," the graph would move down. Therefore he was able to graph and write $y = x^2 - 3$ quite easily. Joseph further discussed his thoughts about Task 2c:

> J: So, now when we try to find the equation, we know that the form for this be $y$ equals $a$ $x$ squared plus $b$ $x$ plus $c$ [*writes* $y = ax^2 + bx + c$]. The simplest way to actually write this out would be, umm.
>
> R: Could you think out loud?
>
> J: I just had it in my head. I wasn't quite sure. Umm.
>
> R: What was it you weren't quite sure about? I'd like to know.
>
> J: Right now, I'm just trying to figure out what I can do to write the equation. I do have my $x$ and y-intercepts, so I'm just going to write that down. This is the x-intercept [*writes* $(3, 0)$ *and then* $(-4, 0)$]. The y-intercept would be somewhere around two, ok umm [*writes* $(0, -2.2)$].
>
> R: Alright, so, what is the, what are the ideas that come, came to your mind when you read the question?
>
> J: Hmm. Well, at first I was thinking if it would be easier if I would started off with finding the slope of this. But, umm it's not a straight line so. That's what I was thinking about first. That's what confused me a little bit.
>
> R: Where would the slope be here?

J: [*Long pause*] Umm. I think we do have a slope here. I'm just not quite sure where it would be. Umm. [*Long pause*]

R: Ok, what are you thinking?

J: Umm right now I'm just trying to think what's an easy way to start off writing an equation. So I know the actual equation to start off with the quadratic formula, $y$ equals $x$ squared, very basic, which gives you the parabola with the y-intercept of zero [*touches the tip of his pencil on the origin* $(0, 0)$ *and draws the graph of* $y = x^2$ *with his pencil in the air*].

R: And that's you call that the quadratic formula?

J: Uh-huh.

R: Ok, and then after the quadratic formula the basic one $y$ is $x$ squared, what else?

J: Umm, I do believe the easiest way to do it is by looking at which way the graph moves. So, the graph didn't actually move left to right, but it does move down by two spaces.

R: Ok.

J: So, umm I would say somewhere close to $x$ squared minus two. Cause whenever you subtract from the base it would either go down or, if you add it goes up [*pointing at the equation* $y = x^2$].

As seen in the above excerpt, Joseph used his understandings of 'whenever you subtract from the base it would either go down, or if you add it goes up' and 'the base is the basic equation $y = x^2$' and then wrote $y = x^2 - 2$ as his answer to Task 2c. Joseph thought that using estimates of the intercepts (or the vertex) is acceptable in finding the answer. Although his initial choice of vertex was $(-1,-2)$, he seemed to have abandoned that point and considered $(0,-2.2)$ to be the vertex due to an error in choosing the second x-intercept, $(3,0)$. It is unclear what he would have done had he chose $(2,0)$ as the second vertex instead. Thus, it seems that Joseph held an understanding of symmetry in parabolas (which was evident in his explanations of his rationale for strategically choosing two x-intercepts when drawing a quadratic graph so that he keeps "the function balanced." This conception of 'balance' is inferred to be his concept image for symmetry (Tall and Vinner, 1981).

Joseph's basis of understanding of *quadratic functions* can thus be summarized as follows:

- When drawing a quadratic graph, keep the actual function balanced so that you can come up with what a good estimate of the equation would be
- The graph is quadratic because it has two roots
- The x-intercepts and y-intercept are important because they help find the equation of the graph
- Using estimates of the intercepts (or the vertex) is acceptable in finding the equation of a given quadratic graph
- Whenever you subtract from the base it would either go down, or if you add it goes up
- The base is the basic equation $y = x^2$, which can also be called the quadratic formula

These quadratic function understandings can be further organized into: equation understandings, graph understandings, understanding of related concepts, and connections.

1) Equation Understandings:

- The base is the basic equation $y = x^2$, which can also be called the quadratic formula

2) Graph Understandings:

- When drawing a quadratic graph, keep the actual function balanced so that you can come up with what a good estimate of the equation would be
- Using estimates of the intercepts (or the vertex) is acceptable in finding the equation of a given quadratic graph
- Whenever you subtract from the base it would either go down, or if you add it goes up

3) Connections:

- The graph is quadratic because it has two roots
- The x-intercepts and y-intercept are important because they help find the equation of the graph

When we examine Seth and Joseph inferred bases of understandings of quadratic functions, neither of them relied on the equation that Ken and Sarah did—the quadratic equation $ax^2 + bx + c = 0$. Unlike Ken or Sarah, Seth's act of understanding the important parts of a quadratic graph resembled what Bruner (1973) called an iconic mental representation that is mediated only through pictures. Thus far in the interview, Seth seemed to only look at the graph he drew (based on memory) and identify it as a picture rather than something that stands for a certain relationship between $x$ and $y$ values. Joseph on the other hand, saw using estimates of the location of important points such as the $x$ and y-intercepts as acceptable in finding the equation of a given quadratic graph.

In Tasks 3 and 7, Seth chose several points and plotted them on his graph paper; and related his acts of understanding to symmetry. In the same tasks, Joseph instead only created tables of $x$ and $y$ values, and then plotted and connected points. Furthermore, neither of them seemed to have a basis of understanding of the prototypical equation $ax^2 + bx + c = 0$ that the other students had. Only Joseph mentioned this equation in the very beginning of Task 1, and he never returned to it. Both students seemed to have common understandings related to function translations, especially the vertical translation of simple forms such as $y = x^2$. They both demonstrated some components of the action view of functions.

Cross Case Analysis of All Four Cases.

Research on students' understanding of the concept of function has revealed common student understandings such as: (1) *function as a machine that produces an output number when an input number is supplied*, (2) *function as an equation involving $x$ and $y$*, and (3) *function as a graph that passes the vertical line test* (Vinner, 1992; Sand, 1996; Clement, 2001). Students who hold the conception of a function machine consider functions as an input-output box, where an output [$y$] value is found by substituting an input [$x$] value in some equation. On the other hand, students who hold the understanding of function as a graph that passes the vertical line test may also associate the graph with an equation, and students who hold that function is an equation involving $x$ and $y$ values may link the equation with a corresponding graph.

All four cases of this study, Ken, Sarah, Seth, Joseph, revealed bases of understanding of functions that were compatible with these findings from the research literature. Ken seemed to have a conception of function as a machine where one solves for $y$. Sarah demonstrated the understanding that function is a graph that passes the vertical line test. Both Seth and Joseph viewed functions as equations that involve two variables $x$ and $y$.

Researchers studying students' conceptions of functions have also identified some common student beliefs regarding certain characteristics of functions. According to Vinner (1992), many students believe that a function has a single rule or expression. In other words, if there are two rules or expressions for the independent variable $x$, students believe that there are two functions. In the first case of this study, Ken generated a piece wise defined function with two rules, $x^2 + 2$ and $x$ (Figure 1.14). Despite the fact that he only wrote the $f(x)$ symbol once in labeling the two rules, he explicitly referred to the existence of two functions. Research has also suggested that many students believe that a function must have an analytic or symbolic expression and its graph has to be continuous (Vinner, 1992).

While the four participants of the study each seemed to hold an *action view of functions* (Dubinsky, 1992; Carlson et al., 2010), relying solely on computational reasoning, none of the four participants seemed to demonstrate function understandings that were compatible with what Thompson and Carlson called a *covariational view of functions* (Thompson, 1994; Carlson, 1998, Carlson et al., 2010). Recall that in referring to this type of functional reasoning, Carlson et al. (2010) found that "the ability to interpret the meaning of a function modeling a dynamic situation also requires attention to how the output values of a function are changing while imagining changes in a function's input values" (p. 115).

Furthermore, O'Callaghan (1998) characterized students' understanding of the concept of function (by assessing their knowledge using a function test that he developed) in a slightly different way. The function test that he used was designed to assess four theoretical component understandings of function. These understandings are: (a) modeling a real world situation using a function, (b) interpreting a function in terms of a realistic situation, (c) translating among different representations of function, and (d) reifying functions. O'Callaghan (1998) found that the most difficult component understanding for students was seeing functions as single entities (reifying them into single objects and operating on them as wholes). The analyses of the four cases of this study suggest that none of the students were operating on functions as single entities—instead they were reasoning computationally and exploring only the geometric or algebraic properties of the various representations that they generated. Furthermore, as discussed earlier, Ken, Sarah and Joseph were seeing various function representations as collections of things. These students believed that this collection was the function itself; and they did not act on the different representations of a function on the basis of an understanding of invariance among them. Thus, the current study confirmed Thompson's (1994) hypothesis that: "Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students… Put another way, the core concept of 'function' is not represented by any of what are commonly called multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance" (p. 24). It is inferred from the students' acts of understanding that neither Ken, Sarah, or Joseph seemed to have had such representational activities in which the different representations emerge as necessary communication tools for the intended object of understanding of the invariant relationship between varying quantities.

Chazan (2000) also discussed in detail three canonical (or most referred to) representations related to functions. One of those representations focuses on the procedural aspect of getting from particular inputs to associated outputs (the *symbolic* or *analytic* representation), other focuses more on the values of inputs and outputs and less on how one gets the outputs from the inputs (the *tabular* representation), and the third focuses on the dynamics of the two sets of numerical values that are treated as coordinate points (the *graphical* representation).

Other researchers also reported that there are many students at the high school and college level, who view functions as: "*plug the number in the equation and get the answer*." Schwingendorf, Hawks and Beineke (1992) identified five different understandings of the concept of function. These understandings ranged from lesser degrees of sophistication to higher degrees. When a student views functions merely as equations and does not display any usage of a process (e.g., taking something and transforming it to something else) they called it a *"pre-function"* understanding. At this level, a student is said to have no understanding of function, because equations are not related to sets of objects or description of some rules. When a student displays a usage of an overall process of transforming a number to obtain another number (e.g., "plug the number in the equation and get the answer") then Schwingendorf, Hawks and Beineke called it an "*action*" understanding of function (as did Dubinsky, 1992, and Carlson et al., 2010). When a student coherently uses an input-output relationship in which there is only one output, and he or she holds an understanding of some systematic handling of pairs of input and output values in which different pairs are coordinated) then the student is said to have a "*process*" understanding of functions. When a student sees function as a single entity of a correspondence or dependence between two variables, then Schwingendorf, Hawks and Beineke called it a "*correspondence/dependence*" understanding.

In a study of ten secondary teachers' understanding of functions, Norman (1992) also characterized various understandings such as: the ability to define, exemplify, and characterize functions; having valid illustrations of definitions; recognizing the conditions that are necessary and/or sufficient for determining the functionality of a relation; and relating the applications of the concept of function in a variety of situations. To define the notion of functional reasoning, Norman (1992) stated: "This sort of reasoning includes the ability to deduce properties or generalizations related to functions, to use one's knowledge of functions to analyze and interpret mathematical situations involving graphical or algebraic presentations of functionally related information, to communicate about functional situations, and to use functions to extend one's knowledge about a mathematical concept, process, or situation" (p. 218). Norman's notion of functional reasoning includes the aforementioned covariational view; and none of the current study participants' bases of understanding of functions was compatible with this type of reasoning. For example, referring to Skemp's (1976) theory of instrumental and relational understanding, Norman (1992) explains how a student may operate with the vertical line test, while holding the understanding of functions as graphs, as Sarah did: "… Consider the determination of functionality of a graphical representation of a relation via the standard vertical line test. It is not difficult for students at an instrumental level to learn to identify whether or not a particular graph is a function. Relational understanding, however, requires an understanding of how the vertical lines used in the test are actually related to the definition of a function and how the graph represents that function. A student exhibiting relational understanding can explain why and in which situations the vertical line test works. In general, relational understanding comes from an understanding of deeper relationships among the concepts and processes associated with a particular concept or situation" (p. 215).

Sand (1996) asked in his classroom with first year high school calculus students to complete the sentence: "a function is…" The most common response that students gave was: "An equation with $x$ and $y$." According to Sand (1996), most students viewed function as either an equation (with $x$ and $y$) or as a machine where there is an input and an output: "'An equation with $x$ and $y$' was a common response, with several including details about a unique $y$ for each $x$. Two main points struck me about their responses. First, all students thought of a function as a process, or operation, that inputs an $x$ value, or domain value, and later outputs a $y$ value, or range value. None viewed a function as an object, something that has properties and can be manipulated in well-defined ways" (p. 468). Moreover, his students thought that functions act on numbers only. Although some type of functions have indeed numerical domain and range (such as continuous, increasing, or quadratic functions), many other functions act on different types of objects. None of the four participants of the current study made reference to the concepts of domain and range. There were no instances in any of the interviews where these two concepts were brought to discussion. Moreover, none showed a basis of understanding that included a view of functions as single objects.

As described earlier, Schwarz and Hershkowitz (1999) attempted to characterize the nature of students' "concept images" of functions (Tall and Vinner, 1981). They suggested three aspects of concept images of the function concept: (1) prototypicality, (2) part-whole reasoning, and (3) attribute understanding. While most of the participants demonstrated understandings resembling the use of prototypicality and part-whole reasoning, none displayed an attribute understanding.

Finally, neither Ken's, Sarah's, Seth's or Joseph's fabric of understandings included the critical components of the formal set-theoretic definition of function. Only Sarah held that 'for each $x$ value there is only one corresponding $y$ value.' The others made no reference or use of this property. The formal set-theoretic definition of function can be written as follows: Let A and B be two sets; a function $f$ from set A to set B is a relation between A and B such that for each element $a$ in set A there is one and only one element $b$ in set B; and set A is called the domain of the function and set B is called the range of the function; and a function is often denoted as $y = f(x)$ indicating a special relation $\{(x, f(x))\}$.

In sum, as seen in Table 1 below, the students in the current study acted on functions as either computational procedures, with an action view, or equations in which one solves for $y$, or equations that simply have $x$ and $y$ variables, or graphs that pass the vertical line test where every value of $x$ has only one $y$ value on the graph. None of the students demonstrated a basis of understanding of functions that included a dependency between two quantities, a relationship between two varying quantities, or a relationship or correspondence between two sets of objects. Table 1 summarizes all four participants' bases of understanding of the word function, the underlying rationales for these bases, as well as the emergent themes for ways of understanding functions. And Table 2 summarizes their bases of understanding of quadratic function, the underlying rationales for these bases, and the emergent themes for ways of understanding quadratic functions.

Table 1: Ways of Understanding Function

| Case | Basis of understanding of the word *function* / Fabric of understandings | Underlying rationales for the basis of understanding | Themes for ways of understanding *function* |
|---|---|---|---|
| Ken | • A function of something<br>• The equation $y = f(x)$<br>• The dependent variable $y$ is a function of the independent variable $x$<br>• A formula<br>• Solve for $y$<br>• Plug in $x$ values in the formula<br>• A table of $x$ and $y$ values<br>• A set of plotted coordinate points on a graph paper | A consistent view of functions as formulas<br><br>The formula $y = f(x)$ is a computational procedure for finding an answer [$y$] for a specific value of $x$ | *Function as a unique type of equation where one solves for y*<br><br>*Function as a collection of things: Graphs, tables, equations and x and y variables*<br><br>*Action view of function*<br><br>*Compartmentalization of understandings within representations (Gerson, 2008)* |
| Sarah | • $f$ of $x$ is equal to something<br>• The equation $f(x) =$ ... ... ...<br>• A graph<br>• A line or shape<br>• The vertical line test<br>• A graph that passes the vertical line test<br>• Can't have more than one $y$ | Graphs must pass the vertical line test | *Function as a unique type of graph where every value of x has only one y value on the graph*<br><br>*Function as a collection of things: Graphs, tables, equations and x and y variables*<br><br>*Action view of function* |

| | | | |
|---|---|---|---|
| | value for a single value of $x$ ("by definition")<br><br>• A table of $x$ and $y$ values<br><br>• Equation as the numerical representation of function or a way to write it<br><br>• Graph as a visual representation of the shape or whatever that equation makes<br><br>• Both equation and graph are functions<br><br>• Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value<br><br>• Solve for $y$ values<br><br>• Take a $y$ value on the graph and plug it into the equation and you will get the corresponding $x$ value<br><br>• The relationship that they have | | *Compartmentalization of understandings within representations* |
| Seth | • $f$ of $x$ equals<br><br>• A function of a particular equation equals something<br><br>• The graph of the line $y = x$ or $f(x) = x$<br><br>• $f(x) = y$<br><br>• $f$ of $x$ is another way of saying $y$<br><br>• $f(x+1) = 7$ and $f(x) = 6$ are examples of function | Recalling that the equation $f(x) = y$ represents a function | *Action view of function*<br><br>*Function as an equation with x and y*<br><br>*Compartmentalization of understandings within representations* |

| | | | |
|---|---|---|---|
| Joseph | • You have an equation, such as $f(x) = ax^2 + bx + c$, and it always has its own graph and table of coordinates<br><br>• There are several different types of forms for functions that you have to solve for some type of variable<br><br>• Coordinates would be used to draw the graph | An equation with $x$ and $y$ are used to generate a table of coordinates, and these coordinates are used to draw graphs | *Action view of function*<br><br>*Function as an equation with x and y*<br><br>*Function as a collection of things: Graphs, tables, equations and x and y variables*<br><br>*Compartmentalization of understandings within representations* |

In terms of their self-generated responses to the questions about quadratic functions in Task 2, the four students of the study showed the following fabrics of understanding various aspects of quadratic functions:

Table 2: Ways of Understanding Quadratic Function

| Case | Basis of understanding of *quadratic function* / Fabric of understandings | Underlying rationales for the basis of understanding | Themes for ways of understanding *quadratic function* |
|---|---|---|---|
| Ken | • The quadratic formula, the original formula, or the quadratic equation: $ax^2 + bx + c = 0$<br><br>• A parabola<br><br>• A graph or equation is quadratic if it fits the quadratic formula: $ax^2 + bx + c = 0$<br><br>• The graph of a quadratic is the shape of a parabola<br><br>• There is only one parabola<br><br>• The graph has either a minimum | Strong emphasis on the words 'quadratic' and 'parabola' and no reference to functions<br><br>Little use of basis of understanding for *functions*<br><br>Identification of the overall shape of a parabola and the prototypical equation $y = x^2$ as main examples | *The equation $ax^2 + bx + c = 0$ as a prototype quadratic function*<br><br>*Compartmentalization of understandings within concepts (of function and quadratic function)* |

| | | | |
|---|---|---|---|
| | or a maximum point<br><br>• The ends of the graph (the parabola) either go to positive infinity or negative infinity<br><br>• The parabola is a "U-shaped" parabola<br><br>• The U-shaped parabola can be upside down, i.e. it can be inverse if there is a negative sign in front of its equation<br><br>• The U-shaped parabola cannot be sideways because it would then have a different type of equation<br><br>• $f(x) = 3x^2 + 2x + 4 = 0$ is an example of an equation of a quadratic graph | of quadratics | |
| Sarah | • The quadratic equation: $ax^2 + bx + c = 0$<br><br>• $y = x^2$ is a quadratic equation<br><br>• Quad means four<br><br>• The reason it might be four is that the quadratic equation $ax^2 + bx + c = 0$ can be factored into (  )(  )<br><br>• (  )(  ) can be written as ( + )( + )<br><br>• ( + )( + ) can be written as $(\_ + \_)(\_ + \_)$<br><br>• A graph or equation is quadratic if it fits the quadratic equation, or the definition: $ax^2 + bx + c = 0$ | Strong emphasis on the words 'quadratic' and 'parabola' and no reference to functions<br><br>Little use of basis of understanding for *functions*<br><br>Identification of the overall shape of a parabola and the prototypical equation $y = x^2$ as main examples of quadratics<br><br>Consistency in graphing quadratics other than $y = x^2$ by | *The equation $ax^2 + bx + c = 0$ as a prototype quadratic function*<br><br>*Compartmentalization of understandings within concepts (of function and quadratic function)* |

| | | | |
|---|---|---|---|
| | • The reason it might be four is that there are quadrants and it's the same on both sides of two of them<br><br>• The graph is quadratic because $x$ is squared<br><br>• In $ax^2 + bx + c = 0$, when $a = 1$, $b = 0$, and $c = 0$, the resulting equation $x^2 = 0$ fits the definition of quadratic equation<br><br>• The graph of $y = x^2$ is symmetric about the origin $(0, 0)$ and therefore origin is important for this graph<br><br>• The points on the left hand side of the vertex are symmetric with the points on the right hand side of the vertex<br><br>• The vertex of the graph is either the minimum point (and no $y$ value can go below that point) or the maximum point (and no $y$ value can go above that point)<br><br>• To generate an equation of a quadratic, one could choose four terms and fill in $y = (\_ + \_)(\_ + \_)$<br><br>• $-b/2a$ gives the $x$ value of the minimum or the maximum value<br><br>• There is a vertex form of the equation $ax^2 + bx + c = 0$ ("but cannot remember")<br><br>• Quadratics are always parabolas | using two x-intercepts<br><br>The symmetric nature of a parabola about the line of symmetry where the vertex lies | |

| | | | |
|---|---|---|---|
| | • Take any $x$ value and plug it into the equation and you will get the corresponding $y$ value<br><br>• Points where $y = 0$ are important because they help in graphing the equation<br><br>• Points where $y = 0$ are zeros<br><br>• In order to graph a quadratic equation one could find two x-intercepts and a vertex point (using $(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$), which is the midpoint between the two x-intercepts, and connect these three points so that the shape come out as that of a parabola<br><br>• $(x + 3)(x + 4) = y$ or $x^2 + 7x + 12 = y$ is an example of an equation of a quadratic graph | | |
| Seth | • $x$ squared is a parabola<br><br>• $y$ equals $x$ squared<br><br>• The $x$ value is squared so all $y$ results are positive, resulting in a parabola shape<br><br>• In the graph of $y = x^2$ the first and second quadrants are the most important because there are more values (or points) in those quadrants | Identification of the overall shape of a parabola and the prototypical equation $y = x^2$ as main examples of quadratics | *An iconic view of function graphs as pictures* |
| Joseph | • When drawing a quadratic graph, one should keep the actual function balanced so that one can come up with what a good estimate of the equation would be | Identification of the overall shape of a parabola and the prototypical equation $y = x^2$ as main examples of quadratics | *Compartmentalization of understandings within representations* |

| | | | |
|---|---|---|---|
| | • The graph is quadratic because it has two roots<br><br>• The x-intercepts and y-intercept are important because they help find the equation of the graph<br><br>• Using estimates of the intercepts (or the vertex) is acceptable in finding the equation of a given quadratic graph<br><br>• Whenever you subtract from the base it would either go down, or if you add it goes up<br><br>• The base is the basic equation $y = x^2$, which can also be called the quadratic formula | The symmetric nature of a parabola about the line of symmetry where the vertex lies | |

In summary, all four students displayed function understandings that were compatible with the well known action view of functions. All students in the current study also demonstrated compartmentalized knowledge. In her analysis of one pre-calculus student's understanding of functions, Gerson (2008) states that "translation from one representation to another includes skills like plotting points from a table of data, finding an equation for a graph, and creating a table of data from an equation" (p. 28). All participants demonstrated this skill in one or more of the interview tasks. See for example Joseph's and Seth's solutions to Task 1, Ken's and Sarah's solutions to Tasks 2. Gerson also writes: "In each representation of a given function, different mathematical features of the function may be apparent. Transferring between representations means carrying those mathematical features through the translation process" (p. 28). However, none of the participants of the study showed evidence of a basis of understanding that enabled them to do so. Gerson's (2008) research report reveals one student's, pseudo named

David, understandings of functions. David demonstrated an action view of functions and his understandings were compartmentalized. Note that David "was one of the three A students studied and therefore represents capable, procedurally fluent students in a traditional pre-calculus class" (p. 20). Gerson's larger study included nine high school students enrolled in a pre-calculus course at the time of the study. Action view of functions and compartmentalization in the context of quadratic functions were observed in all four cases of the current study. Only these two types of understandings seemed to capture the commonalities across all four cases.

# CHAPTER 5: DISCUSSION

## Discussion of Results

As von Glasersfeld (1990) argued, students develop their conceptions and understandings by constructing them. As students reflect on their activities that make sense to them, they relate their various understandings and build unique conceptual structures. Perceptual experiences and conceptual operations on existing understandings shape their developing conceptions. And learning is viewed as the organization of these conceptual structures, or schemes. von Glasersfeld (1990) also wrote: "No schemes could be developed if the organism [the individual] could not isolate situations in which a certain action leads to a desirable result. It is the focus on the result that distinguishes a scheme from a reflex and makes possible the form of learning that Piaget called accommodation. It takes place when a scheme does not lead to the expected result" (p. 24; parenthesis added).

Although the current study did not attempt to address how students develop conceptual structures with regard to quadratic functions, it did address three research questions that aimed at explicating students' current conceptual structures. The research questions of the study were: What are students' understandings of quadratic functions? How do individual students understand and organize various aspects and properties of quadratic functions? How are these understandings constituted within situations involving quadratic functions and their properties?

The detailed bases of understanding for selected tasks and concepts provided some answers to these questions. The connections that four students made among their understandings as well as their compartmentalized use of various parts of bases of understanding helped explicate how they understand quadratic functions and their properties. The bases of understanding that are presented in the study are hypothetical, and not definitive. They serve as working models of how students understand various aspects of quadratic functions. One important contribution of this study has been to give specificity and interpretation to these conceptions. The study also yielded results that suggest additional research studies.

By explicating a small group of students' current conceptions and individual ways of acting on quadratic function problem situations, the study generated several findings. Ken's case yielded an understanding of *quadratic function as a unique type of equation where one "solves for y."* The analysis of Sarah's case led to the emergence of an understanding of *quadratic function as a unique type of graph where every value of x has only one y value on the parabola shaped graph.* And, three of the four cases suggested a way of understanding *quadratic functions as a collection of things that are compartmentalized in multiple ways*. In addition, all four cases confirmed some of the major findings in the literature on students' understandings of functions. All four cases were compatible with both the action view of functions and the compartmentalization of function knowledge. They also added to these existing findings in the literature by providing holistic pictures or fabrics of common ways of understanding *quadratic* functions.

These findings emerged through several cross analyses between and among the multiple cases of the study. The design of the study allowed this multiple layers of analyses, while yielding rich descriptions and explanations throughout.

Implications for Teaching

In a typical mathematics classroom, it is likely to observe a teacher focusing on helping his or her students carry out a certain solution method or algorithm correctly. This method is likely to be presented in its entirety at once and demonstrated several times until most students seem to have mastered its correct execution. Computation is likely to be the overarching mathematical process; and obtaining right answers to the computational procedures is likely to be considered the manifestation of the mastery of the lesson objectives. Vis a vis these typical practices, Ball (1991) writes: "When we hear right answers simply as representing understanding, we miss opportunities to gain insight into students' thinking" (p. 45).

Several professional educational organizations published documents that encouraged the inclusion of additional mathematical processes and lesson objectives in attempts to increase students' ability and knowledge in mathematics. NCTM (2000) recommended the inclusion of five content and five process standards at each level of schooling, including individual classrooms. The recommended content standards are: number and operations, algebra, geometry, measurement, and data analysis and probability. And the recommended processes are: problem solving, reasoning and proof, connections, communication, and representation. National Research Council (NRC) published a book entitled "Adding It Up," which combined the competencies proposed by the reform movement of the 1980s and 1990s, i.e., reasoning, solving problems, connecting mathematical ideas, and communicating mathematics to others, with the proposals from the critics of the movement, i.e., emphasis on memorization, facility in computation, and being able to prove mathematical assertions, in a research based practical guide for mathematics educators (Kilpatrick et al., 2001). These different goals for school mathematics were synthesized into five mathematical proficiency strands that the authors "believe is necessary for anyone to learn mathematics successfully" (Kilpatrick et al., 2001, p. 116). They stated: "Mathematical proficiency, as we see it, has five components, or *strands: conceptual understanding*: comprehension of mathematical concepts, operations, and relations; *procedural fluency*: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately; *strategic competence*: ability to formulate, represent, and solve mathematical problems; *adaptive reasoning*: capacity for logical thought, reflection, explanation, and justification; *productive disposition*: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (Kilpatrick et al., 2001, p. 116).

The authors further argued that these five strands should be viewed as interwoven and complementary to one another. Thus, there has been a strong emphasis on changing the typical, traditional mathematical classrooms to include the development of a variety of knowledge, skills, abilities and mathematical competencies. Conceptual understanding is one of these espoused goals.

The ways in which students reason and think about a given mathematical situation are crucial for teachers to know because, based on constructivist theories, students' existing knowledge and ways of thinking shape their current learning. Even if a student gives a correct answer to a question, the meanings and understandings that the student holds should be known to educators in order to promote conceptual development.

To that end, Simon et al. (2004) suggested a conception based teaching approach, which focuses on identifying students' current understandings, articulating crucial desired understandings within students' process of conceptual development, and designing activity sequences that help them attain the desired conceptions. The instructional goals are not articulated around getting correct answers by routine application of rules or algorithms. The traditional telling and showing method is considered to be insufficient in developing students' mathematical proficiency. Instead of teaching students how to solve different types of problems, conceptual learning based teaching aims at teaching how to think and reason mathematically and how to problem solve in mathematics. Simon et al.'s (2004) conception based perspective to teaching is based on three principles: (1) Mathematics is a human activity and it is created by humans. Humans have no access to a mathematics that is independent from their ways of knowing (2) What individuals currently know afford and constrain what they can understand (3) Learning mathematics is a transformation process in which learners' current conceptions, ways of thinking and acting change into new conceptions (Simon et al., 2004).

"In guiding our students towards a generalization, very often we forget that the object to generalize may not yet be an object for them" (Sierpinska, 1994, p. 59). If we want our students to have strong knowledge of quadratic functions, we need to help them acquire or develop certain ways of understanding or understandings that we want them to have. In order for them to develop the desired ways of understanding, they need to experience acts of understanding in a very rich set of situations involving quadratic functions. Teachers who are equipped with the knowledge of students' current understandings can better design those learning situations. Indeed, teachers can communicate their own understandings to their students, but it is the students themselves who will attend and identify certain objects (i.e., objects that they identify as objects of their understanding) and relate them to their existing bases of understanding. "The concept of function has many aspects and grasping as many of them as possible in teaching should probably be aimed at. The problem is however, that when we use ordinary language to say something about a function, we necessarily focus the listener's attention on one possible understanding of functions. This means that 'whatever we say a function is, it isn't.'" (Sierpinska, 1994, p. 39; Korzybski, 1950). In other words, whatever a teacher says a quadratic function is, in fact it isn't because the students will map that assertion onto their own understandings that are most likely very different from what is presented to them.

"We want to make the students acquire certain ways of understanding, certain 'understandings,' certain knowledge, of course, but we cannot do this other than by helping them to experience acts of understanding" (Sierpinska, 1994, p. 27). "Some teachers, and education researchers, believe that a teaching approach which places the study of equations, including quadratic equations, within the study of functions—the so-called 'functions approach'—is far more likely than traditional elicitation/exposition approaches to solving equations to induce relational understanding within students" (Vaiyavutjamai and Clements, 2006, p. 73).

For example, none of the participants of this study seemed to conceptualize or use a mathematical expression with multiple terms as a single entity. For instance, whereas a mathematician or a mathematics educator may see $(x-3)(x-5)$ as a single expression, which is a product of two binomial expressions with the same variable $x$, and see $(x-3)(x-5)=0$ as an equation with one variable, students may perceive $(x-3)$ and $(x-5)$ in $(x-3)(x-5)=0$ as two 'equations' or two 'problems.' There is some evidence that Ken sees expressions such as $(x-3)$ and $(x-5)$ as two 'equations.' He frequently refers to expressions as equations. It is useful for teachers to know these results and start thinking about effective lessons that provide rich experiences.

None of the participants of the current study mentioned or used the notions of domain and range of a function. Domain and range, as a topic, may have been compartmentalized into a certain set of problems that students have memorized. It is imperative that teachers help their students make sense of quadratic function situations in terms of covarying quantities or variables with corresponding values that belong to different sets called domain and range. Sand (1996) also suggested the use of a real-world example of a function: What a mail carrier does. Each letter (each domain value) is placed in only one mailbox (range value). Such kinds of problems and real world situations of functions could be developed with regard to quadratic functions.

Another recommendation based on the results of the current study is that teachers would better serve their students if they provide them with meaningful experiences with quadratic functions that have either one or no x-intercept. All four students demonstrated an expectation for the need to have two x-intercepts on a quadratic graph. Recall that Sarah heavily relied on the existence of two x-intercepts in graphing quadratic functions.

Also, three of the four participants of the current study seemed to conceive function representations as the necessary parts of a certain collection. They merely accepted the existence of a collection (of graphs, tables, equations, etc.) without investigating quadratic relations between varying quantities.

Although these results provide these practical suggestions, they are not expected to suggest the design of specific activities. Instead, they provide a holistic sense of how similar students may think about quadratic functions and their properties. By making explicit what students know about this concept, teachers and curriculum developers could better diagnose other students, and design learning experiences and relevant assessments. The results thus provide more possibilities for teachers to find ways of helping their students to construct the desired understandings. As mentioned above, teachers could be better equipped in decomposing the necessary quadratic function concepts for their students instead of transmitting information that students memorize and take as true without developing conceptually connected understandings.

## Possible Limitations of the Study

Although the current study provided insights into how students may understand various aspects of quadratic functions, it did not include participants who hold completely mathematical understandings that resemble formal definitions. It would be beneficial to extend the current range of student understandings to include such wider range of conceptions. Such knowledge could help teachers in serving more advanced students.

While the task instrument did not include any task in which a quadratic function is embedded in real life context as a model of a quantitative situation, it was believed that this may confuse the issues that the study intended to address. Instead, it included a sufficiently conceptual and unfamiliar set of tasks that mostly asked students to self-generate a variety of responses and solutions regarding quadratic functions. Moreover, with the exception of a few tasks (i.e., Task 5 and Task 9), none of the tasks were approached by any of the students as "problems that they know or remember how to solve."

Recommendations for Future Research

The three hypotheses that were generated in the current study could be tested with both qualitative and quantitative methods.

Students' quadratic function conceptions can be further investigated within real-world contextual application problem situations. Such a study could provide more insight into this phenomenon by exploring possible ways of understanding, for instance, projectile motions.

Gerson's (2008) research questions could also be further investigated within quadratic function understandings. For example, are students' concept images of quadratic functions compartmentalized within representations and/or within related concepts? This question can be studied by choosing students with very different mathematical backgrounds.

Further studies could be conducted that give students two points and ask them to find a third point on the quadratic graph. These studies could explore the use of linear interpolations and extrapolations. In addition, students' understanding of quadratic growth could be investigated more in-depth.

# REFERENCES


Ajdukiewicz, K. (1974). *Pragmatic logic.* Boston, MA: D. Reidel Publishing Co.

Baker, B., Hemenway, C., & Trigueros, M. (2000). On transformations of basic functions. In: H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference on the future of the teaching and learning of algebra, 1*, 41-47. University of Melbourne.

Ball, D. L. (1991). What's all this talk about "Discourse." *Arithmetic Teacher*, 44-48.

Bosse, M. J., & Nandakumar, N. R. (2000). Extemporaneous problem development: Quadratic functions. *Mathematics Teacher, 93*(3), 243-244.

Bruner, J. S. (1973). The growth of representational processes in childhood. In Anglin, J. M. (Ed) *Beyond the information given. Studies in the psychology of knowing.* New York, NY: W. W. Norton and Co.

Buck, J. C. (1995). Fostering connections between classes of polynomial functions. In *Paper presented at the annual meeting of the North American chapter of the international group for the psychology of mathematics education.* Columbus, OH, October 1995.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment (PCA) instrument: A tool for assessing reasoning patterns, understandings and knowledge of precalculus level students. *Cognition and Instruction, 28*(2), 113-145.

Carlson, M. (2002). Physical enactment: A powerful representational tool for understanding the nature of covarying relationships. In Fernando Hitt (Ed.), *Representations and mathematics visualization. Psychology of mathematics education*, 63–77. Mexico: Cinvestav-IPN.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education III*, 114-162. Providence, RI: American Mathematical Society.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom.* New York, NY: Teachers College Press.

Clement, L. L. (2001). What do students really know about functions? *Mathematics Teacher, 94*(9), 745-748.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In Lester, F. K. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning*, 3-38. Greenwich, CT: Information Age Publishing.

Cobb, P., Yackel, E., & Wood. T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2-33.

Confrey, J. (1991). Steering a course between Piaget and Vygotsky. *Educational Researcher, 20*(2), 29-32.

Craine, T. V. (1996). Activities: A graphical approach to the quadratic formula. *Mathematics Teacher, 89*(1), 34-38, 44-46.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks, CA: Sage.

Davydov, V.V., & Radzikhovskii, L.A. (1985). Vygotsky's theory and the activity-oriented approach in psychology. In J.V. Wertsch (Ed.), *Culture, communication, and cognition: Vygotskian perspectives,* 35-65. New York: Cambridge University Press.

Dewey, J. (1971). *How we think.* Chicago: Regnery.

Dubinsky, E., & Harel, G. (Eds.) (1992). *The concept of function: Epistemological and pedagogical aspects*. Washington, DC: Mathematical Association of America.

Dugopolski, M. (2008). *Pre-calculus: Functions and graphs.* Addison Wesley.

Edwards, T. G., & Ozgun-Koca, S. A. (2010). A historical perspective from "a" to "c". *Mathematics Teaching in the Middle School, 15*(8), 458-465.

Edwards, T. G. (1996). Exploring quadratic functions: From a to c. *Mathematics Teacher, 89*(2), 144-146.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1*, 45-68. Providence, RI: American Mathematical Society.

Ellis, A. B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. *The Journal of Mathematical Behavior, 27*, 277-296.

Eraslan, A. (2008). The notion of reducing abstraction in quadratic functions. *International Journal of Mathematical Education in Science and Technology, 39(8)*, 1051-1060.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Gallagher, J. M., & Reid, D. K. (1981). *The learning theory of Piaget and Inhelder* (pp. 1-37). Monterey, CA: Brooks/Cole Publishing.

Gerson, H. (2008). David's understanding of functions and periodicity. *School Science and Mathematics, 108*(1), 28-38.

Gibbs, M. J. (2006). Patterns of interactions as affected by graphing software: Developing a theoretical framework. *International Journal for Technology in Mathematics Education, 13*(3), 139-149.

Guba, E. G., & Lincoln, Y. (1989). *Fourth generation evaluation.* Newbury Park, CA: Sage.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71-90.

Heid, M. K. (2010). The task of research manuscripts—Advancing the field of mathematics education. *Journal for Research in Mathematics Education, 41*(5), 434-437.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chui, A. M. Y., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). Teaching mathematics in seven countries: Results from the TIMMS 1999 Video Study. Washington, DC: National Center for Educational Statistics (U.S. Department of Education).

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics,* 1-27. Hillsdale, NJ: Erlbaum.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Korzybski, A. (1950). *Science and sanity.* Lakeville, Conn.: International Neo-Aristotelian Publishing Co.

Larson, R., Hostetler, R. P., & Edwards, B. H. (2001). *Precalculus with limits: A graphing approach.* Houghton Mifflin.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Leont'ev, A. N. (1981). The problem of activity in psychology. In Wertsch, J. V. (Ed) The Concept of activity in Soviet psychology. Armonk, NY: Sharpe.

Maxwell, J. A. (1996). *Qualitative research design: An interactive approach.* Thousand Oaks, CA: Sage.

Merriam, S. B. (1988). *Case study research in education: A qualitative approach.* San Francisco: Jossey-Bass Publishers.

Moore-Russo, D., & Golzy, J. B. (2005). Helping students-connect functions and their representations. *Mathematics Teacher, 99*(3), 156-160.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions*, 69–100. Hillsdale, NJ: Lawrence Erlbaum Associates.

Movshovitz-Hadar, N. (1993). A constructive transition from linear to quadratic functions. *School Science and Mathematics, 93*(6), 288-298.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics, Journal for Research in Mathematics Education Monograph No. 4*, 7-18. Reston, VA: National Council of Teachers of Mathematics.

Norman, A. (1992). Teachers' mathematical knowledge of the concept of function. In E. Dubinsky, & G. Harel (Eds.) *The concept of function: Epistemological and pedagogical aspects*, 215-232. Washington, DC: Mathematical Association of America.

O'Callaghan, B. R. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education, 29*(1), 21-40.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426.

Olmstead, E. (1995). A mathematics classroom for the twenty-first century: Exploring the complete story of turning points of quadratics. *New York State Mathematics Teachers' Journal, 45*(1), 25-32.

Patton, M. Q. (1990). *Qualitative evaluation and research methods.* Newbury Park, CA: Sage.

Piaget, J. (1978). *Success and understanding.* London: Routledge and Kegan Paul.

Piaget, J. (1970). *Genetic epistemology.* New York: Columbia University Press.

Sand, M. (1996). A function is a mail carrier. *Mathematics Teacher, 89*(6), 468-469.

Schoenfeld, A. H. (1985). Mathematical problem solving. Orlando, FL: Academic Press.

Schwarz, B. B., & Hershkowitz, R. (1999). Prototypes: Brakes or levers in learning the function concept? The role of computer tools. *Journal for Research in Mathematics Education, 30*(4), 362-389.

Schwingendorf K., Hawks J., & Beineke J. (1992). Horizontal and vertical growth of the students' conception of function. In E. Dubinsky, & G. Harel (Eds.) *The concept of function: Epistemological and pedagogical aspects*, 133-149. Washington, DC: Mathematical Association of America.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press Ltd.

Simon, M., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305-329.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching 77*, 20-26.

Stake, R. (1995). *The art of case study research.* Thousand Oaks, CA: Sage.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior, 20*, 267-307.

Steffe, L. P., & Kieren, T. (1994). Radical constructivism and mathematics education. *Journal for Research in Mathematics Education, 25*(46), 711-733.

Sullivan, M. (2008). *Pre-calculus: Enhanced with graphing utilities.* Prentice Hall.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with special reference to limits and continuity, *Educational Studies in Mathematics*, *12*, 151-169.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In A. H. Schoenfeld & J. J. Kaput (Eds.), *CBMS Issues in mathematics education: Research in collegiate mathematics education, 4,* 21–44.

Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal, 18*(1), 47-77.

Vinner, S. (1992). The function concept as a prototype for problems in mathematics learning. In E. Dubinsky, & G. Harel (Eds.) *The concept of function: Epistemological and pedagogical aspects*, 195-214. Washington, DC: Mathematical Association of America.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

von Glasersfeld, E. (1991). *Radical constructivism in mathematics education.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical? In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics, Journal for Research in Mathematics Education Monograph No. 4*, 19-29. Reston, VA: National Council of Teachers of Mathematics.

Winicki-Landman, G. (2001). Searching families as a source of surprise. *Mathematics Teacher, 94*(6), 468-478.

Yin, R. K. (1989). *Case study research: Design and method.* Newbury Park, CA: Sage.

Zaslavsky, O. (1997). Conceptual obstacles in the learning of quadratic functions. *Focus on Learning Problems in Mathematics, 19*(1), 20-44.

Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Conceptions of function translation: obstacles, intuitions, and rerouting. *The Journal of Mathematical Behavior, 22*, 437-450.

# APPENDIX A: INTERVIEW PROTOCOL

This session is an experiment to observe different ways that students solve math problems. I would like you to solve some pre-calculus problems in the next 75 minutes or so. I am not interested in whether you got the correct answer or the wrong answer. I won't even look at that. What I am interested in is what you think about and how you think about the problems as well as the steps of your solution methods. Therefore, do all of your thinking out loud. That is, tell everything you are thinking about while you solve every single step of each problem. What you say and do will be both videotaped and audio-taped. If you work silently, I will remind you to think aloud by saying something such as, "what are you thinking?"

Please put all your written work on the problem sheets that I provide. Do not erase any of your work. Since we will not be focusing on the correctness of your work, you can just ignore the work that you are not satisfied with. If you want to disregard something, draw a single line through it. Do not erase it.

You are not expected to solve the problems quickly. Take your time and think thoroughly about each problem. Some problems might make you think for a while (Again, while you are thinking, please think aloud). You can comment on the problems however you like. You may use as much time as you like.

Here is the procedure that we will follow for each problem:

- For each problem, as you read it, read out loud. Whenever you reread all or part of the problem, please read aloud.
- Talk out loud in your normal tone of voice as you work through the problems, saying everything you are thinking about. Speak clearly enough to be understood.

- Write all of your work, including scratch work, on the papers provided.
- Do not erase anything.
- You can quit anytime you want.

# APPENDIX B: INTERVIEW TASKS

1) In your own words, please explain what you think a "function" is. Feel free to write as much as you like. You may also draw graphs, diagrams or tables.

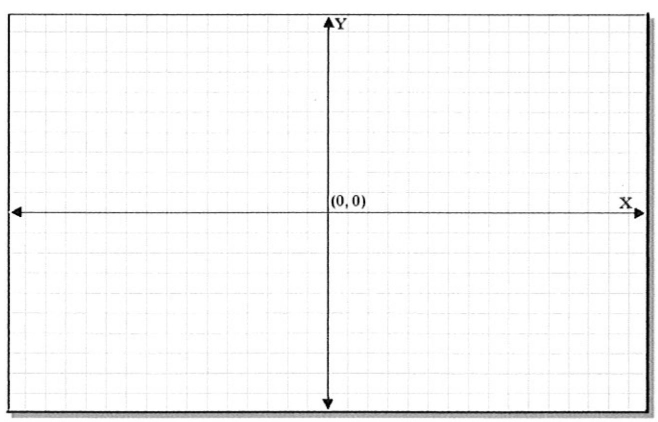
2) Please draw a quadratic function in the box below:

    a) What makes this graph quadratic?

    b) Which parts of this graph are important or special? Why?

    c) Can you give an equation for your graph?

3) Graph $y = -(x-4)^2 + 16$




4) Suppose a friend of yours missed the class that was on quadratic functions. If you were to teach quadratic functions to your friend, what would you tell him or her?

5) In the below equation, find the values of $x$ that make the equation a true equation. In other words, solve for $x$.

$$2x^2 - 7x + 3 = 0$$

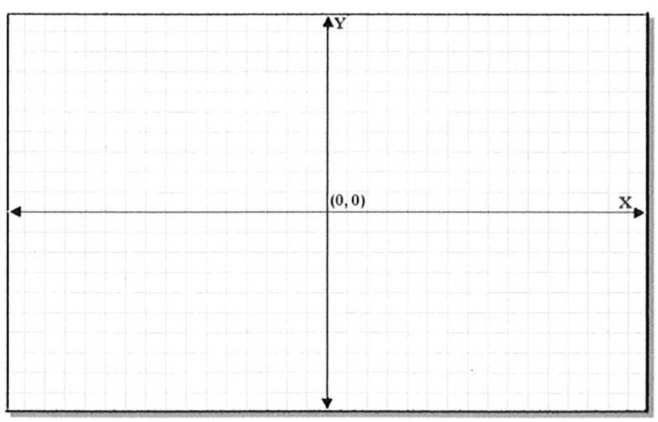
How many other different ways you think you can solve the above equation? List as many ways you know. Just list the steps of the strategies; you don't have to complete the solutions all the way.

Solve $x^2 - 5x + 6 = 0$ for $x$, in a different way than the way you solved the above equation.

6) In mathematics, what do you think is the difference between a "function" and an "equation?"

7) a) Find the vertex of the quadratic function below. Represent the vertex as a coordinate point $(x, y)$.

$$f(x) = 6x - x^2$$




b) Can you make up a quadratic function with no vertex?

8) If we want to find the x-intercepts (if any), the y-intercept, the line of symmetry and the vertex of the below quadratic functions; which one of them you think is the

easiest to solve? Which one is the most difficult? Pick one and write "Easiest" next to it. Pick one and write "Most difficult" right next to it. You do not need to solve... (Explain whether it is the easiest or most difficult in terms of finding the x-intercepts, y-intercept, line of symmetry, or the vertex)

$f(x) = 4x - x^2$

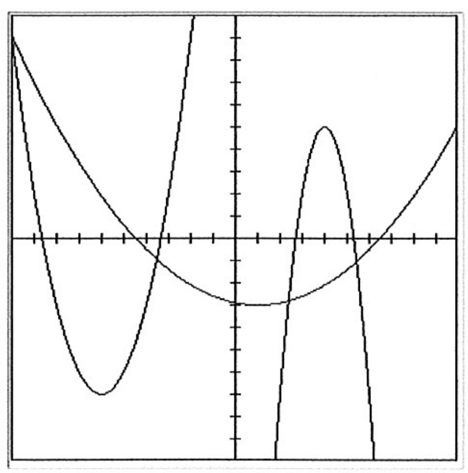
$g\ (x) = (6 - \mathrm{x})^2$

$h(x) = -6x - x^2$

$k(x) = 5(x - 3)(x + 2)$

9) Find the quadratic function that has a vertex at $(-2, 5)$ and whose graph passes through the point $(0, 9)$.

10) Look at the three graphs below. Choose a graph that you think is the easiest to represent with an equation. Write the function you chose in the form $f(x) = ax^2 + bx + c$. Explain the reason for your choice.



11) $f(x) = 4x - x^2$

$g\ (x) = (6 - \mathrm{x})^2$

$h(x) =- 6x - x^2$

$k(x) = 5(x-3)(x+2)$

Look at the above quadratic functions. Can you make up a similar function (to $f$, $g$, $h$, or $k$) that has no x-intercept?

12) Do you think $f(x) = px^2 - qx + 3$ and $g(x) = px^2 - qx + 6$ have the same x-intercepts? Or do they have different x-intercepts? Explain your reasoning.

Compare the graphs of $f$ and $g$.

Do you think $f(x) = px^2 - qx + 3$ and $g(x) = px^2 - qx + 6$ have the same y-intercept? Or do they have different y-intercepts? Explain your reasoning.

Do you think $f(x) = px^2 - qx + 3$ and $g(x) = px^2 - qx + 6$ have the same vertex? Or do they have different vertices? Explain your reasoning.

# APPENDIX C: INDIVIDUAL MATHEMATICS BACKGROUND SURVEY

Please feel free to write, draw or scribble anything you like in response to these questions. Your answers will be kept confidential.

1) How would you characterize your knowledge of mathematics?
2) How would you characterize your experiences in your past and current mathematics classes?
3) Which high school and college mathematics classes have you taken? How well did you do in them? You can write down the grade you earned (if you remember and prefer to share it).
4) What aspects of mathematics do you like the most? You can give a particular mathematics class and explain why you do like it the most, or you can give particular mathematics topics and explain why you like them the most.
5) What aspects of mathematics do you dislike the most? You can give a particular mathematics class and explain why you do dislike it the most, or you can give particular mathematics topics and explain why you do dislike them the most.
6) If you were to rate yourself on a scale between 1 (weak) to 5 (strong), where would you place yourself in:

   Arithmetic (basic mathematics, numbers, fractions, etc.): ____

   Algebra: ____

   Geometry: ____

   Upper level mathematics (pre-calc, calculus, etc.): ____

   Other mathematical topics (discrete mathematics, statistics, etc.): ____

7) Among these, was there any particular topic or subject that caused difficulty for you?

8) Is there anything else that you want to share about your math background?